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On Eisenbud's and Wigner's R-matrix: A general approach

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1 Scattering

1.1 Wave operators

Pair of self-adjoint operators $\{L, L_0\}$ in some separable Hilbert space $\mathfrak{L}$.
Wave operators:

$$W_\pm(L, L_0) = s - \lim_{t \to \pm\infty} e^{itL} e^{-itL_0} P^{ac}(L_0)$$

$P^{ac}(L_0)$ is the projection onto the absolutely continuous subspace $\mathfrak{L}^{ac}(L_0)$ of L_0.

$$\mathrm{ran}(W_\pm(L, L_0)) \subseteq \mathfrak{L}^{ac}(L_0).$$

We say the scattering system is complete if

$$\mathrm{ran}(W_\pm(L, L_0)) = \mathfrak{L}^{ac}(L).$$

1.2 Scattering operator

$S : \mathfrak{L}^{ac}(L_0) \longrightarrow \mathfrak{L}^{ac}(L_0)$

$$S(L, L_0) := W_+(L, L_0)^* W_-(L, L_0).$$

Interwining property:

$$e^{-itL_0} S(L, L_0) = S(L, L_0) e^{-itL_0}, \quad t \in \mathbb{R},$$

which is equivalent to

$$E_0(\Delta) S(L, L_0) = S(L, L_0) E_0(\Delta), \quad \Delta \in \mathfrak{B}(\mathbb{R}).$$

If $\{L, L_0\}$ is a complete scattering system, then $S(L, L_0)$ is unitary on $\mathfrak{L}^{ac}(L_0)$, that is,

$$S(L, L_0)^* S(L, L_0) = S(L, L_0) S(L, L_0)^* = I_{\mathfrak{L}^{ac}(L_0)}.$$

1.3 Scattering matrix

There is direct integral representation of $\mathfrak{L}^{ac}(L_0)$,

$$\mathfrak{L}^{ac}(L_0) \cong \int^{\oplus} \mathfrak{Q}_\lambda d\mu(\lambda),$$

where $\{\mathfrak{Q}_\lambda\}_{\lambda\in\mathbb{R}}$ is family of Hilbert spaces and $\mu(\cdot)$ is a Borel measure on $\mathbb{R}$ which is absolutely continuous with respect to the Lebesgue measure $d\lambda$ on $\mathbb{R}$, such that

$$L_0^{ac} \cong \lambda$$

Such a representation is called a spectral representation of L_0^{ac}.

Since $S(L, L_0)$ commutes with L_0^{ac}, there is a measurable family of operators $\{S(\lambda)\}_{\lambda\in\mathbb{R}}$, $S(\lambda) : \mathfrak{Q}_\lambda \longrightarrow \mathfrak{Q}_\lambda$, such that

$$S(L, L_0) \cong S(\lambda)$$

$\{S(\lambda)\}_{\lambda\in\mathbb{R}}$ is called the scattering matrix of the scattering system $\{L, L_0\}$.

2 Example

Perturbed operator:

$$Lf = -\frac{1}{2}\frac{d}{dx}\frac{1}{M}\frac{d}{dx}f + Vf, \quad f \in \mathrm{dom}(L) = \{f \in W^{1,2}(\mathbb{R}) : \frac{1}{M}f \in W^{1,2}(\mathbb{R})\}.$$

where

$$M(x) := \begin{cases} m_l, & x \in (-\infty, x_l] \\ m(x), & x \in (x_l, x_r) \\ m_r, & x \in [x_r, \infty) \end{cases} \qquad V(x) := \begin{cases} v_l, & x \in (-\infty, x_l] \\ v(x), & x \in (x_l, x_r) \\ v_r, & x \in [x_r, \infty). \end{cases}$$

Unperturbed operator:

$$L_0 := -\frac{1}{2m_l}\frac{d^2}{dx^2} + v_l \oplus -\frac{1}{2}\frac{d}{dx}\frac{1}{m}\frac{d}{dx} + v(x) \oplus -\frac{1}{2m_r}\frac{d^2}{dx^2} + v_r \qquad \text{Dirichlet b. c.}$$

$$L^2(\mathbb{R}) = L^2((-\infty, x_l)) \oplus L^2((x_l, x_r)) \oplus L^2((x_r, \infty)).$$

