Classes of infinite order pseudodifferential operators

Pseudodifferential operators:

-$\sigma_{KN}(x,D)\varphi(x) = (2\pi)^{-n}\int e^{i<x,\xi>}\sigma_{KN}(x,\xi)\hat{\varphi}(\xi)\mathrm{d}\xi$

where $\hat{\varphi}$ is the Fourier transform of the function (distribution) φ and σ_{KN} is the Kohn Nirenberg symbol of the operator) or

-$\sigma_W(x,D)\varphi(x) = (2\pi)^{-n}\int\int e^{i<x-y,\xi>}\sigma_W(\frac{x+y}{2},\xi)\varphi(y)\mathrm{d}y\mathrm{d}\xi$,

where σ_W is the Weyl symbol of the operator.

The conditions imposed on the symbols are of local nature – polynomial growth with respect to the phase variable ξ of the symbol and of its derivatives uniform with respect to the space variable x in a compact set K, where K is an arbitrary compact set included in an open set Ω or of global nature - polynomial growth with respect to the phase variable ξ of the symbol and of its derivatives uniform with respect to the space variable x in $\boldsymbol{R}^n$.

The local conditions and the Kohn-Nirenberg quantization are especially used in the study of partial differential equations, the global conditions and the Weyl quantization – in quantum mechanics.

Local conditions

1) Boutet de Monvel's analytical pseudodifferential operators of infinite order – *A.I.F.*, 223, (1972), 229 -268.

Weight functions: $\Lambda:[0,\infty)\to(0,\infty)$ continuous, increasing function such that

$$\lim_{r\to\infty} e^{-\varepsilon r}\Lambda(r)=0,\ \lim_{r\to\infty} e^{\varepsilon r}\Lambda(r)=+\infty,\ (\forall)\varepsilon>0\,.$$

Analytical symbols: $\sigma:\Omega\times \boldsymbol{R}^n\to \boldsymbol{C}$ analytical function such that for every compact set $K\subset\Omega$ there exist $\varepsilon, c>0$ such that σ is holomorphic in

$$K(\varepsilon)=\left\{(x,\xi)\in \boldsymbol{C}^n\times\boldsymbol{C}^n;\ d(x,K)<\varepsilon,\ (\mathrm{Im}\xi)^2<\varepsilon\left[(\mathrm{Re}\xi)^2+1\right]\right\}$$

and

$$\left|\sigma(x,\xi)\right|\le c\Lambda(\left|\xi\right|),\ (\forall)\ (x,\xi)\in K(\varepsilon)\,.$$

Then $\sigma_{KN}(x,D)u$ can be defined as a hyperfunction for $u\in C_0^\infty(\Omega)$. A composition formula can be obtained.

2) Ultradifferential operators – H. Komatsu, *J. Fac. Sci Tokyo*, 1A, 20 (1973) 25 -105.

Let $(M_p)_p$ be a sequence of positive numbers. An infinitely differentiable function φ defined on an open set Ω is called an ultradifferentiable function of class (M_p) (Beurling), respectively $\{M_p\}$ (Roumieu), if for every compact set $K\subset\Omega$ and every *h* > 0 there exists a positive constant *C* (respectively if for every compact set $K\subset\Omega$ there exist positive constants *C* and *h*) such that

$$\left|D^\alpha\varphi(x)\right|\le Ch^{|\alpha|}M_{|\alpha|},\ (\forall)\,x\in K,\ (\forall)\alpha\in\boldsymbol{N}^n\,.$$

Let us assume that the sequence $(M_p)_p$ satisfy the following conditions:

- logarithmic convexity

$$M_p^2 \le M_{p-1} M_{p+1},\ (\forall) p > 0;$$

- stability under ultradifferential operators – there exist two constants *A* and *H* such that

$$M_p \le AH^p M_q M_{p-q},\ (\forall) p \ge 0,\ 0 \le q \le p;$$

- non-quasi-analiticity

$$\sum_{p \ge 1} \frac{M_{p-1}}{M_p} < \infty .$$

A formal sum

$$P(D) = \sum_{|\alpha| \ge 0} a_\alpha D^\alpha,\ a_\alpha \in \boldsymbol{C},\ (\forall) |\alpha| \ge 0$$

is called un ultradifferential operator of class (M_p), respectively $\{M_p\}$ if there exist two positive constants *L* and *C* (respectively if for every *L* > 0 there exists *C* > 0) such that

$$|a_\alpha| \le \frac{CL^{|\alpha|}}{M_{|\alpha|}},\ (\forall) |\alpha| \ge 0 .$$

