A Beals type criterion for pseudidifferential operators with a magnetic field

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Work done in collaboration with
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Introduction

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Purpose

To elaborate a completely covariant functional calculus, enough powerful to deal with a large numebr of problems concerning quantum systems in a non-homogenuous magnetic field.

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Main idea

Concentrate on the modified symplectic structure defined by the magnetic field, that is gauge independent, and develop an associated twisted Weyl calculus that allows to work with the quantum observables in a completely representation free way.

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- As a first step towrds a definition of Twisted Fourier Integral Operators (in the spirit of that of Bony) and to study their relation with symplectic transforms with respect to the modified symplectic form.
- As a possible consequence to study the unitary evolution associated to a quantum Hamiltonian with magnetic field and its large time or semiclassical limits.

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   - The magnetic Scrödinger representation
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   - *Magnetic* composition of symbols
   - L^2-continuity
   - *Magnetic* Sobolev spaces
   - An inversion result

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   - Idea of the proof
   - The case $m \neq 0$

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- The magnetic field is described by a closed 2-form B on $\mathcal{X}$.
- To the magnetic field we can canonically associate a perturbation of the canonical symplectic form on Ξ:

$$\sigma_z^B((x, \xi), (y, \eta)) := \sigma((x, \xi), (y, \eta)) + B(z)(x, y), \quad \forall z \in \mathcal{X}$$

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- Gauge transformations: $A \mapsto A' = A + d\Phi$; so that $B = dA = dA'$.

The magnetic canonical variables

- For the Hamiltonian formalism, the Lorentz force is equivalent to the replacement of the usual canonical pair of variables

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  defined once we have chosen a vector potential A for B.
- Appearently this prescription is highly non-unique due to the gauge ambiguity.
- In fact it is only the symplectic form σ^B that is important for the Hamiltonian evolution. But in order to see this fact one has to work directly in the algebra of observables and not in a Hilbertian representation.

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representing the canonical variables in the magnetic field.
- We shall use the unitary groups associated to the above $2n$ self-adjoint operators and define the Magnetic Weyl system:

$$W^A((x,\xi)) := e^{-i<\xi,(Q+x/2)>} e^{-i\int_{[Q,Q+x]} A} e^{i<x,P>}$$

The magnetic Schrödinger representation (2)

- For any test function $f : \Xi \longrightarrow \mathbb{C}$ we define the associated magnetic Weyl operator:

$$\mathfrak{Op}^A(f) := \int_\Xi dX \hat{f}(X) W^A(X) \in \mathbb{B}[\mathcal{H}]$$

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- It defines a linear bijection [M.P., *J. Math. Phys. 04*].

The *magnetic* algebra of quantum observables (1)

The magnetic Moyal product

The above functional calculus induces a *magnetic composition* on the complex linear space of test functions $\mathcal{S}(\Xi)$:

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Explicitely we have:

$$(f \sharp^B g)(X) := 4^n \int_\Xi dY \int_\Xi dZ \, e^{-i \int_{\mathcal{T}_X(Y,Z)} \sigma^B} f(X-Y) \, g(X-Z)$$

where $\mathcal{T}_X(Y,Z)$ is the triangle in Ξ having vertices:

$$X - Y - Z, \quad X + Y - Z, \quad X - Y + Z.$$

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Remark: For any 3 test functions f, g, h we have
$(f, g\sharp^B h) = (f\sharp^B g, h) = (h, f\sharp^B g) = (h\sharp^B f, g) = (g, h\sharp^B f)$.

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The magnetic Moyal algebra

We set:

$$\mathfrak{M}^B(\Xi) := \left\{ F \in \mathcal{S}'(\Xi) \mid F\sharp^B \phi \in \mathcal{S}(\Xi), \phi\sharp^B F \in \mathcal{S}(\Xi), \forall \phi \in \mathcal{S}(\Xi) \right\}$$

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This defines a *-algebra for
the *composition* $\sharp^B$ (that we can extend by duality)
and the usual complex conjugation as *-*conjugation*.