$\{L, L_0\}$ performs a complete scattering system

3 Eisenbud-Wigner representation

Wigner's R-matrix:

$$R(\lambda) := i(I_{\mathfrak{Q}_\lambda} - S(\lambda))(I_{\mathfrak{Q}_\lambda} + S(\lambda))^{-1} \quad \Longrightarrow \quad S(\lambda) := \frac{iI_{\mathfrak{Q}_\lambda} - R(\lambda)}{iI_{\mathfrak{Q}_\lambda} + R(\lambda)}$$

$$R(\lambda) = \sum_{k=1}^{\infty}(\lambda_k - \lambda)^{-1}\left(\cdot, \begin{pmatrix} \sqrt[4]{\frac{\lambda - v_l}{2m_l}}\psi_k(x_l) \\ \sqrt[4]{\frac{\lambda - v_r}{2m_r}}\psi_k(x_r) \end{pmatrix}\right)\begin{pmatrix} \sqrt[4]{\frac{\lambda - v_l}{2m_l}}\psi_k(x_l) \\ \sqrt[4]{\frac{\lambda - v_r}{2m_r}}\psi_k(x_r) \end{pmatrix}, \qquad \lambda > v_l,$$

where $\{\lambda_k\}$ and ψ_k, $k = 1, 2, \ldots$, are the eigenvalues and eigenfunctions of the selfadjoint operator

$$A_1 := -\frac{1}{2}\frac{d}{dx}\frac{1}{m(x)}\frac{d}{dx} + v(x), \qquad \text{Neumann b. c.}$$

4 Boundary triplets and scattering

4.1 Boundary triplets

Let A be a closed symmetric operator on $\mathfrak{H}$ and $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet of A^*.

Boundary triplet: $\Gamma_i : \mathrm{dom}(A^*) \longrightarrow \mathcal{H}$,

(i) Green's identity: $(A^* f, g) - (f, A^* g) = (\Gamma_1 f, \Gamma_0 g) - (\Gamma_0 f, \Gamma_1 g), \quad f, g \in \mathrm{dom}(A^*)$,

(ii) surjectivity of map $\Gamma := \begin{pmatrix} \Gamma_0 \\ \Gamma_1 \end{pmatrix} \mathfrak{H} \longrightarrow \begin{matrix} \mathcal{H} \\ \oplus \\ \mathcal{H} \end{matrix}$.

Weyl function $M(\cdot) : \mathcal{H} \longrightarrow \mathcal{H}$,

$$\Gamma_1 f_\lambda := M(\lambda)\Gamma_0 f_\lambda, \quad f_\lambda \in \mathcal{N}_\lambda := \ker(A^* - \lambda).$$

4.2 Extensions

Extension of A are labeled by self-adjoint relations Θ in $\mathcal{H}$,

$$A_\Theta := A^* \upharpoonright \Gamma^{-1}\Theta \tag{1}$$

where Θ is some self-adjoint relation on $\mathcal{H}$.

Two special extensions:

$$\Theta_1 := 0, \qquad A_1 := A^* \upharpoonright \Gamma^{-1}\Theta_1 = A^* \upharpoonright \ker(\Gamma_1),$$
$$\Theta_0 := \begin{pmatrix} 0 \\ \mathcal{H} \end{pmatrix}, \qquad A_0 := A^* \upharpoonright \Gamma^{-1}\Theta_0 = A^* \upharpoonright \ker(\Gamma_0).$$

4.3 Example

In $\mathfrak{H} := L^2((x_l, x_r))$ one defines

$$(Af)(x) := -\frac{1}{2}\frac{d}{dx}\frac{1}{m(x)}\frac{d}{dx}f(x) + v(x)f(x),$$

$$\mathrm{dom}(A) := \left\{ f \in \mathfrak{H} : \begin{array}{l} f, \frac{1}{m}f' \in W^{1,2}((x_l, x_r)) \\ f(x_l) = f(x_r) = 0 \\ \left(\frac{1}{m}f'\right)(x_l) = \left(\frac{1}{m}f'\right)(x_r) = 0 \end{array} \right\}.$$

where $m > 0$ and $m + \frac{1}{m} \in L^\infty((x_l, x_r))$, $v \in L^\infty((x_l, x_r))$.

$$\Gamma_0 f := \begin{pmatrix} f(x_l) \\ f(x_r) \end{pmatrix} \quad \text{and} \quad \Gamma_1 f := \frac{1}{2}\begin{pmatrix} \left(\frac{1}{m}f'\right)(x_l) \\ -\left(\frac{1}{m}f'\right)(x_r) \end{pmatrix},$$

$A_0 \Longleftrightarrow$ Dirichlet boundary conditions $\quad A_1 \Longleftrightarrow$ Neumann boundary conditions.