Then *P*(*D*) can be defined as a continuous operator on $\mathrm{D}^{(M_p)}(\Omega)$ (respectively on $\mathrm{D}^{\{M_p\}}(\Omega)$) (and also on the duals of these spaces called ultradistribution spaces)

Remark. The sequences $(M_p)_p = (p^{pr})_p$, which define the Gevrey spaces of functions, satisfy the conditions from above for *r* > 0.

3) Pseudodifferential operators of infinite order on Gevrey classes – L. Zanghirati, *Ann. Univ. Ferrara*, Sez. VII – Sc. Math, XXXI (1985), 197-219.

The *symbols* are the smooth functions $\sigma : \Omega \times \boldsymbol{R}^n \to \boldsymbol{C}$ which satisfy the following condition: for every compact set $K \subset \Omega$ there exists a positive constant *h* and for every $\varepsilon > 0$ there exists a constant *C* > 0 such that

$$\left| D_\xi^\alpha D_x^\beta \sigma(x,\xi) \right| \le C h^{|\alpha+\beta|} \alpha! \beta!^{r(\rho-\delta)} (1+|\xi|)^{-\rho|\alpha|+\delta|\beta|} e^{\varepsilon |\xi|^{1/r}},$$

for every multiindices α and β, x in Ω and ξ in $\boldsymbol{R}^n$. Here $r > 1,\ 0 \le \delta < \rho \le 1,\ r\rho \ge 1$.

Then $\sigma_{KN}(x, D)$ is a continuous operator defined on the Gevrey space of functions compactly supported $G_0^r(\Omega)$ with values into $G^r(\Omega)$.

One proves a pseudo-locality property, a composition formula and a class of operators which admit parametrices is given.

Remark. For a logarithmic convex sequence $(M_p)_p$ one defines its associated function $M : (0,\infty) \to \boldsymbol{R}$ through the formula

$$M(s) = \sup_{p \ge 0}(p \ln s - \ln M_p),\ (\forall) s > 0.$$

If $\left(M_p\right)_p = \left(p^{pr}\right)_p$, then its associated function is equivalent with $s^{1/r}$.

Global conditions

The Gelfand-Shilov-Roumieu spaces (*S – type spaces).* These are spaces of rapidly decreasing functions. For $(M_p)_p$ and $(N_p)_p$ two logarithmic convex sequences $\mathrm{S}(\{M_p\},\{N_p\})$ is the space of the functions *φ* which have the property that there exist positive constants *C*, *h* and *k* such that

$$\left|x^{\beta}D^{\alpha}\varphi(x)\right| \le Ch^{|\alpha|}k^{|\beta|}M_{|\alpha|}N_{|\beta|},\ (\forall)\,x\in \boldsymbol{R}^n,\ (\forall)\alpha,\beta\in \boldsymbol{N}^n$$

and $\mathrm{S}((M_p),(N_p))$ is the space of the functions *φ* which have the property that for every positive constants *h* and *k* there exists a positive constants *C* such that

$$\left|x^{\beta}D^{\alpha}\varphi(x)\right| \le Ch^{|\alpha|}k^{|\beta|}M_{|\alpha|}N_{|\beta|},\ (\forall)\,x\in \boldsymbol{R}^n,\ (\forall)\alpha,\beta\in \boldsymbol{N}^n.$$

If $(M_p)_p$ = $(N_p)_p$, what we shall assume in what follows, we simply write $\mathrm{S}(\{M_p\},\{M_p\})=\mathrm{S}(\{M_p\})$ and $\mathrm{S}((M_p),(M_p))=\mathrm{S}((M_p))$.(If $(M_p)_p$ = $(N_p)_p$, then the GSR spaces are invariant through the Fourier transform.) The dual spaces are denoted with $\mathrm{S}'(\{M_p\})$, respectively $\mathrm{S}'((M_p))$.

