The Scattering Operator in the Stepwise Waveguides

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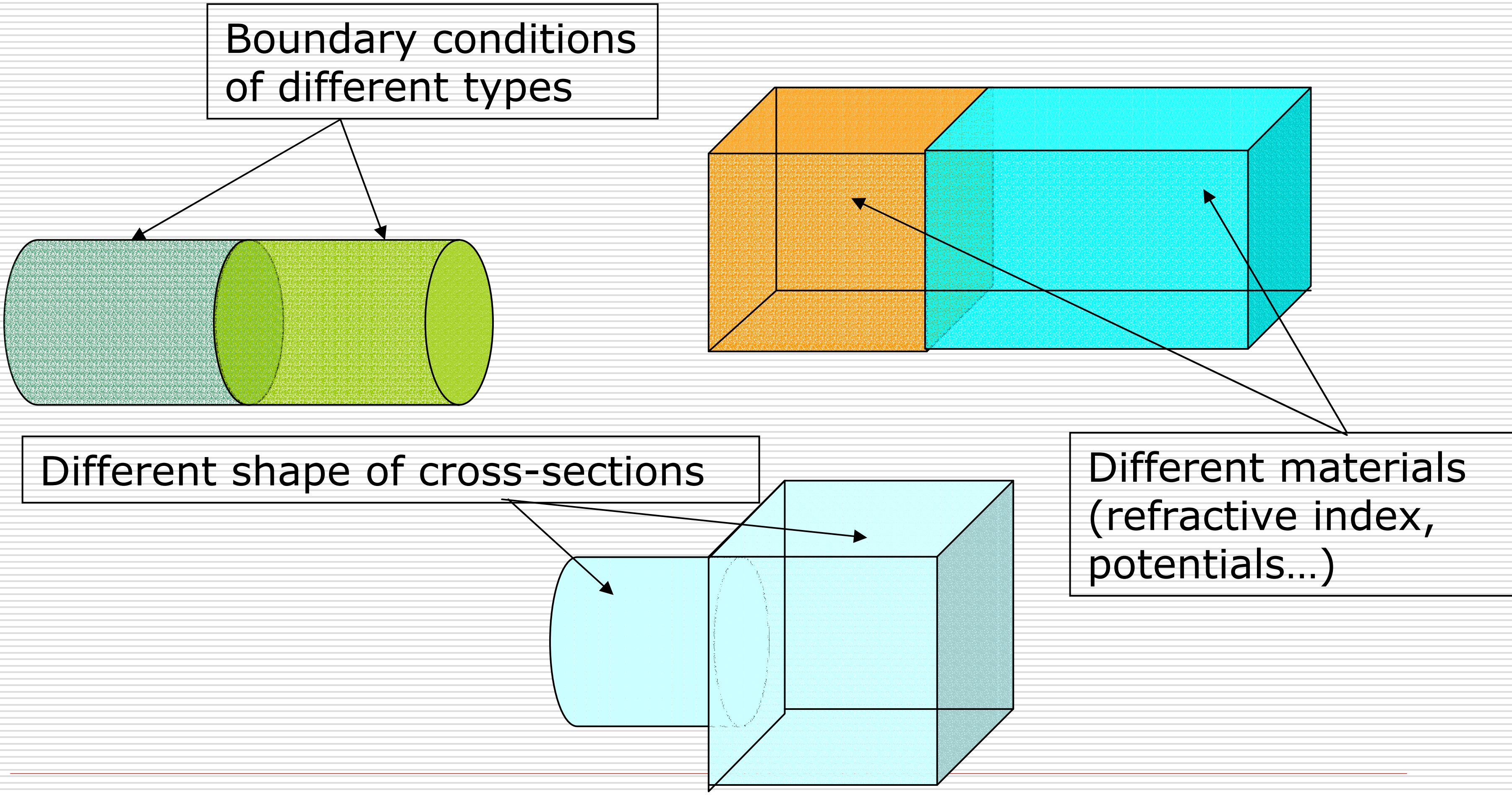
Waveguides…