4.4 Scattering

Let us consider the scattering system $\{A_\Theta, A_0\}$.

THEOREM 1. *Let A be a densely defined closed simple symmetric operator with finite deficiency indices in the separable Hilbert space $\mathfrak{H}$ and let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triple for A^* and $M(\cdot)$ be the corresponding Weyl function. Further, let $A_0 = A^* \upharpoonright \ker(\Gamma_0)$ and let $A_\Theta = A^* \upharpoonright \Gamma^{-1}\Theta$ be a self-adjoint extension of A where Θ is a self-adjoint relation in $\mathcal{H}$. Then the scattering matrix $\{S(\lambda)\}_{\lambda\in\mathbb{R}}$ of the complete scattering system $\{A_\Theta, A_0\}$ admits the representation*

$$S(\lambda) = I_{\mathcal{H}_{M(\lambda)}} + 2i\sqrt{\Im m(M(\lambda))}\big(\Theta - M(\lambda)\big)^{-1}\sqrt{\Im m(M(\lambda))}$$

for a.e. $\lambda \in \mathbb{R}$, where $M(\lambda) := M(\lambda + i0)$.

5 Open quantum systems and coupling

5.1 Open quantum system

Let us consider two symmetric operators A and T in $\mathfrak{H}$ and $\mathfrak{K}$, respectively, with equal finite deficiency indices. Further, let $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ and $\{\mathcal{H}, \Upsilon_0, \Upsilon_1\}$ boundary triplets with Weyl functions $M(\lambda)$ and $\tau(\lambda)$, respectively. Then $\{\widetilde{\mathcal{H}}, \widetilde{\Gamma}_0, \widetilde{\Gamma}_1\}$,

$$\widetilde{\mathcal{H}} := \begin{pmatrix} \mathcal{H} \\ \mathcal{H} \end{pmatrix}, \qquad \widetilde{\Gamma}_0 := \begin{pmatrix} \Gamma_0 \\ \Upsilon_0 \end{pmatrix}, \qquad \widetilde{\Gamma}_1 := \begin{pmatrix} \Gamma_1 \\ \Upsilon_1 \end{pmatrix}$$

performs a boundary triplet for $A^* \oplus T^*$ with Weyl function

$$\widetilde{M}(\lambda) := \begin{pmatrix} M(\lambda) & 0 \\ 0 & \tau(\lambda) \end{pmatrix}$$

The systems $\{\mathfrak{H}, A\}$ and $\{\mathfrak{K}, T\}$ are called open system, $\{\mathfrak{H}, A\}$ is called the inner system, $\{\mathfrak{K}, T\}$ is called the outer system. The observer is in the inner system.

5.2 Unperturbed (decoupled) system

The system $\{\mathfrak{L}, A_0 \oplus T_0\}$,

$$\begin{aligned} A_0 &:= A^* \restriction \ker(\Gamma_0), \\ T_0 &:= T^* \restriction \ker(\Upsilon_0), \end{aligned}$$

is called the decoupled system.

5.3 Perturbed (coupled) system

THEOREM 2 (Derkach, Hassi, M. de Snoo, 2000)**.** *Let $\boldsymbol{A}$ and $\boldsymbol{T}$ be densely defined closed symmetric operators in the Hilbert spaces $\mathfrak{H}$ and $\mathfrak{K}$ which equal deficiency indices. Then the following holds:*

(i) *The closed extension $\boldsymbol{L} := \boldsymbol{A}^* \oplus \boldsymbol{T}^* \upharpoonright \widetilde{\boldsymbol{\Gamma}}^{-1}\widetilde{\boldsymbol{\Theta}}$ corresponding to the relation*

$$\widetilde{\boldsymbol{\Theta}} := \left\{ \begin{pmatrix} (\boldsymbol{v}, \boldsymbol{v})^\top \\ (\boldsymbol{w}, -\boldsymbol{w})^\top \end{pmatrix} : \boldsymbol{v}, \boldsymbol{w} \in \boldsymbol{\mathcal{H}} \right\}$$

is self-adjoint in the Hilbert space $\mathfrak{L} := \mathfrak{H} \oplus \mathfrak{K}$ and is given by