4) Infinite order pseudodifferential operators defined on (ultra)modulation spaces – S. Pilipović, N. Teofanov, *JFA*, 208 (2004), 194-228. Let $\gamma(=1/r)\in[0,1)$. A continuous function $w:\boldsymbol{R}^n\times\boldsymbol{R}^n\to(0,\infty)$ is called a *γ*–exp-type weight if there exist $s\ge 0$ and *C* > 0 such that

$$w(x+y,\xi+\eta)\le C\mathrm{e}^{s(|x|^{\gamma}+|\xi|^{\gamma})}w(y,\eta),\ (\forall)x,y,\xi,\eta\in\boldsymbol{R}^n,$$

i.e. if *w* is moderate with respect to the weight $\mathrm{e}^{s(|x|^{\gamma}+|\xi|^{\gamma})}$. For *γ* < 1 the weight $\mathrm{e}^{s(|x|^{\gamma}+|\xi|^{\gamma})}$ is submultiplicative.

For $1 \le p, q < \infty$, the ultra-modulation space $M_{p,q}^{w,t}$ is the space of the ultradistributions $u \in \mathrm{S}'((p^{pr}))$ such that

$$\left[\int\left(\int\left|\left\langle\overline{T_x M_\xi g}, u\right\rangle\right|^p w(x,\xi)^p \mathrm{e}^{t(|x|^\gamma + |\xi|^\gamma)}\, \mathrm{d}x\right)^{q/p} \mathrm{d}\xi\right]^{1/q} < \infty .$$

Here *g* is an arbitrary window from $\mathrm{S}((p^{pr}))$, T_x is the operator of translation with *x* and M_ξ is the operator of multiplication with $\mathrm{e}^{2\pi i \xi \bullet}$. The function $\left\langle\overline{T_x M_\xi g}, u\right\rangle$ is the short time Fourier transform of *u* of window *g*.

The *symbols* are the smooth functions $\sigma : \boldsymbol{R}^n \times \boldsymbol{R}^n \to \boldsymbol{C}$ which satisfy the following condition: there exist positive constants *h*, *k* and *C* such that

$$\left|D_\xi^\alpha D_x^\beta \sigma(x,\xi)\right| \le C h^{|\alpha|} k^{|\beta|} (\alpha!\beta!)^r \mathrm{e}^{\lambda|x|^\gamma + \tau|\xi|^\gamma} ,$$

for every multiindices α and β, and *x* and ξ in $\boldsymbol{R}^n$.

One proves that if *h* and *k* satisfy some conditions, then $\sigma_W(x,D) : M_{p,q}^{w,0} \to M_{p,q}^{\tilde{w},0}$ is a continuous operator for $\tilde{w}(x,\xi) = w(x,\xi)\mathrm{e}^{-2^\gamma \lambda |x|^\gamma - \tau|\xi|^\gamma}$.

The proof is based on the fact that a Wilson basis of exponential decay is an unconditional basis in $M_{p,q}^{w,t}$. (Wilson basis are orthonormal basis in L^2 which elements are “simple” linear combinations of time – frequency shifts of a fixed function.

A class of elliptic operators is defined, their essential self-adjointness on L^2 is proved and spectral asymptotics for such operators are obtained.

5. Infinite order pseudodifferential operators on general *S* – type spaces.

Let $(M_p)_p$ be a logarithmic convex sequence which satisfies the condition of stability under ultradifferential operators and with the property that there exists a positive constant *C* such that

$$\sqrt{p} M_{p-1} \leq C M_p, \ (\forall) p > 0 .$$

The *symbols* are the smooth functions $\sigma : \boldsymbol{R}^n \times \boldsymbol{R}^n \to \boldsymbol{C}$ which satisfy the following condition: there exist positive constants *h*, *k* such that for every $\varepsilon > 0$ there exists a constant *C* > 0 such that

$$\left| D_\xi^\alpha D_x^\beta \sigma(x, \xi) \right| \leq C h^{|\alpha|} k^{|\beta|} M_{|\alpha|} M_{|\beta|} (1 + |\xi|)^m \mathrm{e}^{M(\varepsilon |\xi|)} ,$$

for every multiindices *α* and *β*, and *x* and *ξ* in $\boldsymbol{R}^n$.

Then $\sigma_{KN}(x, D)$ is a continuous operator in $\mathrm{S}((M_p))$.

Remark. The case $(M_p)_p = (p^{pr})_p$ for some $r \geq 1/2$ is covered.