Mean field limits and semiclassical techniques.

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Work in progress with Z. Ammari.

Outline.

- Introduction.
- Formal correspondence.
- Quantizations. Infinite dimensional difficulties.
- An example. Coherent states and propagation of chaos.
- Wigner measures. An application.
- Summary.

Introduction.

Mean field limit : Bosonic Fock space on $\mathcal{Z}$

$$\mathcal{H} = \bigoplus_{n=0}^{\infty} \bigvee^{n} \mathcal{Z} = \Gamma_s(\mathcal{Z}) \, ,$$

Specific states $\psi = z^{\otimes N}$
Hamiltonian $H_N(a^*, a)$ Wick quantization of $h(z, \overline{z})$.

$$\psi(t) = e^{-itNH_N} \psi .$$

$$\langle \psi(t) \, , \, \mathcal{O} \psi(t) \rangle \overset{N \to \infty}{\sim} \left\langle z_t^{\otimes N} \, , \, \mathcal{O} z_t^{\otimes N} \right\rangle$$

$$i \partial_t z_t = \partial_{\overline{z}} h(z_t) \, .$$

Introduction.

Semiclassical analysis : $a^{\cdots}(x, hD_x)$ acting on $L^2(\mathbb{R}^d)$
Hamiltonian $H^h = p^h(x, hD)$ Wick quantization of $p(z, \overline{z})$.
Egorov theorem

$$e^{i\frac{t}{h}H^h} a(x, hD) e^{-i\frac{t}{h}H^h} \overset{h \to 0}{\longrightarrow} (a \circ \Phi_t)(x, hD_x)$$

- Algebra of semi-classical pseudos preserved by the classical flow.
- Possibility of an asymptotic expansion up to $O(h^\infty)$.
- Weak and very flexible version via Wigner measures.

Formal correspondence.

bosonic QFT	Semi-classical on $\mathbb{R}^d$
$a^*(z), \quad a(z),$	$a(z) = \sum_j \overline{z_j} \frac{(\partial_{x_j} + x_j)}{\sqrt{2}}$
$[a(z_1), a^*(z_2)] = \langle z_1 \, , \, z_2 \rangle$	$a^*(z) = \sum_j z_j \frac{(-\partial_{x_j} + x_j)}{\sqrt{2}}$
$\Phi(z) = \frac{1}{\sqrt{2}}(a(z) + a^*(z))$	$z_R x - z_I D_x$
$W(z) = e^{i\Phi(z)}$	$\tau_{(z_I, z_R)}$
$E(z) = W(\frac{\sqrt{2}}{i} z)\Omega$	$\tau_{(-\sqrt{2}z_R, \sqrt{2}z_I)}(C_d e^{-\frac{x^2}{2}})$
$z^{\otimes n}, \lvert z \rvert = 1$	$Hermite\ functions.$

Formal correspondence.

bosonic QFT	Semi-classical on $\mathbb{R}^d$
$a^*(z), \quad a(z),$	$a(z) = \sum_j \overline{z_j} \frac{\sqrt{\varepsilon}(\partial_{x_j} + x_j)}{\sqrt{2}}$
$[a(z_1), a^*(z_2)] = \varepsilon \langle z_1 \, , \; z_2 \rangle$	$a^*(z) = \sum_j z_j \frac{\sqrt{\varepsilon}(-\partial_{x_j} + x_j)}{\sqrt{2}}$
$\Phi(z) = \frac{1}{\sqrt{2}}(a(z) + a^*(z))$	$z_R \sqrt{\varepsilon} x - z_I \sqrt{\varepsilon} D_x$
$W(z) = e^{i\Phi(z)}$	$\tau_{(\sqrt{\varepsilon} z_I, \sqrt{\varepsilon} z_R)}$
$E(z) = W(\frac{\sqrt{2}}{i\varepsilon} z)\Omega$	$\tau_{(-\sqrt{\frac{2}{\varepsilon}} z_R, \sqrt{\frac{2}{\varepsilon}} z_I)}(C_d e^{-\frac{x^2}{2}})$
$z^{\otimes n}, \lvert z \rvert = 1$	$\varepsilon/2 - Hermite\ functions.$

Formal correspondence.

