μ- scale invariant linear relations

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$$UTU^* = \mu T$$

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The multi-valued operator case

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Outline

Let $\mathfrak{H}$ be a complex Hilbert space.

A linear subspace A in the Cartesian product $\mathfrak{H} \times \mathfrak{H}$ is called a linear relation in $\mathfrak{H}$.

$$\begin{aligned} \operatorname{dom} A &= \{ f \in \mathfrak{H} : \{f, f'\} \in A \text{ for some } f' \in \mathfrak{H} \}, \\ \operatorname{ran} A &= \{ f' \in \mathfrak{H} : \{f, f'\} \in A \text{ for some } f \in \mathfrak{H} \}, \\ \ker A &= \{ f \in \mathfrak{H} : \{f, 0\} \in A \}, \\ \operatorname{mul} A &= \{ f' \in \mathfrak{H} : \{0, f'\} \in A \}. \end{aligned}$$

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A relation A has a formal inverse $A^{-1} = \{ \{f', f\} : \{f, f'\} \in A \}$.

Let A and B be linear relations in $\mathfrak{H}$. Then the product BA is the linear relation defined by

$$BA = \{ \{f, g\} \in \mathfrak{H} \times \mathfrak{H} : \{f, \varphi\} \in A, \{\varphi, g\} \in B \text{ for some } \varphi \in \mathfrak{H} \}.$$

For any $\lambda \in \mathbb{C}$ the relation $A - \lambda$ is defined by
$A - \lambda = \{ \{f, f' - \lambda f\} : \{f, f'\} \in A \}$.

Linear relations in Hilbert spaces

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Let P be the orthogonal projection from $\mathfrak{H}$ onto $(\mathrm{mul}\, A)^{\perp}$.

Then each $\{f,f'\} \in A$ can be uniquely decomposed as

$$\{f,f'\} = \{f, Pf'\} + \{0, (I-P)f'\}.$$

The linear relation

$$A_s = \{\, \{f,f'\} : \{f,f'\} \in A,\, f' = Pf' \,\} = \{\, \{f, Pf'\} : \{f,f'\} \in A \,\}$$

is called the (orthogonal) operator part of A: it is the graph of an operator from $\mathfrak{H}$ to $P\mathfrak{H} \subset \mathfrak{H}$.

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Define the linear relation A_∞ by

$$A_\infty = A \cap (\{0\} \times \mathfrak{H}).$$

Then the linear relation A admits the orthogonal decomposition

$$A = A_s \oplus A_\infty,$$

where the orthogonal sum is with respect to the inner product on $\mathfrak{H} \times \mathfrak{H}$.

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The adjoint A^* of a linear relation A in $\mathfrak{H}$ is the linear relation in $\mathfrak{H}$, defined by

$$A^* = \{\, \{f',f\} \in \mathfrak{H} \times \mathfrak{H} : \langle \{f',f\}, \{h,h'\} \rangle = 0,\ \{h,h'\} \in A \,\},$$

where

$$\langle \{f',f\}, \{h,h'\} \rangle = (f,h) - (f',h'), \quad \{f,f'\}, \{h,h'\} \in \mathfrak{H} \times \mathfrak{H}.$$

The adjoint A^* is automatically closed and linear.

The resolvent set $\rho(A)$ of a closed linear relation A in $\mathfrak{H}$ is defined by:

$$\rho(A) = \{\, \lambda \in \mathbb{C} : (A-\lambda)^{-1} \in [\mathfrak{H}] \,\},$$

where $[\mathfrak{H}]$ denotes the set of all bounded linear operators on $\mathfrak{H}$ and $(A-\lambda)^{-1}$ is identified with its graph.

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$$(A^*)^{-1} = (A^{-1})^*$$

$$B^*A^* \subset (AB)^*$$

Lemma

Assume that A is a linear relation in $\mathfrak{H}$ and U an invertible bounded operator.Then the following two identities hold

$$(UA)^* = A^*U^*, \quad (AU)^* = U^*A^*$$

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Linear relations in Hilbert spaces

A linear relation A in $\mathfrak{H}$ is said to be symmetric if $(f',f) \in \mathbb{R}$ for all $\{f,f'\} \in A$, or, equivalently, if $A \subset A^*$.

The relation A is said to be selfadjoint if $A = A^*$.

If the relation A is selfadjoint, then $\overline{\operatorname{dom} A} = (\operatorname{mul} A)^\perp$ and A_s is a (densely defined) selfadjoint operator in $\overline{\operatorname{dom} A}$.

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μ-scale invariant relations

Let U be a unitary operator in a separable complex Hilbert space $\mathfrak{H}$ and let $\mu \in \mathbb{C}\backslash\{0\}$.