The *magnetic* algebra of quantum observables (3)

Proposition [M.P., *J. Math. Phys. 04*]

The space of indefinitely differentiable functions with uniform polynimial growth on $\mathcal{X}$ is contained in $\mathfrak{M}^B(\Xi)$.

The norm (1)

Observation: *Gauge covariance*

The Schrödinger representations associated to any two gauge-equivqlent vector potentials are unitarily equivqlent:

$$A' = A + d\varphi \quad \Rightarrow \quad \mathfrak{Op}^{A'}(f) = e^{i\varphi(Q)}\mathfrak{Op}^{A}(f)e^{-i\varphi(Q)}.$$

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Thus the familly:

$$\mathfrak{C}^B(\Xi) := \left\{ F \in \mathcal{S}'(\Xi) \mid \mathfrak{Op}^A(F) \in \mathbb{B}[L^2(\mathcal{X})] \right\}$$

(defined once we have chosen a vector potential A for B)

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(defined once we have chosen a vector potential A for B) does not depend on the choice of A
but only on the magnetic field B.

The norm (2)

- On $\mathfrak{C}^B$ we can define the map:

$$\|F\|_B := \|\mathfrak{Op}^A(F)\|_{\mathbb{B}[L^2(\mathcal{X})]}$$

that does not depend on the choice of A and is in fact a C^*-norm on $\mathfrak{C}^B$.

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  that does not depend on the choice of A and is in fact a C^*-norm on $\mathfrak{C}^B$.
- $\mathfrak{C}^B$ is a C^*-algebra isomorphic to $\mathbb{B}[L^2(\mathcal{X})]$.

Structure

Symbols

We shall use the following Hörmander type symbols:

Definition

For $m \in \mathbb{R}$ and $0 \leq \delta \leq \rho \leq 1$ we define:
$\forall F \in C^\infty(\Xi)$ the family of seminorms

$$|F|^{(m;\rho,\delta)}_{(a,\alpha)} := \sup_{(x,\xi)\in\Xi} < \xi >^{-m+\rho|\alpha|-\delta|a|} \left|(\partial_x^a \partial_\xi^\alpha F)(x,\xi)\right|,$$

the Frechet space

$$S^m_{\rho,\delta}(\Xi) := \left\{ F \in C^\infty(\Xi) \mid \forall (a,\alpha),\, |F|^{(m;\rho,\delta)}_{(a,\alpha)} < \infty \right\}.$$

Observables

Hypothesis

The magnetic field B has components of class $C^\infty_{\text{pol}}(\mathcal{X})$.

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By usual oscilatory integrals techniques we prove that the symbols define 'good' quantum observables:

Proposition [I.M.P., *Proc. RIMS 07*]

For $m \in \mathbb{R}$ and $0 \leq \delta \leq \rho \leq 1$ we have $S^m_{\rho,\delta}(\Xi) \subset \mathfrak{M}^B(\Xi)$.

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Definition

Choosing any vector potential A for $B = dA$ we define the associated *magnetic* pseudodifferential operators on $\mathcal{H} := L^2(\mathcal{X})$:

$$\mathbf{\Psi}^m_{\rho,\delta}(A) := \mathfrak{Op}^A[S^m_{\rho,\delta}(\Xi)].$$

Composition of symbols

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Theorem [I.M.P., *Proc. RIMS 07*]

For any m_1 and m_2 in $\mathbb{R}$ and for any $0 \leq \delta \leq \rho \leq 1$ we have:

$$S^{m_1}_{\rho,\delta}(\Xi)\,\sharp^B\,S^{m_2}_{\rho,\delta}(\Xi) \subset S^{m_1+m_2}_{\rho,\delta}(\Xi).$$

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$$S^{m_1}_{\rho,\delta}(\Xi) \sharp^B S^{m_2}_{\rho,\delta}(\Xi) \subset S^{m_1+m_2}_{\rho,\delta}(\Xi).$$

Corollary

Under the above hypothesis on the magnetic field B, for any vector potential A we have that in the Schrödinger representation:

$$\mathbf{\Psi}^{m_1}_{\rho,\delta}(A) \cdot \mathbf{\Psi}^{m_2}_{\rho,\delta}(A) \subset \mathbf{\Psi}^{m_1+m_2}_{\rho,\delta}(A).$$

Composition of symbols

REMARK:
In fact, for magnetic fields with components of class $BC^\infty(\mathcal{X})$ and for $\delta < \rho$ we have a much stronger result giving an asymptotic development of the composed symbol [I.M.P., *Proc. RIMS 07*].