- Cross-section
- Boundary conditions
- Refractive index
- …
- Electrodynamics
- Acoustics
- Quantum mechanics
- Optics…

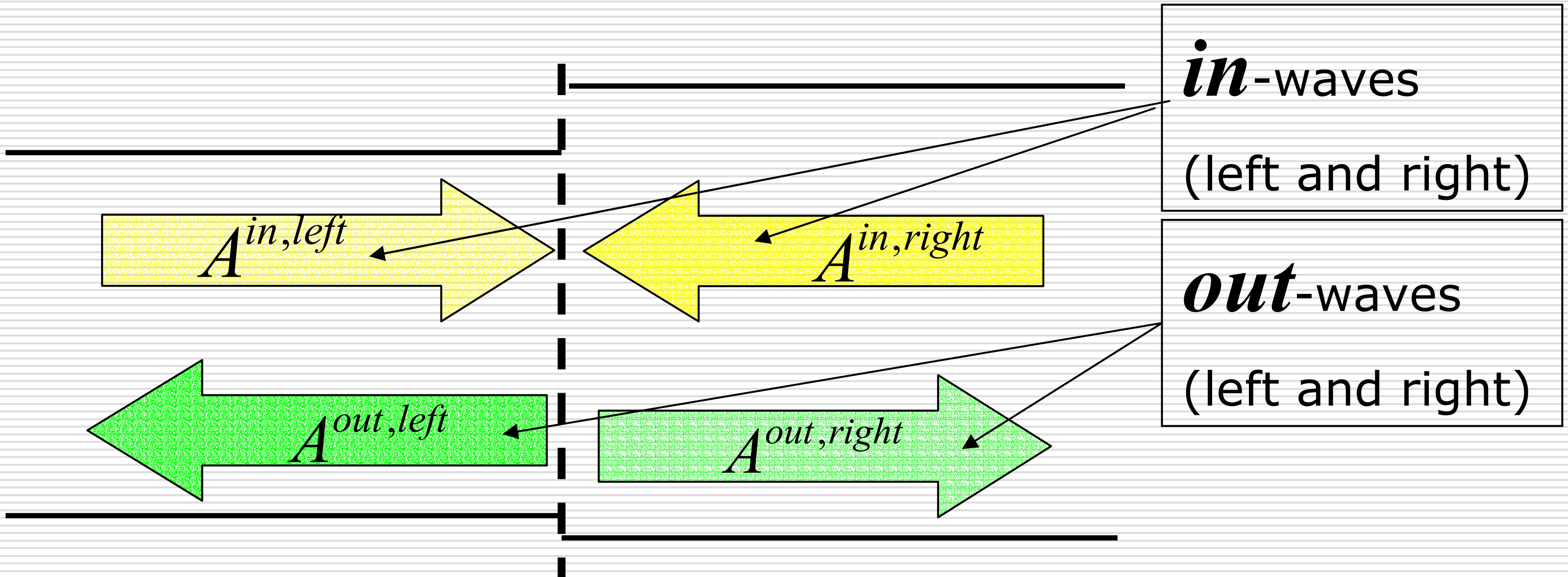
Waveguides interfaces…



Scattering operator

- In- and out- waves: what they are?
- Scattering matrix and scattering operator
- What are mathematic definition?
- What are properties?

S-operator in waveguide…



$$\begin{pmatrix} A^{out,left} \\ A^{out,right} \end{pmatrix} = \hat{S} \begin{pmatrix} A^{in,left} \\ A^{in,right} \end{pmatrix}$$

Abstract waveguide operator

- Homogeneous waveguide: Operator, resolvent, limiting amplitude principle.
- Stepwise waveguide: Operator, selfadjoint extensions, scattering operator in the resolvent point
- Domain and properties

Homogeneous waveguide: mathematical model

Space: $H_W = H \otimes L_2(\mathbb{R}), \qquad \dim H \le \infty$

Operator:

$$W = I \otimes \frac{d^2}{dz^2} + A \otimes I, \quad D_{ess}(W) = D(A) \otimes C_0^\infty(\mathbb{R}), \quad \dim H \le \infty$$

where $\hat{A}$ is ***self-adjoint*** non-positive operator in H with ***compact*** resolvent.