Definition

A linear relation S is said to be μ- scale invariant with respect to U if the following identity is satisfied:

$$USU^* = \mu S.$$

$$U^*(\operatorname{dom} S) \subset \operatorname{dom} S$$

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Assume that S is a linear relation in $\mathfrak{H}$ which is μ- scale invariant with respect to U. Then

1. the inverse relation S^{-1} is μ^{-1}- scale invariant with respect to U;
2. the relation S is also μ- scale invariant with respect to the unitary transformation U^n, $n \in \mathbb{N}$. That is $U^n S U^{*n} = \mu^n S$, for all $n \in \mathbb{N}$;
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(ii)

This follows by induction on $n \in \mathbb{N}$.

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$$\begin{aligned} US^*U^* &= (U^*)^* S^* U^* = (SU^*)^* U^* \\ &= (USU^*)^* = (\mu S)^* = \bar{\mu} S^*. \end{aligned}$$

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Closed nonnegative forms

Let $\mathfrak{t} = \mathfrak{t}[\cdot,\cdot]$ be a nonnegative form in the Hilbert space $\mathfrak{H}$ with domain $\operatorname{dom}\mathfrak{t}$.

The inclusion $\mathfrak{t}_1 \subset \mathfrak{t}_2$ for nonnegative forms $\mathfrak{t}_1$ and $\mathfrak{t}_2$ is defined by

$$\operatorname{dom}\mathfrak{t}_1 \subset \operatorname{dom}\mathfrak{t}_2, \quad \mathfrak{t}_1[h] = \mathfrak{t}_2[h], \quad h \in \operatorname{dom}\mathfrak{t}_1 .$$

The nonnegative form $\mathfrak{t}$ is closed if

$$h_n \to h, \quad \mathfrak{t}[h_n - h_m] \to 0, \quad h_n \in \operatorname{dom}\mathfrak{t}, \quad h \in \mathfrak{H}, \quad m,n \to \infty,$$

imply that $h \in \operatorname{dom}\mathfrak{t}$ and $\mathfrak{t}[h_n - h] \to 0$.

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$\mathfrak{t}_1 \subset \mathfrak{t}_2$ implies $\mathfrak{t}_1 \geq \mathfrak{t}_2$

There is a one-to-one correspondence between all closed nonnegative forms $\mathfrak{t}$ in $\mathfrak{H}$ and all nonnegative selfadjoint relations A in $\mathfrak{H}$ via

$$\operatorname{dom} A \subset \operatorname{dom} \mathfrak{t},$$

and

$$\mathfrak{t}[f,g] = (A_s f, g), \quad f \in \operatorname{dom} A, \quad g \in \operatorname{dom} \mathfrak{t}.$$

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Closed nonnegative forms

let the nonnegative form $\mathfrak{t}$ and the nonnegative selfadjoint relation A be connected as above. If $\mathfrak{t} \geq 0$ or, equivalently, $A \geq 0$, then

$$\operatorname{dom} \mathfrak{t} = \operatorname{dom} A_s^{1/2},$$

and

$$\mathfrak{t}[f, g] = (A_s^{1/2} f, A_s^{1/2} g), \quad f, g \in \operatorname{dom} \mathfrak{t}.$$

Nonnegative selfadjoint extensions of nonnegative relations

Let S be a not necessarily closed nonnegative relation in a Hilbert space $\mathfrak{H}$.

One nonnegative selfadjoint extension can be constructed as follows:

Let $\{f, f'\}, \{h, h'\} \in S$ and define $\mathfrak{s}[f, h] = (f', h)$, so that $\mathfrak{s}$ is a nonnegative form on $\operatorname{dom} \mathfrak{s} = \operatorname{dom} S$. The closure $\mathfrak{t}$ of the form $\mathfrak{s}$ is nonnegative (and is equal to the form obtained by starting with the closure of S) and gives rise to a nonnegative selfadjoint relation which is called the Friedrichs extension S_F of S.

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The so-called Kreĭn-von Neumann extension S_N of S is defined by

$$S_N = ((S^{-1})_F)^{-1}.$$

Theorem

Assume that S is a nonnegative linear relation in $\mathfrak{H}$ which is μ- scale invariant with respect to U. Then

1. the Friedrichs extension S_F of S is μ- scale invariant with respect to U;
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Proof

Let $\{f,f'\} \in S_F$. Then there exists a sequence $(\{f_n,f'_n\}) \subset S$ such that $f_n \to f$, and

$$(f'_n - f'_m, f_n - f_m) \to 0, \quad \text{as} \quad m,\, n \to \infty.$$

It follows from $\{f_n, \mu f'_n\} \in \mu S = USU^*$ that

$$\{U^* f_n, \mu U^* f'_n\} \in S. \tag{1}$$

Furthermore,

$$U^* f_n \to U^* f, \tag{2}$$

and

$$(\mu U^* f'_n - \mu U^* f'_m, f_n - f_m) \to 0, \quad \text{as} \quad m,\, n \to \infty. \tag{3}$$

Proof

Let $\{f, f'\} \in S_F$. Then there exists a sequence $(\{f_n, f_n'\}) \subset S$ such that $f_n \to f$, and