L^2-continuity

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Theorem [I.M.P., *Proc. RIMS 07*]

In any Schrödinger representation of the form $\mathfrak{Op}^A$,
the operator corresponding to an observable F of class $S^0_{\rho,\rho}(\Xi)$,
with $0 \leq \rho < 1$, defines a bounded operator
and there exist two constants $c(n) \in \mathbb{R}_+$ and $p(n) \in \mathbb{N}$, depending only on the dimension n of the space $\mathcal{X}$, such that we have the estimation:

$$\|\mathfrak{Op}^A(F)\|_{\mathbb{B}(\mathcal{H})} \leq c(n)|F|_{(p(n),p(n))}.$$

L^2-continuity

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Corollary

Taking into account the obvious inclusion $S^0_{\rho,\delta}(\Xi) \subset S^0_{\delta,\delta}(\Xi)$ we deduce that the previous Theorem remains true for F of class $S^0_{\rho,\delta}(\Xi)$ for $0 \leq \delta < \rho \leq 1$.

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Corollary

For a magnetic field B with components of class $BC^\infty(\mathcal{X})$, any function of class $BC^\infty(\Xi)$ defines a bounded observable, i.e. a bounded operator in any representation of the algebra of quantum observables.

Sobolev spaces

Hypothesis

The magnetic field B has components of class $BC^\infty(\mathcal{X})$.

We shall define the scale of Sobolev spaces starting from a special set of symbols; for any $m > 0$ we define:

$$\wp_m(x,\xi) :=< \xi >^m \equiv (1+|\xi|^2)^{m/2}$$

so that $\wp \in S^m_{1,0}(\Xi) \subset \mathfrak{M}^B(\Xi)$ and for any potential vector A we can define:

$$\mathfrak{p}^A_m := \mathfrak{Op}^A(\wp_m).$$

Sobolev spaces

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Definition

Suppose chosen a vector potential A for it. For any $m > 0$ we define the complex linear space:

$$\mathcal{H}^m_A(\mathcal{X}) := \left\{ u \in L^2(\mathcal{X}) \mid \mathfrak{p}^A_m u \in L^2(\mathcal{X}) \right\}.$$

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Proposition [I.M.P., *Proc. RIMS 07*]

The space $\mathcal{H}^m_A(\mathcal{X})$ is a Hilbert space for the scalar product:

$$< u, v >_{(m,A)} := (\mathfrak{p}^A_m u, \mathfrak{p}^A_m v)_2 + (u, v)_2.$$

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Definition

Suppose chosen a vector potential A. For any $m > 0$ we define the space $\mathcal{H}_A^{-m}(\mathcal{X})$ as the dual space of $\mathcal{H}_A^m(\mathcal{X})$ with the dual norm:

$$\|\phi\|_{(-m,A)} := \sup_{u \in \mathcal{H}_A^m(\mathcal{X}) \setminus \{0\}} \frac{| < \phi, u > |}{\|u\|_{(m,A)}}$$

that induces a scalar product.

We also denote $\mathcal{H}_A^0(\mathcal{X}) := L^2(\mathcal{X})$.

Elliptic symbols

Definition

For $m > 0$ a symbol $F \in S^m_{\rho,\delta}(\Xi)$ is said to be elliptic if there exist two positive constants R and C such that for any $(x, \xi) \in \Xi$ with $|\xi| \geq R$ one has that

$$|F(x, \xi)| \geq C < \xi >^m$$

.