$$h = \frac{\varepsilon}{2}, \qquad \varepsilon = \frac{1}{N}$$

$a(\sqrt{h}x, \sqrt{h}D_x)$ unitarily equivalent to $a(x, hD_x)$

$$\tau^h_{x_0,\xi_0} = e^{\frac{i}{h}(\xi_0.\sqrt{h}x - x_0\sqrt{h}D_x)}, \qquad z_0 = -x_0 + i\xi_0 \, .$$

$$E(z) = W(\frac{\sqrt{2}}{i\varepsilon}z)\Omega = e^{\frac{1}{\varepsilon}[a^*(z)-a(z)]}\Omega = e^{-\frac{|z|^2}{2\varepsilon}} \sum_{n=0}^{\infty} \varepsilon^{-n/2} \frac{z^{\otimes n}}{\sqrt{n!}}.$$

$$a(\xi)E(z) = \langle \xi \, , \, z \rangle \, E(z).$$

Known difficulties

REF: ...Bogolubov, Berezin,Segal, Kree, Lascar, L. Gross...
$p\mathcal{Z}$ finite dimensional (sub)space. L_p Leb. meas. on $p\mathcal{Z}$.
Weyl quantization :

$$\mathcal{F}[f](z) = \int_{p\mathcal{Z}} f(\xi)\ \ e^{-2\pi i\, S(z,\xi)}\ L_p(d\xi) \qquad S(z_1, z_2) = \ \mathrm{Re}\ \langle z_1, z_2\rangle$$

$$b^{Weyl} = \int_{p\mathcal{Z}} \mathcal{F}[b](z) \quad W(\sqrt{2}\pi z)\ L_p(dz)\,.$$

A-Wick quantization :

$$b^{A-Wick} \quad = \quad \int_{p\mathcal{Z}} b(\xi)\ \ P_\xi^\varepsilon \quad \frac{L_p(d\xi)}{(\pi\varepsilon)^{\mathrm{dim} p\mathcal{Z}}}$$

$$= \quad \int_{p\mathcal{Z}} \mathcal{F}[b](\xi)\ \ W(\sqrt{2}\pi\xi)\ \ e^{-\frac{\varepsilon\pi^2}{2}|\xi|^2_{p\mathcal{Z}}}\ L_p(d\xi)\,.$$

Known difficulties

Infinite dimensional integration. Quasi-equivalent Gaussian measures, Shale's theorem $\rightarrow$ Hilbert-Schmidt condition

$$\int A(x_1, \ldots, x_m, y_1, \ldots, y_n)$$
$$a^*(x_1) \ldots a^*(x_m) a(y_1) \ldots a(y_n)\; dx_1 dx_2 \ldots dx_m dy_1 \ldots dy_n$$

can be considered as Weyl-pseudos when A is Hilbert-Schmidt.

Pb : The class nonlinear flows which preserve the quasi-equivalence with a fixed gaussian measure is very restricted and non-physical.

Wick quantization

REF:. . . Derezinski-Gérard, Fröhlich-Graffi-Schwartz, Fröhlich-Knowles-Pizzo . . .

$$(b(z) \in \mathcal{P}_{p,q}(\mathcal{Z})) \Leftrightarrow \begin{cases} \tilde{b} = \frac{1}{p!}\frac{1}{q!}\partial_z^p \partial_{\bar{z}}^q b(z) \in \mathcal{L}(\bigvee^p \mathcal{Z}, \bigvee^q \mathcal{Z}) \,, \\ b(z) = \left\langle z^{\otimes q} \,,\, \tilde{b} z^{\otimes p} \right\rangle \,. \end{cases}$$

$$b^{Wick} : \mathcal{H}_{fin} \to \mathcal{H}_{fin}, \qquad \mathcal{H}_{fin} = \oplus_n^{alg} \bigvee^n Z$$

$$b^{Wick}_{|\bigvee^n \mathcal{Z}} = 1_{[p,+\infty)}(n) \frac{\sqrt{n!(n+q-p)!}}{(n-p)!} \, \varepsilon^{\frac{p+q}{2}} \left(\tilde{b} \bigvee I_{\bigvee^{n-p} \mathcal{Z}} \right)$$

$$\in \mathcal{L}(\bigvee^n \mathcal{Z}, \bigvee^{n+q-p} \mathcal{Z}) \,,$$

Wick quantization

REF:... Derezinski-Gérard, Fröhlich-Graffi-Schwartz, Fröhlich-Knowles-Pizzo ...