$$\boldsymbol{L} = \boldsymbol{A}^* \oplus \boldsymbol{T}^* \upharpoonright \left\{ \boldsymbol{f}_1 \oplus \boldsymbol{f}_2 \in \mathrm{dom}(\boldsymbol{A}^* \oplus \boldsymbol{T}^*) : \begin{array}{l} \boldsymbol{\Gamma}_0 \boldsymbol{f}_1 - \boldsymbol{\Upsilon}_0 \boldsymbol{f}_2 = 0 \\ \boldsymbol{\Gamma}_1 \boldsymbol{f}_1 + \boldsymbol{\Upsilon}_1 \boldsymbol{f}_2 = 0 \end{array} \right\}.$$

(ii) *The Strauss family $\boldsymbol{A}_{-\boldsymbol{\tau}(\boldsymbol{\lambda})} := \boldsymbol{A}^* \upharpoonright \ker(\boldsymbol{\Gamma}_1 + \boldsymbol{\tau}(\boldsymbol{\lambda})\boldsymbol{\Gamma}_0)$, $\boldsymbol{\lambda} \in \mathbb{C}_+$, satisfies*

$$(\boldsymbol{A}_{-\boldsymbol{\tau}(\boldsymbol{\lambda})} - \boldsymbol{\lambda})^{-1} = \boldsymbol{P}_{\mathfrak{H}} \big(\boldsymbol{L} - \boldsymbol{\lambda}\big)^{-1} \upharpoonright \mathfrak{H}, \qquad \boldsymbol{\lambda} \in \mathbb{C}_+.$$

The system $\{\mathfrak{L}, \boldsymbol{L}\}$ is called the coupled system.

5.4 Strauss family

Let $\tau(\cdot) : \mathcal{K} \longrightarrow \mathcal{K}$ be a Nevanlinna function.

$$A_{-\tau(\lambda)} := A^* \upharpoonright \{f \in \mathrm{dom}(A^*) : \Gamma_1 f = -\tau(\lambda)\Gamma_0 f\}, \quad \lambda \in \mathbb{C}_+,$$

$\{A_{-\tau(\lambda)}\}_{\lambda \in \mathbb{C}_+}$ is called a Strauss family.

Since $\dim(\mathcal{H}) < \infty$ the family admits an extension to a.e. $\lambda \in \mathbb{R}$, i.e.

$$\tau(\lambda) := \lim_{\epsilon \to +0} \tau(\lambda + i\epsilon).$$

In general, the Strauss family consists of maximal dissipative operators. The characteristic function of $A_{-\tau(\lambda)}$ are given by

$$\Theta_{A_{-\tau(\lambda)}}(\mu) = I_{\mathcal{Q}_\lambda} + 2i\sqrt{\Im\mathrm{m}(\tau(\lambda))}\left(\tau(\lambda)^* + M(\overline{\mu})^*\right)^{-1}\sqrt{\Im\mathrm{m}(\tau(\lambda))}, \quad \mu \in \mathbb{C}_-,$$

where

$$\mathcal{Q}_\lambda := \mathrm{clo}\{\mathrm{ran}(\Im\mathrm{m}\tau(\lambda))\}.$$

5.5 Scattering

THEOREM 3. *Let A and T be densely defined closed simple symmetric operators in $\mathfrak{H}$ and $\mathfrak{K}$, respectively, with equal finite deficiency indices such that A_0 is discrete. Then*

(i) *The wave operators*

$$W_\pm(L, L_0) = s - \lim_{t\to\pm\infty} e^{itL}e^{-itL_0}P^{ac}(L_0) = s - \lim_{t\to\pm\infty} e^{itL}e^{-itT_0}P^{ac}(T_0)$$

exist and are complete.

(ii) *The scattering matrix $\{S(\lambda)\}_{\lambda\in\mathbb{R}}$ of the scattering system $\{L, L_0\}$ admits the representation*

$$S(\lambda) = I_{\mathfrak{Q}_\lambda} - 2i\sqrt{\Im m \tau(\lambda)}\big(\tau(\lambda) + M(\lambda)\big)^{-1}\sqrt{\Im m \tau(\lambda)}$$

for a.e. $\lambda \in \mathbb{R}$, where $\tau(\lambda) = \tau(\lambda + i0)$ and $M(\lambda) = M(\lambda + i0)$.

