Scattering theory for Jacobi operators with quasi-periodic background

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Outline of the talk

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2. Quasi-periodic background
Scattering theory (existence of Jost solutions, transformation operators, scattering data, Gel'fand-Levitan-Marchenko equation)
Inverse scattering theory (reconstruction, existence & uniqueness)
3. 'Step-like' quasi-periodic background

PART II : **Initial value problem of the Toda Hierarchy**

4. The Toda Hierarchy (finite-gap solutions)
5. Inverse scattering transform (time evolution of scattering data)

PART I

Scattering theory for Jacobi operators

Jacobi operators

For $u \in \ell^2(\mathbb{Z}, \mathbb{C})$ the Jacobi operator H is defined by

$$Hu(n) = a(n)u(n+1) + b(n)u(n) + a(n-1)u(n-1),$$

where a, b are bounded and real valued sequences.

H is bounded, self-adjoint, and is associated to the real tridiagonal infinite symmetric matrix

$$\begin{pmatrix} \ddots & \ddots & & & \\ \ddots & b(n-1) & a(n-1) & & \\ & a(n-1) & b(n) & a(n) & \\ & & a(n) & b(n+1) & \ddots \\ & & & \ddots & \ddots \end{pmatrix}.$$

Jacobi operators

appear in a variety of applications:

- discrete analogue of Sturm-Liouville operators
- orthogonal polynomials on the real line
- play a fundamental role in investigation of completely integrable non-linear lattices - Lax pair

Classical scattering theory

Given a Jacobi operator H which is a short range perturbation

$$\sum_{n\in\mathbb{Z}} |n|\Big(\Big|a(n)-\frac{1}{2}\Big| + \Big|b(n)-0\Big|\Big) < \infty$$

of the free Jacobi operator H_0 associated with $a_0(n)=\frac{1}{2}$, $b_0(n)=0$, can one find "scattering data" which determine H uniquely?

We want to replace the free Jacobi operator by a quasi-periodic one (which include periodic ones as a special case).

Can we even replace the free one by two quasi-periodic ones with different asymptotics on each side, that is, with

$$\sum_{n=0}^{\pm\infty} |n|\Big(\Big|a(n)-a_q^{\pm}(n)\Big| + \Big|b(n)-b_q^{\pm}(n)\Big|\Big) < \infty?$$

Can we find scattering data which uniquely determine H in these cases?

What is known?

Scattering theory for Jacobi operators:
constant background: Case 1973, Guseinov 1976, Teschl ’00
step-like constant bg: Egorova ’02
first results for periodic bg: Volberg-Yuditskii ’02, Boutet de Monvel-Egorova ’04
first results for step-like periodic bg: Bazargan-Egorova ’03

Scattering theory for Sturm-Liouville operators:
constant bg: Gel’fand, Levitan, Marchenko 1950
periodic bg: Firsova 1987, Gesztesy-Nowell-Pötz 1997

Quasi-periodic Jacobi operators (I)

Let H_q be a quasi-periodic Jacobi operator associated with the Riemann Surface

$$y^2 = R_{2g+2}(z) = \prod_{j=0}^{2g+1} (z - E_j), \qquad E_0 < E_1 < \cdots < E_{2g+1},$$

that is,

$$\begin{aligned} a_q(n)^2 &= \tilde{a}^2 \frac{\theta(\underline{z}(n+1))\theta(\underline{z}(n-1))}{\theta(\underline{z}(n))^2}, \\ b_q(n) &= \tilde{b} + \sum_{j=1}^{g} c_j(g) \frac{\partial}{\partial w_j} \ln \left(\frac{\theta(\underline{w} + \underline{z}(n))}{\theta(\underline{w} + \underline{z}(n-1))} \right) \Big|_{\underline{w}=0}. \end{aligned}$$

Here θ is the Riemann theta function and

$$\underline{z}(p, n) = \underline{\hat{A}}_{p_0}(p) - \underline{\hat{\alpha}}_{p_0}(\mathcal{D}_{\underline{\hat{\mu}}(n)}) - \underline{\hat{\Xi}}_{p_0} \in \mathbb{C}^g, \quad \underline{z}(n) = \underline{z}(\infty_+, n),$$

where $\underline{\hat{A}}_{p_0}$, $\underline{\hat{\alpha}}_{p_0}$, and $\underline{\Xi}_{p_0}$ are Abel map for points, divisors, and the vector of Riemann constants, respectively.