$$\left(b(z) \in \mathcal{P}_{p,q}(\mathcal{Z})\right) \Leftrightarrow \begin{cases} \tilde{b} = \frac{1}{p!}\frac{1}{q!}\partial_z^p \partial_{\bar{z}}^q b(z) \in \mathcal{L}(\bigvee^p \mathcal{Z}, \bigvee^q \mathcal{Z}), \\ b(z) = \left\langle z^{\otimes q}, \tilde{b} z^{\otimes p} \right\rangle . \end{cases}$$

Ex: $a^*(\xi) = \langle z, \xi\rangle^{Wick}$, $a(\xi) = \langle \xi, z\rangle^{Wick}$,
$\Phi(\xi) = \sqrt{2} S(\xi, z)^{Wick}$.
$d\Gamma(A) = (\langle z\,, Az\rangle)^{Wick}$.

Replace the variable z by a_x and $\bar{z}$ by a_x^* while keeping the a^* on the left-hand side.

Wick quantization

The formulas

$$b_1^{Wick}\ b_2^{Wick} \quad = \quad \left(\sum_{p=0}^{\min\{p_1,q_2\}} \frac{\varepsilon^p}{p!} \quad \partial_z^p b_1 . \partial_{\bar z}^p b_2 \right)^{Wick}$$

$$= \quad \left(e^{\varepsilon \langle \partial_z , \partial_{\bar\omega} \rangle} b_1(z) b_2(\omega) \,|_{z=\omega} \right)^{Wick} ,$$

$$[b_1^{Wick}, b_2^{Wick}] = \left(\sum_{p=1}^{\max\{\min\{p_1,q_2\}\,,\min\{p_2,q_1\}\}} \frac{\varepsilon^p}{p!} \quad \{b_1, b_2\}^{(p)} \right)^{Wick} ,$$

hold for any $b_\ell \in \mathcal{P}_{p_\ell,q_\ell}(\mathcal{Z})$, $\ell = 1, 2$.

An example.

Quantized

$$H^{\varepsilon} = d\Gamma(-\Delta) + \frac{1}{2}\int_{\mathbb{R}^d} V(x_1 - x_2) a^*(x_1) a^*(x_2) a(x_1) a(x_2)\, dx_1 dx_2$$

$$i\varepsilon\partial_t \psi = H^{\varepsilon}\psi, \quad \psi \in \mathcal{H}$$

$$Q(z) = \langle z^{\otimes 2}, V(x_1 - x_2) z^{\otimes 2}\rangle\,, \quad V \in L^{\infty}(\mathbb{R}^d)\,.$$

Classical

$$i\partial_t z = -\Delta + (V * |z_t|^2) z_t$$

An example.

Propagation of chaos: Empirical distribution (bosonic)

$$\langle \psi(t)\,,\,\mathcal{O}\psi(t)\rangle \overset{\varepsilon\to 0}{\sim} \left\langle z_t^{\otimes k}\,,\,\mathcal{O}z_t^{\otimes k}\right\rangle$$

with $\kappa\varepsilon \to 1$ and

$$\mathcal{O} = d\Gamma(A(x,D_x)) = (\langle z\,,\,A(x,D_x)z\rangle)^{Wick}\,.$$

REF: Bardos-Golse-Mauser, Erdös-Yau,
Graffi-Martinez-Pulvirenti, Fröhlich-Graffi-Schwartz,
Fröhlich-Knowles-Pizzo

An example.

Hepp method: Coherent state.
REF : Hepp, Ginibre-Velo.

With $\psi = E(z)$, $z \in \mathcal{Z}$, the solution $\psi(t) = e^{-i\frac{t}{\varepsilon}H^\varepsilon}\psi$ satisfies

$$\sup_{t\in[-T,T]} \left\| \psi(t) - e^{\frac{i}{\varepsilon}\omega(t)}\hat{E}(z_t) \right\| \overset{\varepsilon\to 0}{\longrightarrow} 0\,.$$

$$\omega(t) = \int_0^t \left(\int_{\mathbb{R}^{2d}} V(x_1 - x_2)\, |z_s(x_1)|^2\, |z_s(x_2)|^2 \; dx_1 dx_2 \right) \, ds\,.$$

$\hat{E}(z_t)$ squeezed state computed explicitly after considering the quadratic approximation of H_ε at the point z_t

An example.

