Schrödinger operators with random δ magnetic fields

by

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Schrödinger operators with random magnetic fields on $\mathbf{R}^2$

We consider the magnetic Schrödinger operators on $\mathbf{R}^2$

$$\mathcal{L}_\omega = \left(\frac{1}{i}\nabla + \mathbf{a}_\omega\right)^2,$$

where $\mathbf{a}_\omega = (a_{\omega,x}, a_{\omega,y})$ is the magnetic vector potential, $\omega \in \Omega$ is a random parameter. The function

$$\operatorname{rot} \mathbf{a}_\omega = \partial_x a_{\omega,y} - \partial_y a_{\omega,x}$$

denotes the magnetic field perpendicular to the plane.

Poisson-Anderson type δ-fields

We assume

$$\mathrm{rot}\,\mathbf{a}_\omega(z) = B + \sum_{\gamma \in \Gamma_\omega} 2\pi \alpha_\gamma(\omega)\delta(z - \gamma). \tag{1}$$

Here,

- B is a positive constant independent of ω,
- Γ_ω is the Poisson configuration (the support of the Poisson point process) with intensity $\rho\, dxdy$ ($\rho > 0$ is a constant),

- $\{\alpha_\gamma\}_{\gamma\in\Gamma}$ is a sequence of i.i.d. random variables independent of Γ_ω, satisfying
$$0 \le \alpha_\gamma < 1$$
for any $\gamma \in \Gamma_\omega$. We denote
$$p = \mathbf{P}\{\alpha_\gamma \neq 0\}, \quad \bar{\alpha} = \mathbf{E}[\alpha_\gamma].$$

Eliminating a set of measure zero from Ω, we may assume $0 \notin \Gamma_\omega$.

Poisson point process

Put $N_\omega(E) = \#(E \cap \Gamma_\omega)$. Then,

1) For every disjoint measurable sets $E_1, \ldots, E_n$, the random variables $N_\omega(E_1), \ldots, N_\omega(E_n)$ are independent.

2) For every measurable set $|E| < \infty$,

$$P\{N_\omega(E) = k\} = e^{-\rho|E|}\frac{(\rho|E|)^k}{k!} \quad (k = 0, 1, 2, \ldots).$$

From this, we have $\mathbf{E}[N_\omega(E)] = \mathbf{V}[N_\omega(E)] = \rho|E|$.

Construction of $\mathbf{a}_\omega$

Identify a vector $z = (x, y)$ with a complex number $z = x + iy$. Put

$$\zeta_\omega(z) = \sum_{\gamma \in \Gamma_\omega} \alpha_\gamma \left(\frac{1}{z - \gamma} + \frac{1}{\gamma} + \frac{z}{\gamma^2} \right).$$

The sum converges locally uniformly in $\mathbf{C} \setminus \Gamma_\omega$ with probability one. Put $\phi_\omega(z) = \dfrac{B\bar{z}}{2} + \zeta_\omega(z)$ and put

$$\mathbf{a}_\omega(z) = (\operatorname{Im} \phi_\omega(z), \operatorname{Re} \phi_\omega(z)).$$

Then, $\mathbf{a}_\omega$ satisfies (1).

Self-adjoint extension

We denote the Friedrichs extension of $\mathcal{L}_\omega|_{C_0^\infty(\mathbf{R}^2\setminus\Gamma_\omega)}$ by H_ω. A function $u \in L^2(\mathbf{R}^2)$ belongs to $D(H_\omega)$ if and only if

- $\mathcal{L}_\omega u \in L^2(\mathbf{R}^2)$,
- $\limsup_{z\to\gamma} |u(z)| < \infty, \quad \forall \gamma \in \Gamma_\omega$.

Problem

When $\mathrm{rot}\,\mathbf{a} = B$ (constant magnetic field), it is known that the n-th Landau level $E_n = (2n-1)B$ is an eigenvalue of multiplicity ∞ for every $n = 1, 2, \ldots$. Does E_n remain eigenvalue of multiplicity ∞, even if random δ magnetic fields are added?