(iii) *The scattering matrix $\{S(\lambda)\}_{\lambda\in\mathbb{R}}$ of the scattering system $\{L, L_0\}$ admits the representation*

$$S(\lambda) = \Theta_{A_{-\tau(\lambda)}}(\lambda - i0)^* \qquad (2)$$

for a.e. $\lambda \in \mathbb{R}$ where $\Theta_{A_{-\tau(\lambda)}}(\cdot)$, $\lambda \in \mathbb{R}$, are the characteristic functions of the the Strauss family $\{A_{-\tau(\lambda)}\}_{\lambda\in\mathbb{R}}$.

6 R-matrix

6.1 R-matrix and Weyl functions

One introduces the R-matrix

$$R(\lambda) := i(I_{\mathcal{H}_{\tau(\lambda)}} - S(\lambda))(I_{\mathcal{H}_{\tau(\lambda)}} + S(\lambda))^{-1},$$

which is a bounded operator acting in $\mathcal{H}_{\tau(\lambda)}$. Conversely, one has

$$S(\lambda) = \frac{iI_{\mathcal{H}(\tau(\lambda))} - R(\lambda)}{iI_{\mathcal{H}(\tau(\lambda))} + R(\lambda)}$$

A straightforward calculation shows that

$$R(\lambda) = -\sqrt{\Im\mathrm{m}(\tau(\lambda))}\,(M(\lambda) + (\Re\mathrm{e}(\tau(\lambda)))^{-1})\sqrt{\Im\mathrm{m}(\tau(\lambda))}.$$

If $\Re\mathrm{e}(\tau(\lambda)) = 0$, then

$$R(\lambda) = -\sqrt{\Im\mathrm{m}(\tau(\lambda))}M(\lambda)^{-1}\sqrt{\Im\mathrm{m}(\tau(\lambda))}.$$

6.2 Eigenfunction representation

Let us introduce the self-adjoint extensions

$$A_{-\Re e(\tau(\lambda))} := A^* \restriction \ker(\Gamma_1 + \Re e(\tau(\lambda))\Gamma_0).$$

PROPOSITION 4. *Let A, $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$, $M(\cdot)$ and T, $\{\mathcal{H}, \Upsilon_0, \Upsilon_1\}$, $\tau(\cdot)$ be as above and assume $\sigma(A_0) = \sigma_p(A_0)$ and that A is semibounded from below. If $A_{-\Re e(\tau(\lambda))} \leq A_0$, then the R-matrix admits the representation*

$$R(\lambda) = \sum_{k=1}^{\infty} (\lambda_k[\lambda] - \lambda)^{-1} \left(\cdot, \sqrt{\Im m(\tau(\lambda))}\Gamma_0 \psi_k[\lambda]\right) \sqrt{\Im m(\tau(\lambda))}\Gamma_0 \psi_k[\lambda],$$

where $\{\lambda_k[\lambda]\}$, $k = 1, 2, \ldots$, are the eigenvalues of the selfadjoint extension $A_{-\Re e(\tau(\lambda))}$ in increasing order and $\psi_k[\lambda]$ are the corresponding eigenfunctions.

6.3 Wigner-Eisenbud representation

COROLLARY 5 (Wigner-Eisenbud '46–'47). *If in addition* $\Re\mathrm{e}(\boldsymbol{\tau}(\boldsymbol{\lambda})) = \mathbf{0}$ *and* $\boldsymbol{A}_1 \leq \boldsymbol{A}_0$, *then the* $\boldsymbol{R}$*-matrix admits the representation*

$$\boldsymbol{R}(\boldsymbol{\lambda}) = \sum_{k=1}^{\infty} (\boldsymbol{\lambda}_k - \boldsymbol{\lambda})^{-1} \left(\cdot, \sqrt{\Im\mathrm{m}(\boldsymbol{\tau}(\boldsymbol{\lambda}))}\boldsymbol{\Gamma}_0 \boldsymbol{\psi}_k\right) \sqrt{\Im\mathrm{m}(\boldsymbol{\tau}(\boldsymbol{\lambda}))}\boldsymbol{\Gamma}_0 \boldsymbol{\psi}_k,$$

where $\{\boldsymbol{\lambda}_k\}$, $\boldsymbol{k} = \mathbf{1}, \mathbf{2}, \ldots$, *are the eigenvalues of the selfadjoint extension* $\boldsymbol{A}_1 := \boldsymbol{A}^* \restriction \ker(\boldsymbol{\Gamma}_1)$ *in increasing order and* $\boldsymbol{\psi}_k$ *are the corresponding eigenfunctions.*

In particular, if $\boldsymbol{A}_0 := \boldsymbol{A}^* \restriction \ker(\boldsymbol{\Gamma}_0)$ is the Friedrichs extension, then the condition $\boldsymbol{A}_{-\Re\mathrm{e}(\tau(\lambda))} \leq \boldsymbol{A}_0$ or $\boldsymbol{A}_1 \leq \boldsymbol{A}_0$ is always satisfied.