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Proposition [I.M.P., *Proc. RIMS 07*]

- Suppose given a real symbol $F \in S^m_{\rho,\delta}(\Xi)$, where $m \geq 0$ and F elliptic if $m > 0$,
  with either $0 \leq \delta < \rho \leq 1$ or $\delta = \rho \in [0, 1)$.
  Then for any vector potential A defining B the operator

$$\mathfrak{Op}^A(F) : \mathcal{H}^m_A(\mathcal{X}) \to L^2(\mathcal{X})$$

  is self-adjoint.

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  is self-adjoint.
- If $F \geq 0$ then $\mathfrak{Op}^A(F)$ is lower semibounded.

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- Let us denote by $F_a^{-1}(\xi) := 1/F_a(\xi)$ its usual inverse (for pointwise multiplication).

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- Let us denote by $F_a^{-1}(\xi) := 1/F_a(\xi)$ its usual inverse (for pointwise multiplication).
- We define: $\mathfrak{r}_a^B[F] := F_a \sharp^B F_a^{-1} - 1 \in \mathfrak{M}^B(\Xi)$.

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Theorem [M.P.R., *J.Func. Anal. 07*]

For $m > 0$ and $F \in S^m_{1,0}(\Xi) \cap C^\infty(\mathcal{X}')$ elliptic, we have that:

1. for $-a < \inf_{\xi \in \mathcal{X}'} F(\xi)$, the symbol $\mathfrak{r}_a^B[F]$ has strictly negative order and belongs to $\mathfrak{C}^B(\Xi)$.

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1. for $-a < \inf_{\xi \in \mathcal{X}'} F(\xi)$, the symbol $\mathfrak{r}_a^B[F]$ has strictly negative order and belongs to $\mathfrak{C}^B(\Xi)$.
2. For $a \in \mathbb{R}_+$ large enough we have: $\|\mathfrak{r}_a^B[F]\|_B < 1$.

An inversion result

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Corollary

For $m > 0$, $F \in S^m_{1,0}(\Xi) \cap C^\infty(\mathcal{X}')$ elliptic and $a \in \mathbb{R}_+$ large enough F_a is invertible in $\mathfrak{C}^B(\mathcal{X})$ and its inverse F_a^- is given by the formula

$$F_a^- = F_a^{-1} \sharp^B \left(\sum_{k \in \mathbb{N}} (\mathfrak{r}_a^B[F])^{\sharp^B k} \right)$$

with the series converging in the C^*-norm $\|.\|_B$.

Structure

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MOTIVATION

Let us very briefly recall the Beal's criterion in the usual pseudodifferential calculus,
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MOTIVATION

Let us very briefly recall the Beal's criterion in the usual pseudodifferential calculus,
that may be obtained from our formalism by taking $B = 0$
(and $A = 0$ evidently).

Let us recall the following notations:

$$\mathfrak{ad}_{Q_j} T := Q_j T - T Q_j, \quad \mathfrak{ad}_{D_j} T := D_j T - T D_j, \quad \forall T \in \mathbb{B}[L^2(\mathcal{X})]$$

as sesquilinear forms on the domain of Q_j, resp. D_j.

Motivation

Beal's criterion

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Beal's criterion

$T \in \mathbb{B}[L^2(\mathcal{X})]$ has the form $T = \mathfrak{Op}(F_T)$, with $F_T \in S^0_{0,0}(\Xi)$, if and only if for any family $\{\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n\} \in \mathbb{N}^{2n}$

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Beal's criterion

$T \in \mathbb{B}[L^2(\mathcal{X})]$ has the form $T = \mathfrak{Op}(F_T)$, with $F_T \in S^0_{0,0}(\Xi)$, if and only if for any family $\{\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n\} \in \mathbb{N}^{2n}$ the sesquilinear form $\mathfrak{ad}^{\alpha_1}_{Q_1}[\ldots \mathfrak{ad}^{\alpha_n}_{Q_n}[\mathfrak{ad}^{\beta_1}_{D_1}[\ldots \mathfrak{ad}^{\alpha_n}_{D_n}[T]\ldots]$ defined on an evident dense domain in $L^2(\mathcal{X})$,

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Given a magnetic field B with components of class $BC^\infty(\mathcal{X})$ our purpose is to formulate a similar criterion for a bounded operator T to be in $\mathbf{\Psi}^0_{0,0}(A)$.