Answer Yes, if the magnetic fields are sufficiently strong (the threshold value depends on n). The opposite is (almost) true for the lowest Landau level E_1.

Known results

Nambu '00, Exner-Šťovíček-Vytřas'02

Assume $\Gamma = \{0\}$ (one point) and $\alpha_0 = \alpha$, $0 < \alpha < 1$. Then,

$$\begin{aligned}\sigma(H) &= \{E_n \mid n = 1, 2, \ldots\} \\ &\qquad \cup \ \{E_n + 2\alpha B \mid n = 1, 2, \ldots\}, \\ \mathrm{mult}(E_n; H) &= \infty, \\ \mathrm{mult}(E_n + 2\alpha B; H) &= n.\end{aligned}$$

(Moreover, Exner et al. investigates the spectrum of all the self-adjoint extensions of $\mathcal{L}|_{C_0^\infty(\mathbf{R}^2 \setminus \{0\})}$.)

M-N '06

Let $n \in \mathbf{N}$. Assume Γ is a non-random lattice, $(\alpha_\gamma)_{\gamma \in \Gamma}$ is periodic, and $0 < \alpha_\gamma < 1$ for all γ. Put $\Phi = \dfrac{B}{2\pi}|\mathcal{D}| + \bar{\alpha}$, where $\mathcal{D}$ is the fundamental domain of Γ. Then,

1) If $\Phi > n$, then $\mathrm{mult}(E_n; H) = \infty$.

2) If $\Phi \leq 1$, then $\mathrm{mult}(E_1; H) = 0$.

(The value $2\pi\Phi$ denotes the average of the magnetic flux per one δ obstacle.)

M-N, at OTQP' 06 in Czech republic

Assume Γ_ω is the Poisson configuration with intensity $\rho dxdy$, $\alpha_\gamma = \alpha$ (cosntant) for every γ, $0 < \alpha < 1$. Then, for every $n_0 \in \mathbf{N}$, there exists a constant $C = C(\alpha, n_0) \geq 0$ such that

$$B/\rho > C \ \Rightarrow \ \mathrm{mult}(E_n; H_\omega) = \infty \quad (n = 1, \ldots, n_0)$$

almost surely.

Today's result

Theorem 1. Let $n \in \mathbf{N}$ and put $\Phi = \dfrac{B}{2\pi\rho} + \bar{\alpha}$. Then,

1) If $\Phi > np$, then $\mathrm{mult}(E_n; H_\omega) = \infty$ almost surely.

2) If $\Phi < p$, then $\mathrm{mult}(E_1; H_\omega) = 0$ almost surely.

(The value $2\pi\Phi$ also denotes the average of the magnetic flux per one δ obstacle, since $\mathbf{E}[\#(\Gamma_\omega \cap \mathcal{D})] = 1$ if $|\mathcal{D}| = 1/\rho$.)

Remark

1) When Γ is a non-random lattice and $\{\alpha_\gamma\}$ is i.i.d. (Anderson type), the same conclusion also holds if we put

$$\Phi = \frac{B}{2\pi}|\mathcal{D}| + \bar{\alpha},$$

where $\mathcal{D}$ is the fundamental domain of the lattice Γ. This is an extension of the result M-N '06 in the periodic case.

2) Nothing is known in the threshold case $\Phi = p$, at present.

Related results

Geyler-Grishanov '02, Geyler-Šťovíček '04, Rozenblum-Shirokov '06 Zero-modes for the Pauli operators with (a constant magnetic field $+$) δ magnetic fields in various cases (periodic, etc.). Rozenblum and Shirokov also investigates the case the magnetic field is a signed measure.

Remark. The zero-mode of a component of the Pauli operator corresponds to the lowest Landau level of the Schrödinger operator.