If the condition $\boldsymbol{A}_{-\Re\mathrm{e}(\tau(\lambda))} \leq \boldsymbol{A}_0$ or $\boldsymbol{A}_1 \geq \boldsymbol{A}_0$ is not satisfied, then Wigner-Eisenbud representation is not true.

7 Example

7.1 Left-hand side outer system

In $\mathfrak{K}_l = L^2((-\infty, x_l))$ one defines

$$(T_l f)(x) := -\frac{1}{2m_l}\frac{d^2}{dx^2} f(x) + v_l f(x),$$

$$\mathrm{dom}(T_l) := \left\{ f \in \mathfrak{K}_l : \begin{array}{l} f, \frac{1}{m_l} f' \in W^{1,2}((-\infty, x_l)) \\ f(x_l) = \left(\frac{1}{m_l} f'\right)(x_l) = 0 \end{array} \right\}.$$

Boundary triplet:

$$\Upsilon^l_0 f := f(x_l) \quad \text{and} \quad \Upsilon^l_1 f = -\left(\frac{1}{2m_l} f'\right)(x_l), \quad f \in \mathrm{dom}(T_l^*),$$

Weyl function: $\quad \tau_l(\lambda) := i\sqrt{\frac{\lambda - v_l}{2m_l}}, \qquad \lambda \in \mathbb{C}_+.$

7.2 Right-hand side outer system

In $\mathfrak{K}_r = L^2((x_r,\infty))$ one defines

$$(T_r f)(x) := -\frac{1}{2m_r}\frac{d^2}{dx^2}f(x) + v_r f(x),$$

$$\mathrm{dom}(T_r) := \left\{ f \in \mathfrak{K}_r : \begin{array}{l} f, \frac{1}{m_r}f' \in W^{1,2}((x_r,\infty)) \\ f(x_r) = \left(\frac{1}{m_r}f'\right)(x_r) = 0 \end{array} \right\}.$$

Boundary triplet

$$\Upsilon_0^r f := f(x_r) \quad \text{and} \quad \Upsilon_1^r f = \left(\frac{1}{2m_r}f'\right)(x_r), \quad f \in \mathrm{dom}(T_r^*),$$

Weyl function:

$$\tau_r(\lambda) := i\sqrt{\frac{\lambda - v_r}{2m_r}}, \qquad \lambda \in \mathbb{C}_+.$$

7.3 Full outer system

Full outer system

$$\boldsymbol{L}^2(\mathbb{R} \setminus (\boldsymbol{x}_l, \boldsymbol{x}_r)) = \boldsymbol{L}^2((-\infty, \boldsymbol{x}_r)) \oplus \boldsymbol{L}^2((\boldsymbol{x}_r, \infty)),$$

$$\boldsymbol{T} = \boldsymbol{T}_l \oplus \boldsymbol{T}_r$$

Boundary triplet:

$$\Upsilon_0 := \Upsilon_0^l \oplus \Upsilon_0^r \quad \text{and} \quad \Upsilon_1 := \Upsilon_1^l \oplus \Upsilon_1^r$$

Weyl function:

$$\tau(\lambda) := \begin{pmatrix} \tau_l(\lambda) & 0 \\ 0 & \tau_r(\lambda) \end{pmatrix}$$

