The Scattering Operator in the Stepwise Waveguides

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Waveguides...

- Cross-section
- Boundary conditions
- Refractive index
- ... 
- Electrodynamics
- Acoustics
- Quantum mechanics
- Optics...
Waveguides interfaces...

Boundary conditions of different types

Different shape of cross-sections

Different materials (refractive index, potentials...)

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Scattering operator

- In- and out-waves: what they are?
- Scattering matrix and scattering operator
- What are mathematic definition?
- What are properties?
$S$-operator in waveguide...

\[
\begin{pmatrix}
A_{\text{out},\text{left}} \\
A_{\text{out},\text{right}}
\end{pmatrix}
= \hat{S}
\begin{pmatrix}
A_{\text{in},\text{left}} \\
A_{\text{in},\text{right}}
\end{pmatrix}
\]

\text{in-waves} (left and right)

\text{out-waves} (left and right)
Abstract waveguide operator

- Stepwise waveguide: Operator, selfadjoint extensions, scattering operator in the resolvent point
- Domain and properties
Homogeneous waveguide: mathematical model

Space: \( H_W = H \otimes L_2(\mathbb{R}), \quad \dim H \leq \infty \)

Operator:
\[
W = I \otimes \frac{d^2}{dz^2} + A \otimes I, \quad D_{\text{ess}}(W) = D(A) \otimes C_0^\infty(\mathbb{R}), \quad \dim H \leq \infty
\]

where \( \hat{A} \) is self-adjoint non-positive operator in \( H \) with compact resolvent.

Resolvent:
\[
\left( R_W(\lambda)f \right)(z) = -\frac{1}{2} \int_{-\infty}^{\infty} dy \left( \lambda - A \right)^{-1/2} e^{-A^{1/2}|z-y|} f(y)
\]

\( \text{Im}\lambda \neq 0, \quad \text{Re}\left(\lambda - \hat{A}\right)^{1/2} > 0 \)
Homogeneous waveguide: Limiting amplitude principle

\[ \left( R_w \left( -\omega^2 + i0 \right) f \right)(z) = \frac{i}{2} \int_{-\infty}^{\infty} dy \left( \omega^2 + \hat{A} \right)^{-1/2} e^{i \left( \omega^2 + \hat{A} \right)^{1/2} |z-y|} f(y) \]

\(-\omega^2 \notin \text{spec } \hat{A}\)

\[ D \left( R_w \left( -\omega^2 + i0 \right) \right) \supset H \otimes C_0^\infty (\mathbb{R}), \]

\[ R \left( R_w \left( -\omega^2 + i0 \right) \right) = \left( H \otimes L_2 (\mathbb{R}) \right) \bigoplus \left( V_\mu \otimes \left\{ e^{i (\omega^2 - \mu)^{1/2} |z|} \right\} \right), \]

This is “propagating” “Out”-states
Stepwise waveguide: mathematical model

Space: \[ H_w = H_+ \otimes L_2(\mathbb{R}_+) \oplus H_- \otimes L_2(\mathbb{R}_-) \supset \]

\[ \supset D(\hat{\mathcal{W}}_0) = H_+ \otimes C_0^\infty(\mathbb{R}_+) \oplus H_- \otimes C_0^\infty(\mathbb{R}_-) \]

Operator: \[ \hat{\mathcal{W}}_0 = \left( I_+ \otimes \frac{d^2}{dz^2} + \hat{A}_+ \otimes I \right) \oplus \left( I_- \otimes \frac{d^2}{dz^2} + \hat{A}_- \otimes I \right) \]

The models of the stepwise waveguide are the self-adjoint extensions \( \hat{\mathcal{W}}_V \) of \( \hat{\mathcal{W}}_0 \)

This operator is not self-adjoint!

\[ V \text{ is a parameter of extension} \]

Notations:

\[ B_\pm(\lambda) = (\lambda - A_\pm)^{1/2}, \quad \text{Im} B_\pm(\lambda) > 0, \quad B(\lambda) = B_+(\lambda) \oplus B_-(\lambda) \]
Stepwise waveguide: Self-adjoint extensions, von Neumann approach