There are similar results for Schrödinger operators with a constant magnetic field plus point interactions (not δ-magnetic fields); e.g.,

- Geĭler '92,
- Avishai-Redheffer-Band '92, Avishai-Redheffer '93, Avishai-Azbel-Gredeskul '93,
- Dorlas-Macris-Pulé '99.

Strategy

In the rest of time, we present an outline of the proof of Theorem 1. The main strategy is the following:

1) Construct eigenfunctions explicitly, using canonical commutation relations.

2) Estimate the growth order of the eigenfunctions by (an extension of) the entire function theory.

The argument similar to 2) above is used in Chistyakov-Lyubarskii-Pastur '01 "On completeness of random exponentials in the Bargmann-Fock space".

Canonical commutation relations

Put

$$\mathcal{A} = 2\partial_z + \phi_\omega(z), \quad \mathcal{A}^\dagger = -2\partial_{\bar{z}} + \overline{\phi_\omega(z)}.$$

These operators satisfy the canonical commutation relations

$$\mathcal{A}\mathcal{A}^\dagger = \mathcal{L} + B, \quad \mathcal{A}^\dagger\mathcal{A} = \mathcal{L} - B,$$

on $\mathbf{C} \setminus \Gamma$. By the above relations, we can (formally) show

$$\begin{aligned} \mathcal{A}u = 0 \qquad &\Rightarrow \qquad \mathcal{L}u = Bu, \\ \mathcal{L}u = Eu \qquad &\Rightarrow \qquad \mathcal{L}\mathcal{A}^\dagger u = (E + 2B)\mathcal{A}^\dagger u. \end{aligned}$$

Multi-valued canonical product

Definition 2. Let Γ be a discrete subset of $\mathbf{C}$ satisfying

$$\#(\Gamma \cap \{|z| < R\}) = O(R^2) \quad R \to \infty. \tag{2}$$

Let $\alpha = (\alpha_\gamma)_{\gamma \in \Gamma}$ be a sequence of bounded non-negative numbers. Define

$$\sigma_{\Gamma,\alpha}(z) \;=\; \prod_{\gamma \in \Gamma} \left(1 - \frac{z}{\gamma}\right)^{\alpha_\gamma} e^{\alpha_\gamma \left(\frac{z}{\gamma} + \frac{z^2}{2\gamma^2}\right)}.$$

We call the multi-valued function $\sigma_{\Gamma,\alpha}$ the (multi-valued) canonical product for (Γ, α) (this definition is a natural generalization of the one in the entire function theory). We denote $\alpha(\omega) = (\alpha_\gamma(\omega))_{\gamma \in \Gamma_\omega}$ and

$$\sigma_\omega = \sigma_{\Gamma_\omega, \alpha(\omega)}, \quad \widetilde{\sigma_\omega} = \sigma_{\Gamma_\omega, \widetilde{\alpha(\omega)}},$$

where $\widetilde{\alpha(\omega)} = (\widetilde{\alpha_\gamma}(\omega))_{\gamma \in \Gamma_\omega}$,

$$\widetilde{\alpha_\gamma}(\omega) = \begin{cases} 1 & (\text{if } 0 < \alpha_\gamma(\omega) < 1), \\ 0 & (\text{if } \alpha_\gamma(\omega) = 0). \end{cases}$$

The function σ_ω satisfies

$$\sigma'_\omega(z) = \sigma_\omega(z)\zeta_\omega(z).$$

Explicit solutions

Proposition 3. Let n be a positive integer and f an entire function. Put

$$u(z) = {\mathcal{A}^{\dagger}}^{n-1}\left(e^{-\frac{B}{4}|z|^2}|\sigma_\omega(z)|^{-1}\overline{\widetilde{\sigma_\omega}(z)^n f(z)}\right).$$

If $u \in L^2$, then $u \in D(H_\omega)$ and $H_\omega u = E_n u$. Moreover, all the solution of $H_\omega u = E_1 u$ is written in the above form with $n = 1$.