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$$\{U^* f_n, \mu U^* f_n'\} \in S. \tag{1}$$

Furthermore,

$$U^* f_n \to U^* f, \tag{2}$$

and

$$(\mu U^* f_n' - \mu U^* f_m', f_n - f_m) \to 0, \quad \text{as} \quad m, n \to \infty. \tag{3}$$

Proof

Let $\{f,f'\} \in S_F$. Then there exists a sequence $(\{f_n,f_n'\}) \subset S$ such that $f_n \to f$, and

$$(f_n' - f_m', f_n - f_m) \to 0, \quad \text{as} \quad m,\, n \to \infty.$$

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Since $\{f, \mu f'\} \in \mu S_F \subset \mu S^* = US^* U^*$ it follows that

$$\{U^* f_n, \mu U^* f_n'\} \in S^*. \tag{4}$$

A combination of (2), (3) and (4) leads to $\{U^* f, \mu U^* f'\} \in S_F$, so that $\{f, \mu f'\} \in US_F U^*$. This implies that $\mu S_F \subset US_F U^*$. Since both μS_F and $US_F U^*$ are selfadjoint linear relations it follows that $\mu S_F = US_F U^*$.

Since $\{f, \mu f'\} \in \mu S_F \subset \mu S^* = U S^* U^*$ it follows that

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Extremal extensions

A nonnegative selfadjoint extension $\widetilde{A}$ of S is called *extremal* when

$$\inf\{\, (f' - h', f - h) : \, \{h, h'\} \in S \,\} = 0 \quad \text{for all} \quad \{f, f'\} \in \widetilde{A}.$$

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The invariance of nonnegative selfadjoint relations

Hassi, Sandovici, de Snoo, Winkler – 2006

Let S be a nonnegative relation in a Hilbert space $\mathfrak{H}$. Then the following statements are equivalent:

1. $\widetilde{A}$ is an extremal extension of S;
2. $\widetilde{A} = R_{\mathfrak{L}}^{*} R_{\mathfrak{L}}^{**}$ for some subspace $\mathfrak{L}$ such that $\operatorname{dom} S \subset \mathfrak{L} \subset \operatorname{dom} S_N^{1/2}$;
3. $\widetilde{A}$ is a nonnegative selfadjoint extension of S whose corresponding form $\widetilde{\mathfrak{t}}$ satisfies $\widetilde{\mathfrak{t}} \subset \mathfrak{t}_N$.

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The invariance of the extremal extensions

Assume that S is a nonnegative linear relation in $\mathfrak{H}$ which is μ- scale invariant with respect to U.

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Example A

A purely multi-valued relation

Assume that U is a unitary operator in the Hilbert space $\mathfrak{H}$ such that $U^*(\mathfrak{K}) = \mathfrak{K}$, where $\mathfrak{K}$ is a not necessarily closed subspace of $\mathfrak{H}$.

Consider the purely multi-valued relation S in $\mathfrak{H}$ defined by

$$S = \{0\} \times \mathfrak{K}.$$

Then S is closed if and only if $\mathfrak{K}$ is closed, and it is μ invariant with respect to U for any $\mu > 0$.

The adjoint S^* is given by

$$S^* = \mathfrak{K}^\perp \times \mathfrak{H},$$

so that $\operatorname{mul} S^* = \mathfrak{H}$.

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The Kreĭn-von Neumann extension S_N and the Friedrichs extension S_F are given by

$$S_N = \mathfrak{K}^\perp \times \overline{\mathfrak{K}}, \quad S_F = \{0\} \times \mathfrak{H},$$

There exists a one to-one-correspondence between the class of all extremal extensions $\widetilde{A}$ of S and the set of all closed subspaces $\mathfrak{L}$ of $\mathfrak{K}^\perp$. The correspondence is given by

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Example B

Assume that $\mu > 0$, $\mu \neq 1$, and that U is the scaling transformation on the Hilbert space $\mathfrak{H} = L^2(0, \infty)$ defined by

$$(Uf)(x) = \mu^{-\frac{1}{4}} f\left(\mu^{-\frac{1}{2}} x\right), \quad f \in L^2(0, \infty).$$

Consider T the maximal operator on the Sobolev space $H^{2,2}(0, \infty)$ defined by

$$T = -\frac{d^2}{dx^2}, \quad \operatorname{dom} T = H^{2,2}(0, \infty).$$

The linear operator S defined by

$$S = T^* \restriction_{\operatorname{dom} S}, \operatorname{dom} S = \{f \in \operatorname{dom} T \ : f(0) = f'(0) = 0\}$$

is a closed nonnegative operator with deficiency indices $(1, 1)$.

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All the operators S, S^*, S_F and S_N are μ- scale invariant with respect to the transformation U.

Any other nonnegative selfadjoint extension of S different from the extremal ones can be obtained by the restriction of T to the domain

$$\operatorname{dom} A_s = \{f \in \operatorname{dom} T \; : \; f'(0) = sf(0)\}$$

for some $s > 0$.

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