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- Replace the operators $\{D_j\}_{1\leq j\leq n}$ with the *'magnetic moments'* $\{\Pi_j^A\}_{1\leq j\leq n}$
- Try to formulate the criterion in a gauge invariant way by using the algebraic framework developped above.

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The magnetic field B has components of class $BC^\infty(\mathcal{X})$.

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In any Schrödinger representation associated to a vector potential A for B, an operator $T \in \mathbb{B}[L^2(\mathcal{X})]$ has the form $T = \mathfrak{Op}^A(F_T)$, with $F_T \in S^0_{0,0}(\Xi)$,

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Main Result

In fact the above theorem is the evident *'represented version'* of a result concerning the algebra $\mathfrak{C}^B(\Xi)$ that we shall now present.

The magnetic action of Ξ on $\mathfrak{C}^B(\Xi)$

- In order to define the *'linear monomimals'* on Ξ we shall use the canonical symplectic form σ on Ξ and consider for any $X \in \Xi$ the function: $\mathfrak{l}_X : \Xi \ni Y \mapsto \sigma(X, Y) \in \mathbb{R}$

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- Then we can introduce *the algebraic Weyl system*: $\mathfrak{e}_X := \exp\{-i\mathfrak{l}_X\}$ indexed by $X \in \Xi$.
- We define the following *twisted action* of Ξ on $\mathfrak{C}^B(\Xi)$:

$$\Xi \ni X \mapsto \mathcal{E}^B_X \in \mathbb{A}\mathrm{ut}[\mathfrak{C}^B(\Xi)], \quad \mathcal{E}^B_X[F] := \mathfrak{e}_{-X} \sharp^B F \sharp^B \mathfrak{e}_X$$

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- Some calculus gives: $-i\partial_t \mathcal{E}^B_{tX}[F]\big|_{t=0} = \mathfrak{ad}^B_X[F]$ where $\mathfrak{ad}^B_X[F] := \mathfrak{l}_X \sharp^B F - F \sharp^B \mathfrak{l}_X$.

The magnetic action of Ξ on $\mathfrak{C}^B(\Xi)$

The space of $\mathcal{E}^B$-regular vectors at the origin

$$\mathfrak{V}^B_{\infty,0} := \left\{ F \in \mathfrak{C}^B(\Xi) \mid \mathfrak{ad}^B_{X_1}[\ldots \mathfrak{ad}^B_{X_N}[F]\ldots] \in \mathfrak{C}^B(\Xi) \right\}$$

where $N \in \mathbb{N}$ and $\{X_1, \ldots, X_N\} \subset \Xi$ are arbitrary.

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where $N \in \mathbb{N}$ and $\{X_1, \dots, X_N\} \subset \Xi$ are arbitrary.

This space is endowed with the family of seminorms:

$$|F|_{X_1,\dots,X_N} := \|\mathfrak{ad}^B_{X_1}[\dots \mathfrak{ad}^B_{X_N}[F]\dots]\|_B$$

indexed by all the families $\{X_1, \dots, X_N\} \subset \Xi$ with $N \in \mathbb{N}$ arbitrary

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The above seminorms define on $\mathfrak{V}^B_{\infty,0}$ a Frechet space structure.

Let us recall the usual action through translations of Ξ on the C^*-algebra $BC_{\mathrm{u}}(\Xi)$ (endowed with the usual norm $\|.\|_\infty$):

$$\Xi \ni X \mapsto \mathcal{T}_X \in \mathbb{A}\mathrm{ut}[BC_{\mathrm{u}}(\Xi)], \quad \mathcal{T}_X[F](Y) := F(Y+X)$$

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The space of associated $\mathcal{T}$-regular vectors (in $BC_{\mathrm{u}}(\Xi)$) is $BC^\infty(\Xi)$ with the family of seminorms

$$|F|_{(N)} := \max_{|a|+|\alpha|\leq N} \|\partial_x^a \partial_\xi^\alpha F\|_\infty$$

indexed by $N \in \mathbb{N}$,
that also induce a Frechet space structure on $BC^\infty(\Xi)$

The main result

Theorem

If the magnetic field B has components of class $BC^\infty(\mathcal{X})$, then the two Frechet spaces $\mathfrak{V}^B_{\infty,0}$ and $BC^\infty(\Xi)$ coincide (as subspaces of $\mathcal{S}'(\Xi)$).