7.4 Strauss family

$$\mathrm{dom}(\boldsymbol{A}_{-\tau(\lambda)}) := \left\{ \boldsymbol{f} \in \mathfrak{H} : \begin{array}{c} \boldsymbol{f}, \frac{1}{m}\boldsymbol{f}' \in \boldsymbol{W}^{1,2}((\boldsymbol{x}_l, \boldsymbol{x}_r)) \\ (\frac{1}{2m}\boldsymbol{f}')(\boldsymbol{x}_l) = -\boldsymbol{\tau}_l(\boldsymbol{\lambda})\boldsymbol{f}(\boldsymbol{x}_l) \\ (\frac{1}{2m}\boldsymbol{f}')(\boldsymbol{x}_r) = \boldsymbol{\tau}_r(\boldsymbol{\lambda})\boldsymbol{f}(\boldsymbol{x}_r) \end{array} \right\}, \quad \boldsymbol{\lambda} \in \mathbb{C}_+,$$

and

$$(\boldsymbol{A}_{-\tau(\lambda)}\boldsymbol{f})(\boldsymbol{x}) = -\frac{\mathbf{1}}{\mathbf{2}}\frac{\boldsymbol{d}}{\boldsymbol{dx}}\frac{\mathbf{1}}{\boldsymbol{m}}\frac{\boldsymbol{d}}{\boldsymbol{dx}}\boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{v}(\boldsymbol{x})\boldsymbol{f}(\boldsymbol{x}), \quad \boldsymbol{x} \in (\boldsymbol{x}_l, \boldsymbol{x}_r),$$

$\boldsymbol{f} \in \mathrm{dom}(\boldsymbol{A}_{-\tau(\lambda)})$, $\boldsymbol{\lambda} \in \mathbb{C}_+$.

Characteristic function:

$$\boldsymbol{\Theta}_{A_{-\tau(\lambda)}}(\boldsymbol{\mu}) = \boldsymbol{I}_{\mathcal{H}_{\tau(\lambda)}} - \boldsymbol{i}\sqrt{\mathbf{2}\,\Im\mathrm{m}(\boldsymbol{\tau}(\boldsymbol{\lambda}))}\Gamma_0(\boldsymbol{A}^*_{-\tau(\lambda)} - \boldsymbol{\mu})^{-1}\Gamma_0^*\sqrt{\mathbf{2}\,\Im\mathrm{m}(\boldsymbol{\tau}_l(\boldsymbol{\lambda}))}, \quad \boldsymbol{\mu} \in \mathbb{C}_-.$$

for $\boldsymbol{\lambda} \in \Sigma^{\tau}$.

7.5 Scattering

Scattering system $\{L, L_0\}$,

$$L_0 f = -\frac{1}{2}\frac{d}{dx}\frac{1}{M}\frac{d}{dx}f + Vf,$$

with domain

$$\mathrm{dom}(L_0) := W_0^{2,2}((-\infty, x_l)) \oplus \left\{ f \in W^{1,2}((x_l, x_r)) : \begin{array}{c} \frac{1}{m}f' \in W^{1,2}((x_l, x_r)) \\ f(x_l) = f(x_r) = 0 \end{array} \right\} \oplus W_0^{2,2}((x_r, \infty)).$$

Notice that L_0 is the Friedrichs extension.

Scattering matrix:

$$S(\lambda) := \Theta_{A_{-\tau(\lambda)}}(\lambda - i0)^*.$$

7.6 R-matrix

We note that

$$\Re\mathrm{e}(\tau(\lambda)) = 0, \qquad \lambda \in (\max\{v_l, v_r\}, \infty).$$

Further

$$A_1 := A^* \upharpoonright \ker(\Gamma_1)$$

with

$$\ker(\Gamma_1) := \left\{ f \in W^{1,2}((x_l, x_r)) : \begin{array}{c} \frac{1}{m} f' \in W^{1,2}((x_l, x_r)) \\ \left(\frac{1}{2m} f'\right)(x_l) = \left(\frac{1}{2m} f'\right)(x_r) = 0 \end{array} \right\}.$$

The $\boldsymbol{R}$-matrix admits the representation

$$R(\lambda) = \sum_{k \in \mathbb{N}} (\lambda_k - \lambda)^{-1} \left\langle \cdot, \begin{pmatrix} \sqrt{\Im\mathrm{m}(\tau_l(\lambda))}\psi_k(x_l) \\ \sqrt{\Im\mathrm{m}(\tau_r(\lambda))}\psi_k(x_r) \end{pmatrix} \right\rangle \begin{pmatrix} \sqrt{\Im\mathrm{m}(\tau_l(\lambda))}\psi_k(x_l) \\ \sqrt{\Im\mathrm{m}(\tau_r(\lambda))}\psi_k(x_r) \end{pmatrix}$$

for $\lambda \in (\max\{v_l, v_r\}, \infty)$ where λ_k and ψ_k are the eigenvalues and eigenfunctions of A_1.