The deficiency spaces: \( N_{\pm} = \ker(W_0^* \pm i) \)

The isomorphism: \( H_{A}^{\pm} \) are the augmentations of the spaces \( H_{\pm} \)

with respect the norm \( \|v\|_{H_{A}^{\pm}} = \|X_{\pm}^{-1}v\|_{H_{\pm}}, \ X_{\pm} = (B_{\pm}(i) + B_{\pm}(-i))^{1/2} \)

\[ N_{\pm} \approx H_{A}^{+} \oplus H_{A}^{-} = H_{A} \]

If \( u \in D(W_0) \), then \( u(0 \pm 0) \in H_{A}^{\pm} \)

The self-adjoint extensions are parameterized by unitary operators

\( V \) in \( H_{A} : \)

\[ u \in D(W_{V}) \Leftrightarrow Q_{0}u_{0} + Q_{1}u_{0}' = 0 \in H_{A}, \ u_{0}^e = u_{0}^e(0 + 0) \oplus u_{0}^e(0 - 0) \]

\[ Q_{0} = (I + V)^{-1}, \ Q_{1} = (B(i) + B(-i)V)^{-1}J, \ J = I_{+} \oplus (-I_{-}) \]
Stepwise waveguide: The resolvent and the scattering operator

\[ \left( R_{W_\nu}(\lambda)f \right)(z) = -\frac{1}{2} \int_0^\infty B_{\varepsilon}^{-1}(\lambda) e^{-|z-\xi|B_{\varepsilon}(\lambda)} f(\varepsilon\xi) d\xi - e^{-\varepsilon z B_{\varepsilon}(\lambda)} u_\varepsilon, \]

\[ \varepsilon = \text{sign } z \in \{+,-\} \]

\[ u_+ \oplus u_- = S(\lambda) \left( F_+^{\text{in}} \oplus F_-^{\text{in}} \right), \]

\[ F_\varepsilon^{\text{in}} = -\frac{1}{2} \int_0^\infty B_{\varepsilon}^{-1}(\lambda) e^{-\xi B_{\varepsilon}(\lambda)} f(\varepsilon\xi) d\xi \]

\[ S(\lambda) \equiv \left( -Q_0 + Q_1 B(\lambda) J \right)^{-1} \left( Q_0 + Q_1 B(\lambda) J \right) \]

This are the In- and Out- states

This is scattering operator
Questions...

\[ \hat{S}(\lambda) = \left( -\hat{Q}_0 + \hat{Q}_1 B(\lambda) J \right)^{-1} \left( \hat{Q}_0 + \hat{Q}_1 B(\lambda) J \right) \]

\[ Q_0 = (V + I)^{-1}, \quad Q_1 = \left( B(-i)V + B(i) \right)^{-1} J \]

The scattering operator is \textit{bounded operator} in the space $H_A$

All is O.K. if $\dim H_\pm < \infty$ ! (quantum mechanics)

1. Does the scattering operator must be bounded in $H$?
2. How to calculate the scattering operator in the infinite dimensional case?
The “wild” scattering operator

**Theorem.** There exists “abstract” infinite dimensional stepwise waveguide such that corresponding scattering operator is unbounded for all $\lambda \in \mathbb{C}$ as operator in the Hilbert space $H_+ \oplus H_-$, and its domain contains all finite elements.