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Our *'magnetic'* version of Beal's criterion (stated before) is a straightforward consequence of the above result.

(Just observe that $S^0_{0,0}(\Xi) = BC^\infty(\Xi)$).

Sketch of the proof

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- For the inclusion $BC^\infty(\Xi) \hookrightarrow \mathfrak{V}^B_{\infty,0}$ (as Frechet spaces), we use our L^2-continuity Theorem above [I.M.P., *Proc. RIMS 07*] and a Lemma estimating the sup-norm of $\mathfrak{ad}^B_{X_1}[\dots\mathfrak{ad}^B_{X_N}[F]\dots]$ by some BC^∞-seminorm.

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- For the inclusion $BC^\infty(\Xi) \hookrightarrow \mathfrak{Y}^B_{\infty,0}$ (as Frechet spaces), we use our L^2-continuity Theorem above [I.M.P., *Proc. RIMS 07*] and a Lemma estimating the sup-norm of $\mathfrak{ad}^B_{X_1}[\ldots\mathfrak{ad}^B_{X_N}[F]\ldots]$ by some BC^∞-seminorm.
- For the other inclusion: $\mathfrak{Y}^B_{\infty,0} \hookrightarrow BC^\infty(\Xi)$ (as Frechet spaces), we shall follow the main steps in [B. Helffer ...] but we shall replace L^2-norm estimations on $Tu \in L^2(\mathcal{X})$, for $u \in L^2(\mathcal{X})$ with Hilber-Schmidt-norm estimations on $TR \in \mathbb{B}_2[L^2(\mathcal{X})]$, for $R \in \mathbb{B}_2[L^2(\mathcal{X})]$.

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An important ingredient of the proof is the following equality [M.P., *J. Math. Phys. 04*]:

$$\|\mathfrak{Op}^A(f)\|_{\mathbb{B}_2} = \|f\|_2, \quad \forall f \in \mathcal{S}(\Xi)$$

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This lat estimation allows us to put into evidence the rather abstract B-norm in inequalities involving usual functional norms.

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- The final step is to obtain $\|F\|_B$ by the trick explained above.

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- We have $\mathfrak{ad}^B_{e_j} f = -i\partial_{x_j} f + \delta^B_j[f]$ where

$$\delta^B_j[f] = \sum_{1 \leq \alpha \leq 2[n/4]+3} c^B_{j,\alpha} \star (-i\partial_\xi)^\alpha f$$

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- $c_{j,\alpha}^B \in L^1(\mathcal{X}'; BC(\mathcal{X}))$ has rapid decay in the $\mathcal{X}'$-variables (due to the assumptions on B having components in BC^∞)

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- $(f \star g)(x,\xi) := \int_{\mathcal{X}'} d\eta\, f(x,\eta) g(x, \xi - \eta)$

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- We shall define

$$\mathfrak{w}_0 := 1$$

$$\mathfrak{w}_m := \wp_{m,a_m}, \quad \text{for } m > 0$$

$$\mathfrak{w}_m := \wp_{|m|,a_{|m|}}^-, \quad \text{for } m < 0$$

The case $m \neq 0$

Theorem

Let us suppose the magnetic field B *has components of class* $BC^\infty(\mathcal{X})$. *A distribution* $F \in \mathcal{S}(\Xi)$ *is a symbol of class* $S^m_{0,0}(\Xi)$ *if and only if for any* $N \in \mathbb{N}$ *and any family* $\{X_1, \ldots, X_N\} \subset \Xi$ *we have that*

$$\mathfrak{w}_m^- \sharp^B \mathfrak{ad}^B_{X_1}[\ldots \mathfrak{ad}^B_{X_N}[F]\ldots] \in \mathfrak{C}^B(\Xi)$$