Propagation Chaos: Truncated Dyson expansion

$$U_\varepsilon(t)^*\mathcal{O}U_\varepsilon(t) = \mathcal{O}_t$$

$$+\sum_{n=1}^{\ell-1}(\frac{i}{2\varepsilon})^n\int_0^t dt_1\cdots\int_0^{t_{n-1}} dt_n[Q_{t_n}^{Wick},\cdots[Q_{t_1}^{Wick},\mathcal{O}_t]\cdots]$$

$$+(\frac{i}{2\varepsilon})^\ell\int_0^t dt_1\cdots\int_0^{t_{\ell-1}} dt_\ell\ \ U_\varepsilon(t_\ell)^*U_\varepsilon^0(t_\ell)[Q_{t_\ell}^{Wick},\cdots[Q_{t_1}^{Wick},\mathcal{O}_t]\cdots]$$

$$U_\varepsilon^0(t_\ell)^*U_\varepsilon(t_\ell).$$

$\mathcal{O}_t(z) = U_\varepsilon^0(t)^*\mathcal{O}U_\varepsilon^0(t)$ $\qquad Q_t(z) = Q(e^{it\Delta}z)$ with
$U_\varepsilon^0(t) = \Gamma(e^{it\Delta})$.
$U_\varepsilon(t)$ preserves the number.

An example.

Propagation Chaos: Truncated Dyson expansion

$$
\begin{aligned}
U_\varepsilon(t)^*\mathcal{O}U_\varepsilon(t) &= \mathcal{O}_t \\
&+\sum_{n=1}^{\ell-1}\left(\frac{i}{2\varepsilon}\right)^n\int_0^t dt_1\cdots\int_0^{t_{n-1}} dt_n [Q^{Wick}_{t_n},\cdots[Q^{Wick}_{t_1},\mathcal{O}_t]\cdots] \\
&+\left(\frac{i}{2\varepsilon}\right)^\ell\int_0^t dt_1\cdots\int_0^{t_{\ell-1}} dt_\ell\ U_\varepsilon(t_\ell)^*U^0_\varepsilon(t_\ell)[Q^{Wick}_{t_\ell},\cdots[Q^{Wick}_{t_1},\mathcal{O}_t]\cdots] \\
&\qquad U^0_\varepsilon(t_\ell)^*U_\varepsilon(t_\ell).
\end{aligned}
$$

$\mathcal{O}_t(z)=U^0_\varepsilon(t)^*\mathcal{O}U^0_\varepsilon(t)$ $\quad Q_t(z)=Q(e^{it\Delta}z)$ with $U^0_\varepsilon(t)=\Gamma(e^{it\Delta})$.

An example.

Propagation Chaos: Truncated Dyson expansion

$$\frac{1}{\varepsilon^n}\,[Q_{t_n}^{Wick},\cdots,[Q_{t_1}^{Wick},b_t^{Wick}]] = \sum_{r=0}^{n}\varepsilon^r\,\left(C_r^{(n)}(t_n,\cdots,t_1,t)\right)^{Wick},$$

holds with $b \in \mathcal{P}_{p,q}(\mathcal{Z})$ and

$$C_r^{(n)}(t_n,\cdots,t_1,t) = \frac{1}{2^r}\sum_{\sharp\{i:\,\varepsilon_i=2\}=r}\{Q_{t_n},\cdots,\{Q_{t_1},b_t\underbrace{\}^{(\varepsilon_1)}\ldots\}^{(\varepsilon_n)}}_{\varepsilon_i\in\{1,2\}}$$

$$\in \mathcal{P}_{p-r+n,q-r+n}(\mathcal{Z}).$$

An example.