**Problem:** What is the condition on the abstract waveguide (spaces, operators, extensions), such that corresponding scattering operator is bounded?
The “wild” scattering operator: construction

\[ W(\beta): \]

\[ H_+ = H_- = \mathbb{C}^2, \quad A_0 = \begin{pmatrix} 1 & 0 \\ 0 & \beta^4 \end{pmatrix}, \quad A_+ = UA_0 U^*, \quad U = U^* = U^{-1} = \frac{1}{1 + \beta^2} \begin{pmatrix} 1 - \beta^2 & 2\beta \\ 2\beta & \beta^2 - 1 \end{pmatrix} \]

\[ u(0^-) = u(0^+), \quad u(0^-) = u'(0^+) \]

\[ S(\beta) = \begin{pmatrix} R(\beta) & T(\beta) \\ UR(\beta) U^* & UR(\beta) U^* \end{pmatrix}, \quad R(\beta) = T(\beta) - E, \quad T_{12}(\beta) = O(\beta) \text{ when } \beta \gg |\lambda| \]

\[ W = \bigoplus_{n=1}^{\infty} W(\beta_n), \quad \beta_n \to \infty \]

\[ S_W = \bigoplus_{n=1}^{\infty} S(\beta_n) \]
The problem of approximation

What is finite dimensional approximation?

For space $H : \{H_n, T_n : H \to H_n\}_{n=1}^{\infty}$, $\|T_n u\|_n \xrightarrow{n \to \infty} \|u\|$

For operator in $H$:

\[
\begin{align*}
    H & \xrightarrow{A} H \\
    \downarrow T_n & \quad \downarrow T_n, \quad \|A_n T_n u - T_n A u\|_n \xrightarrow{n \to \infty} 0 \\
    H_n & \xrightarrow{A_n} H_n
\end{align*}
\]

Definition. Let $\hat{A}$ be the self-adjoint operator with compact resolvent in the Hilbert space $H$. The element $f \in H$ is called finite with respect to the operator $\hat{A}$ if its spectral expansion contains finite number of members.
Approximation and convergence for scattering operator

Construction of approximation. Let $P_n^\pm$ be the orthogonal projectors in $H_\pm$ onto the increasing finite parts of the spectra of operators $\hat{A}_\pm$ such that $P_n^\pm \rightarrow I_\pm$. Let further for any operator $Op^\pm$ in $H_\pm$ 
$Op^\pm_n = P_n^\pm Op^\pm P_n^\pm$ and for any operator $Op$ in $H = H_+ \oplus H_-$ 
$Op_{m,n} = P_{m,n} Op P_{m,n}$
with $P_{m,n} = P_m^+ \oplus P_n^-$

Let $S^{(m,n)}$ be the scattering operator for $A^+_m, A^-_m, Q^{0,1}_{m,n}$

Theorem. For any finite vector $f \in H$ there exists the limit in $H_A$ 
$$\lim_{m,n \rightarrow \infty} S^{(m,n)} P_{m,n} f = S f$$
where $S$ is the scattering operator for $A^+_0, A^-_0, Q^{0,1}_{m,n}$

This is the method for calculating $S$!
Multistep waveguide and composition of scattering operators

\[ \Delta_j = (a_{j-1}, a_j) \]

Mathematics: Multipoint self-adjoint extensions

\[ P_j(t) = e^{-\frac{i}{\hbar}(\lambda - A_j)^{1/2}} \]

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Multistep waveguide: Resolvent via scattering operators for interfaces

\[ u = R_w(\lambda) f : \]

\[ u(z) = -\frac{1}{2} B_j^{-1}(\lambda) \int_{\Delta_j} P_j(|z - \zeta|) f(\zeta) \, d\zeta + P_j(z - a_{j-1}) v_j^+ + P_j(a_j - z) v_j^- \]

\[
\begin{pmatrix}
    v_{j+1}^+ \\
    v_j^-
\end{pmatrix}
= S_j(\lambda)
\begin{pmatrix}
    P_{j+1}(|\Delta_{j+1}|) v_{j+1}^- + \psi_{j+1}^+ \\
    P_j(|\Delta_j|) v_j^+ + \psi_j^-
\end{pmatrix}, \quad v_0^+ = 0, \quad v_{N+1}^- = 0
\]

\[ \psi_j^e = -\frac{1}{2} B_j^{-1}(\lambda) \int_{\Delta_j} P_j \left( \varepsilon \zeta + a_{j+\text{sign} \varepsilon - 1} \right) f(\zeta) \, d\zeta \]

The approximation and these formulas is the background for field calculation in the multistep waveguide.