Resolvent: $$\left(R_W(\lambda) f\right)(z) = -\frac{1}{2}\int_{-\infty}^{\infty} dy\,(\lambda - A)^{-1/2} e^{-(\lambda - A)^{1/2}|z-y|} f(y)$$

$$\operatorname{Im}\lambda \ne 0, \quad \operatorname{Re}\left(\lambda - \hat{A}\right)^{1/2} > 0$$

Homogeneous waveguide: Limiting amplitude principle

$$\left(R_W\left(-\omega^2+i0\right)f\right)(z)=\frac{i}{2}\int_{-\infty}^{\infty}dy\left(\omega^2+\hat{A}\right)^{-1/2}e^{i\left(\omega^2+\hat{A}\right)^{1/2}|z-y|}f(y)$$

$$-\omega^2\notin\operatorname{spec}\hat{A}$$

$$D\left(R_W\left(-\omega^2+i0\right)\right)\supset H\otimes C_0^\infty(\mathbb{R}),$$

$$R\left(R_W\left(-\omega^2+i0\right)\right)=\left(H\otimes L_2(\mathbb{R})\right)\oplus\left(\bigoplus_{-\mu\in spec\,A,\,\mu<\omega^2}\left(V_\mu\otimes\left\{e^{i\left(\omega^2-\mu\right)^{1/2}|z|}\right\}\right)\right)$$

This is "propagating" "Out"-states

Stepwise waveguide: mathematical model

Space: $H_W = H_+ \otimes L_2(\mathbb{R}_+) \oplus H_- \otimes L_2(\mathbb{R}_-) \supset$

$$\supset D(W_0) = H_+ \otimes C_0^\infty(\mathbb{R}_+) \oplus H_- \otimes C_0^\infty(\mathbb{R}_-)$$

Operator: $W_0 = \left(I_+ \otimes \frac{d^2}{dz^2} + \hat{A}_+ \otimes I \right) \oplus \left(I_- \otimes \frac{d^2}{dz^2} + \hat{A}_- \otimes I \right)$

The models of the stepwise waveguide are the self-adjoint extensions $\hat{W}_V$ of $\hat{W}_0$

This operator is not self-adjoint!

V is a parameter of extension

Notations:

$$B_\pm(\lambda) = (\lambda - A_\pm)^{1/2}, \ \operatorname{Im} B_\pm(\lambda) > 0, \ B(\lambda) = B_+(\lambda) \oplus B_-(\lambda)$$

Stepwise waveguide: Self-adjoint extensions, von Neumann approach

The deficiency spaces: $N_{\pm} = \ker\left(W_0^* \pm i\right)$

The isomorphism: $H_A^{\pm}$ are the augmentations of the spaces $H_{\pm}$ with respect the norm $\|v\|_{H_A^{\pm}} = \left\|X_{\pm}^{-1} v\right\|_{H_{\pm}}$, $X_{\pm} = \left(B_{\pm}(i) + B_{\pm}(-i)\right)^{1/2}$

$$N_{\pm} \approx H_A^{+} \oplus H_A^{-} = H_A$$

$$\text{If } u \in D\left(W_0\right), \text{ then } u\left(0 \pm 0\right) \in H_A^{\pm}$$

The self-adjoint extensions are parameterized by unitary operators V in H_A:

$$u \in D\left(W_V\right) \Leftrightarrow Q_0 u_0 + Q_1 u_0' = 0 \in H_A, \quad u_0^{\varepsilon} = u^{\varepsilon}\left(0+0\right) \oplus u^{\varepsilon}\left(0-0\right)$$

$$Q_0 = \left(I + V\right)^{-1}, \quad Q_1 = \left(B(i) + B(-i)V\right)^{-1} J, \quad J = I_+ \oplus \left(-I_-\right)$$