Entire function theory

Theorem 4 (Levin). Let Γ, α given in Definition 2, and assume all α_γ are integers. For $0 \leq \theta_2 - \theta_1 \leq 2\pi$, put

$$n(r, \theta_1, \theta_2) = \sum_{\gamma \in \Gamma,\ 0 < |\gamma| \leq r,\ \theta_1 \leq \arg \gamma < \theta_2} \alpha_\gamma.$$

Assume the limit

$$\Delta(\theta_1, \theta_2) = \lim_{r \to \infty} \frac{n(r, \theta_1, \theta_2)}{r^2}$$

exists for all θ_1, θ_2 except countable values. Assume additionally the limit

$$\delta = \frac{1}{2} \lim_{r\to\infty} \sum_{\gamma\in\Gamma, 0<|\gamma|\le r} \frac{\alpha_\gamma}{\gamma^2}$$

exists and finite. Then, there exists a C^0-set $\mathcal{C}$ such that

$$\lim_{r\to\infty, re^{i\theta}\notin\mathcal{C}} \frac{\log|\sigma_{\Gamma,\alpha}(re^{i\theta})|}{r^2} = H(\theta) \tag{3}$$

holds for every $\theta \in [0, 2\pi)$, where

$$H(\theta) = -\int_{\theta-2\pi}^{\theta} (\psi - \theta) \sin 2(\psi - \theta) d\Delta(\psi) + \mathrm{Re}(e^{2i\theta}\delta).$$

The convergence of (3) is uniform with respect to $\theta \in [0, 2\pi)$.

C^0-set

Definition. A C^0-set is the union of disks $\{|z - z_j| < r_j\}$ satisfying

$$\lim_{r \to \infty} \frac{1}{r} \sum_{|z_j| \le r} r_j = 0.$$

(C^0-set is a set 'rare' at infinity.)

Outline of proof of Theorem 1

Proposition 5. Theorem 4 is true if $\alpha_\gamma \geq 0$ for all $\gamma \in \Gamma$ (not necessarily be integers).

(The proof of Proposition 5 is almost parallel to that of Theorem 4.)

Lemma 6. The assumption of Proposition 5 is satisfied with $\Gamma = \Gamma_\omega$, $\alpha = \beta(\omega) = (n\widetilde{\alpha_\gamma}(\omega) - \alpha_\gamma(\omega))_{\gamma \in \Gamma_\omega}$, and

$$\Delta(\theta_1, \theta_2) = \rho(\theta_2 - \theta_1)(np - \bar{\alpha})/2,$$

with probability one. (Notice that $\sigma_\omega^{-1}\widetilde{\sigma_\omega}^n = \sigma_{\Gamma,\alpha}$, which appeared in the solution u.)

Corollary 7. Let u be the solution given in Proposition 3, where $f = e^{-\delta z^2} g$, δ is the constant given in Proposition 5 for $(\Gamma_\omega, \beta(\omega))$, g is a non-zero polynomial.

(1) For almost all ω, for every $\epsilon > 0$ we have

$$|u(z)| \leq \exp\left(\left(-\frac{B}{4} + \frac{\pi\rho(np - \bar{\alpha})}{2} + \epsilon\right)|z|^2\right)$$

for sufficiently large z.

(2) Assume $n = 1$. Then, for almost all ω, we have

$$|u(z)| \geq \exp\left(\left(-\frac{B}{4} + \frac{\pi\rho(np - \bar{\alpha})}{2} - \epsilon\right)|z|^2\right)$$

for sufficiently large z outside some C^0-set $\mathcal{C}$.

Another topics

- We have the Lifshitz tail

$$N(\lambda) \le e^{-c\lambda^{-1}} \quad \text{for } \lambda > 0$$

  for the Anderson type δ-magnetic fields with $B = 0$, $\operatorname{supp} \mu \ni 0$ and $\operatorname{supp} \mu \ne \{0\}$ (μ is the common distribution measure for α_γ). It is not yet proved for the Poisson type.

- Anderson localization is not proved in both cases.