Propagation Chaos: Truncated Dyson expansion $\varepsilon k \to 1$

$$\langle z^{\otimes k},\ U_\varepsilon(t)^* b^{Wick} U_\varepsilon(t)\, z^{\otimes k}\rangle =$$

$$\sum_{s=0}^{\ell-1} \varepsilon^s \sum_{n=0}^{\infty} \left(\frac{i}{2}\right)^n \int_0^t dt_1 \cdots \int_0^{t_{n-1}} dt_n$$

$$[\sum_{j=0}^{s} \alpha_j^{s-j,n}(k\varepsilon) C_{s-j}^{(n)}(t_n, \cdots, t_1, t; z)] + O(\varepsilon^\ell),$$

$$\sum_{i=0}^{p+n-r-1} \alpha_i^{r,n}(\kappa)\varepsilon^i = \kappa(\kappa-\varepsilon)(\kappa-2\varepsilon)\cdots(\kappa-(p+n-r-1)\varepsilon),$$

$\alpha_s^{r,n} = 0$ when $s > (p+n-r)$ or $r > n$.

An example.

Propagation Chaos: Truncated Dyson expansion $\varepsilon k \to 1$
This implies a similar result for coherent states. (2 proofs)

Not exactly an Egorov Theorem (valid only for some specific states).

Wigner measures

REF: ...Tartar, Helffer-Martinez-Robert, P. Gérard, Lions-Paul, P. Gérard-Mauser-Poupaud...
$a \to a^{A-Wick}(\sqrt{\frac{\varepsilon}{2}}x, \sqrt{\frac{\varepsilon}{2}}D_x)$ is a positive quantization.
In dimension $d < \infty$ and for $a \in S(1, g_0)$

$$\left\| a^{A-Wick}(\sqrt{\frac{\varepsilon}{2}}x, \sqrt{\frac{\varepsilon}{2}}D_x) - a^{Weyl}(\sqrt{\frac{\varepsilon}{2}}x, \sqrt{\frac{\varepsilon}{2}}D_x) \right\| \leq C_d\, p_{k_d}(a)\varepsilon\,.$$

For any family $(\varrho_\varepsilon)_{\varepsilon>0}$ of (normal) states, there exists a subsequence $(\varepsilon_k)_{k\in\mathbb{N}}$ and a measure μ on $T^*\mathbb{R}^d \sqcup \{\infty\}$ s.t.

$$\mathrm{Tr}\ \left[\varrho_{\varepsilon_k} a(\sqrt{\frac{\varepsilon}{2}}x, \sqrt{\frac{\varepsilon}{2}}D_x)\right] \overset{k\to\infty}{\to} \int a(x,\xi)\ d\mu(x,\xi)\,,$$

holds for all $a \in \mathcal{C}^\infty(T^*\mathbb{R}^d \sqcup \{\infty\})$.

Wigner measures

p finite rank projector, $I_{\mathcal{Z}} = p + (1-p)$

$$\Gamma(\mathcal{Z}) = \Gamma(p\mathcal{Z}) \otimes \Gamma((1-p)\mathcal{Z}) \,,$$
$$W(\sqrt{2}\pi\xi_1 + \xi_2) = W_p(\sqrt{2}\pi\xi_1) \otimes W_{1-p}(\sqrt{2}\pi\xi_2) \,,$$

when $\xi_1 \in p\mathcal{Z}$ and $\xi_2 \in (1-p)\mathcal{Z}$.
Characteristic of distribution (A-Wick)

$$G_\varepsilon(\xi) = \operatorname{Tr}\left[\varrho_\varepsilon W(\sqrt{2}\pi\xi)\right] e^{-\frac{\varepsilon\pi^2|\xi|^2}{2}}$$

Wigner measures

Bochner : Characteristic function of distribution iff positive type + continuity on any finite dim. space.

Prokhorov (tightness) : A distribution on $\mathcal{Z}$ separable Hilbert, is a Borel (probability) measure iff it is weakly Radon.