Stepwise waveguide: The resolvent and the scattering operator

$$\left(R_{W_V}(\lambda)f\right)(z) = -\frac{1}{2}\int_0^\infty B_\varepsilon^{-1}(\lambda)\,e^{-|z-\varsigma|B_\varepsilon(\lambda)} f(\varepsilon\varsigma)\,d\varsigma - e^{-\varepsilon z B_\varepsilon(\lambda)}u_\varepsilon,$$

$$\varepsilon = \operatorname{sign} z \in \{+,-\}$$

$$u_+ \oplus u_- = S(\lambda)\left(F_+^{in} \oplus F_-^{in}\right), \qquad F_\varepsilon^{in} = -\frac{1}{2}\int_0^\infty B_\varepsilon^{-1}(\lambda)\,e^{-\varsigma B_\varepsilon(\lambda)} f(\varepsilon\varsigma)\,d\varsigma$$

$$S(\lambda) \equiv \left(-Q_0 + Q_1 B(\lambda) J\right)^{-1}\left(Q_0 + Q_1 B(\lambda) J\right)$$

This are the In- and Out- states

This is scattering operator

Questions…

$$\hat{S}(\lambda) \equiv \left(-\hat{Q}_0 + \hat{Q}_1 \hat{B}(\lambda) J\right)^{-1} \left(\hat{Q}_0 + \hat{Q}_1 \hat{B}(\lambda) J\right)$$

$$Q_0 = (V + I)^{-1}, \; Q_1 = \left(B(-i)V + B(i)\right)^{-1} J$$

The scattering operator is *bounded operator* in the space H_A

All is O.K. if $\dim H_{\pm} < \infty$! (quantum mechanics)

1. Does the scattering operator must be bounded in *H?*
2. How to calculate the scattering operator in the infinite dimensional case?

The “wild” scattering operator

Theorem. *There exists “abstract” infinite dimensional stepwise waveguide such that corresponding scattering operator is unbounded for all* $\lambda \in \mathbb{C}$ *as operator in the Hilbert space* $H_+ \oplus H_-$*, and its domain contains all finite elements.*

(A.V.Lebedev, Master Thesis, Nizhnii Novgogrod State University,
Dept “High School of General and Applied Physics”, 2005)

Problem: *What is the condition on the abstract waveguide (spaces, operators, extensions), such that corresponding scattering operator is bounded?*

The "wild" scattering operator: construction

$W(\beta)$:

$$H_+ = H_- = \mathbb{C}^2,\ A_- = \begin{pmatrix} 1 & 0 \\ 0 & \beta^4 \end{pmatrix},\ A_+ = UA_-U^*,\ U = U^* = U^{-1} = \frac{1}{1+\beta^2}\begin{pmatrix} 1-\beta^2 & 2\beta \\ 2\beta & \beta^2 - 1 \end{pmatrix}$$

$$u(0-0) = u(0+0),\ u(0-0) = u'(0+0)$$

$$S(\beta) = \begin{pmatrix} R(\beta) & T(\beta) \\ UR(\beta)U^* & UR(\beta)U^* \end{pmatrix},\ R(\beta) = T(\beta) - E,\ T_{12}(\beta) = O(\beta) \text{ when } \beta \gg |\lambda|$$

$$W = \bigoplus_{n=1}^{\infty} W(\beta_n),\ \beta_n \to \infty$$

$$S_W = \bigoplus_{n=1}^{\infty} S(\beta_n)$$

The problem of approximation

What is finite dimensional approximation?

For space $H : \{H_n,\ T_n : H \to H_n\}_{n=1}^{\infty},\ \|T_n u\|_n \xrightarrow{n\to\infty} \|u\|$

For operator in H :

$$\begin{array}{ccc} H & \xrightarrow{A} & H \\ \downarrow T_n & & \downarrow T_n \\ H_n & \xrightarrow{A_n} & H_n \end{array}, \quad \|A_n T_n u - T_n A u\|_n \xrightarrow{n\to\infty} 0$$

Definition. *Let $\hat{A}$ be the self-adjoint operator with compact resolvent in the Hilbert space H. The element $f \in H$ is called finite with respect to the operator $\hat{A}$ if its spectral expansion contains finite number of members*