Wigner measures

Consider normal states $(\varrho_\varepsilon)_{\varepsilon>0}$ such that
$\mathrm{Tr}\,\left[\varrho_\varepsilon \langle N\rangle^{\delta>0}\right] \leq C_\delta < \infty$ uniformly in ε with $\delta > 0$
Then there exists $(\varepsilon_k)_{k\in\mathbb{N}}$, $\varepsilon_k \to 0$, and a Borel probabiblity measure on $\mathcal{Z}$, s.t.

$$\lim_{k\to\infty} \mathrm{Tr}\,\left[\varrho_{\varepsilon_k} a^{Weyl\ or\ A-Wick}\right] = \int_{\mathcal{Z}} a(z)\, d\mu(z)$$

for any $a \in \mathcal{C}_0^\infty(p\mathcal{Z})$, any p finite dim. projector.

$$\int_{\mathcal{Z}} \langle z\rangle^\delta\, d\mu(z) \leq C_\delta\,.$$

Wigner measures

Sketch of the proof:

- Ascoli type result for $G_\varepsilon(\xi)$: separability of $\mathcal{Z}$ with

$$\left|[W(z_1) - W(z_2)](N+1)^{-s/2}\right| \leq C_s \ |z_1 - z_2|^s \times [\min(\varepsilon|z_1|, \varepsilon|z_2|)^s + 1]$$

  for $0 < s < 1$.

- Prokhorov (tightness) condition due to

$$\left(N^\delta \geq N_p^\delta \otimes I_{\Gamma((1-p)\mathcal{Z})}\right) \Rightarrow \left(\mathrm{Tr}\ \left[\varrho_\varepsilon \left\langle N_p \right\rangle^\delta\right] \leq C_0\right) ,$$

  with some finite dimensional ε-pseudodiff calculus.

Wigner measures

Examples:

- If $\varrho = |E(z)\rangle\langle E(z)|$ then $\mu = \delta_z$
- If $\varrho = |z^{\otimes n}\rangle\langle z^{\otimes n}|$ with $\lim_{\varepsilon\to 0} n\varepsilon = 1$ and $|z| = 1$, then $\mu = (2\pi)^{-1} \int_0^{2\pi} \delta_{e^{i\theta}z}\, d\theta$
- If $\varrho = \int_{\mathcal{Z}} |E(z)\rangle\langle E(z)|\, d\nu(z)$ then $\mu = \nu$.
- The propagation of chaos in the example implies the propagation of Wigner measure for coherent states and Hermite states (Laguerre connection).
- Again: No Egorov theorem. Cylindrical functions do not remain cylindrical (non linear flow).

Wigner measures, an application

Assume that U_ε is a unitary operator on $\mathcal{H}$ s.t. $U_\varepsilon e^{i\theta N} = e^{i\theta N} U_\varepsilon$ for $\theta \in \mathbb{R}$.
Assume that for $z \in \mathcal{Z}$, the image of the coherent state $U_\varepsilon E(z)$ satisfies

$$\lim_{\varepsilon \to 0} \left\langle U_\varepsilon E(z) \, , a^{Weyl} U_\varepsilon E(z) \right\rangle = a(z_U) \, .$$

Then for any $\varphi \in \mathcal{C}_0^\infty(\mathbb{R})$, $\int \varphi > 0$, the state

$$\varrho_\varepsilon = \frac{1}{\int \varphi} \sum_{n \in \mathbb{N}} \varepsilon^{1/2} \varphi \left(\varepsilon^{1/2} (n - 1/\varepsilon) \right) U_\varepsilon |z^{\otimes n}\rangle\langle z^{\otimes n}| U_\varepsilon^*$$

has a unique Wigner measure $\mu = (2\pi)^{-1} \int_0^{2\pi} \delta_{e^{i\theta} z_U} \; d\theta$.

Wigner measures, an application

Sketch of the proof:

- Gauge invariance of $(2\pi)^{-1}\int_0^{2\pi}|E(e^{i\theta}z)\rangle\langle E(e^{i\theta}z)|\, d\theta$ preserved by the conjugation by U_ε.
- Convex extremality argument.

Summary

- scaling : $W(z) \to$ characteristic function $\to$ stochastic processes.
  $W(z/\varepsilon) \to$ phase space translation $\to$ phase space geometry.
- quantizations :
  - Weyl and Anti-Wick specify some directions in $\mathcal{Z}$
    $\to$ inductive point of view (Hilbert-Schmidt cond $\ldots$).
    $\to$ projective point of view (very weak but geometric).
  - Wick quantization uniform treatment of all the directions in $\mathcal{Z}$.
    $\to$ No Hilbert-Schmidt condition.
    $\to$ Fits directly with the number representation.
    $\to$ Complete expansions of propag of chaos.