Approximation and convergence for scattering operator

Construction of approximation. Let $P_n^{\pm}$ be the orthogonal projectors in $H_{\pm}$ onto the increasing finite parts of the spectra of operators $\hat{A}_{\pm}$ such that $P_n^{\pm} \underset{n\to\infty}{\overset{s-\lim}{\to}} I_{\pm}$. Let further for any operator $Op^{\pm}$ in $H_{\pm}$ $Op_n^{\pm} = P_{\pm}^n Op_{\pm} P_{\pm}^n$ and for any operator Op in $H = H_+ \oplus H_-$ $Op_{m,n} = P_{m.n} Op P_{m.n}$

with $P_{m,n} = P_m^+ \oplus P_n^-$

Let $S^{(m,n)}$ be the scattering operator for $A_m^+, A_n^-, Q_{m,n}^{0,1}$

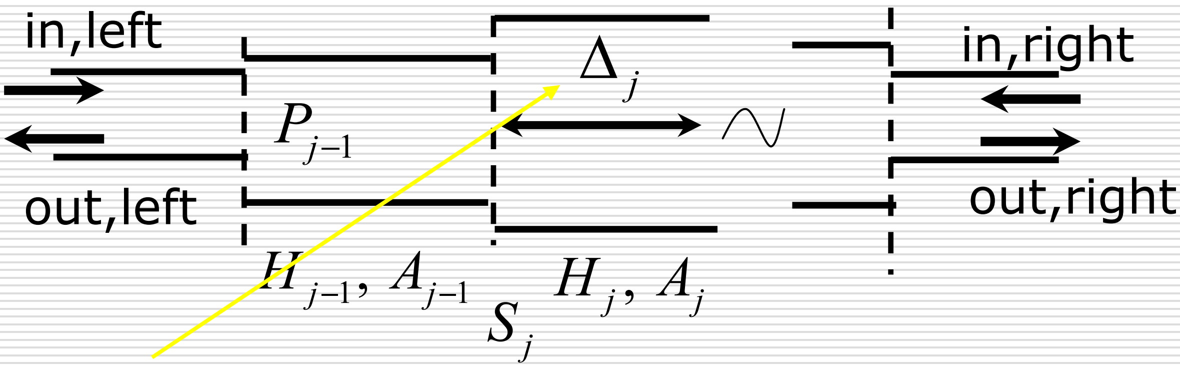
Theorem. *For any finite vector* $f \in H$ *there exists the limit in* H_A

$$\lim_{m,n\to\infty} S^{(m,n)} P_{m,n} f = Sf$$

where S *is the scattering operator for* $A_+, A_-, Q_{0,1}$

This is the method for calculating ***S!***

Multistep waveguide and composition of scattering operators



$$P_j(t) = e^{-|t|(\lambda - A_j)^{1/2}}$$

"propagator"

$$\Delta_j = (a_{j-1}, a_j)$$

Mathematics:
Multipoint self-adjoint extensions

Multistep waveguide: Resolvent via scattering operators for interfaces

$$u = R_W(\lambda) f:$$

$$u(z) = -\frac{1}{2} B_j^{-1}(\lambda) \int_{\Delta_j} P_j(|z-\zeta|) f(\zeta) d\zeta + P_j(z - a_{j-1}) v_j^+ + P_j(a_j - z) v_j^-$$

$$\begin{pmatrix} v_{j+1}^+ \\ v_j^- \end{pmatrix} = S_j(\lambda) \begin{pmatrix} P_{j+1}(|\Delta_{j+1}|) v_{j+1}^- + \psi_{j+1}^+ \\ P_j(|\Delta_j|) v_j^+ + \psi_j^- \end{pmatrix}, \quad v_0^+ = 0, \; v_{N+1}^- = 0$$

Matrix sweep method

$$\psi_j^\varepsilon = -\frac{1}{2} B_j^{-1}(\lambda) \int_{\Delta_j} P_j\left(\varepsilon\zeta + a_{j+\frac{\operatorname{sign}\varepsilon - 1}{2}}\right) f(\zeta) d\zeta$$

The approximation and these formulas is the background for field calculation in the multistep waveguide