A special case of quasi-periodic Jacobi operators are periodic ones,

$$a_q(n+N) = a_q(n), \; b_q(n+N) = b_q(n), \; N \in \mathbb{N}.$$

Quasi-periodic Jacobi operators (II)

The Baker-Akhiezer function is given by

$$\psi_q(p,n) = \sqrt{\frac{\theta(\underline{z}(-1))\theta(\underline{z}(0))}{\theta(\underline{z}(n-1))\theta(\underline{z}(n))}}\frac{\theta(\underline{z}(p,n))}{\theta(\underline{z}(p,0))} \exp\left(n\int_{E_0}^{p}\omega_{\infty_+,\infty_-}\right),$$

where $\omega_{\infty_+,\infty_-}$ is the Abelian differential of the third kind with simple poles at $\infty_\pm$ and residues ± 1. The two branches

$$\psi_{q,\pm}(z,n) = \prod_{j=0}^{n-1}\phi_{q,\pm}(z,j)$$

of the BA function are solutions of $H_q\psi = z\psi$, where

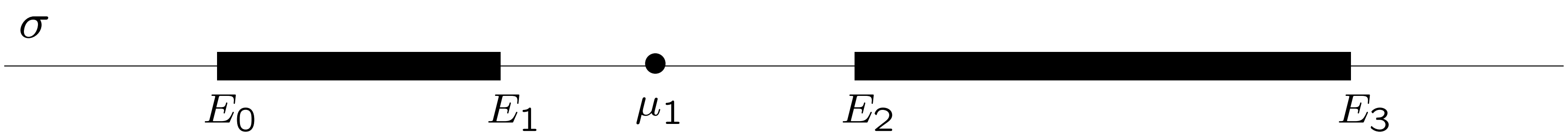
$$\phi_{q,\pm}(z,n) = \frac{1}{2a_q(n)}\left(z - b_q(n) + \sum_{j=1}^{g}\frac{\hat{R}_j(n)}{z-\mu_j(n)} \pm \frac{R_{2g+2}^{1/2}(z)}{\prod_{j=1}^{g}(z-\mu_j(n))}\right).$$

Here $R_j(n) = \frac{R_{2g+1}^{1/2}(\mu_j(n))}{\prod_{k\neq j}(\mu_j(n)-\mu_k(n))}$ and μ_j are the Dirichlet eigenvalues of H_q located in the spectral gaps.

Spectrum of quasi-periodic Jacobi operators

The spectrum of H_q is purely absolutely continuous and consists of the branch cuts of the root $R_{2g+2}^{1/2}(z)$:

$$\sigma(H_q) = \bigcup_{j=0}^{g} [E_{2j}, E_{2j+1}]$$



The quasi-momentum map

The Abelian differential in the Baker-Akhiezer function is given by

$$\omega_{\infty_+,\infty_-} = \frac{\prod_{j=1}^{g}(z-\lambda_j)}{R_{2g+2}^{1/2}(z)}dz,$$

(λ_j are real numbers sitting in the spectral gaps). It is the average of the Green function (spectral theory) and $\int_{E_0}^{z}\widehat{\omega}_{\infty_+,\infty_-}$ is the Green function (potential theory) of the upper sheet Π_+ with pole at ∞_+.

Hence the quasi momentum map

$$w(z) = \exp\left(\int_{E_0}^{z}\widehat{\omega}_{\infty_+,\infty_-}\right)$$

maps the upper sheet Π_+ to the unit circle. Since Π_+ is *not* simply connected, it is only conformal after removing the slits corresponding to the spectral gaps.

Orthonormal basis on the unit circle

By the spectral theorem,

Theorem 1 *The Baker-Akhiezer functions* $\psi_{q,\pm}(\lambda, n)$ *form a complete orthogonal system on the spectrum with respect to the weight*

$$d\omega(\lambda) = \frac{1}{2\pi\mathrm{i}} \frac{\prod_{j=1}^{g}(\lambda - \mu_j)}{R_{2g+2}^{1/2}(z)} d\lambda,$$

namely

$$\oint_{\sigma} \overline{\psi_{q,\pm}(\lambda, m)} \psi_{q,\pm}(\lambda, n) d\omega = \delta(n, m),$$

where

$$\oint_{\sigma} f(\lambda) d\lambda := \int_{\sigma^{\mathrm{u}}} f(\lambda^{\mathrm{u}}) d\lambda - \int_{\sigma^{\mathrm{l}}} f(\lambda^{\mathrm{l}}) d\lambda.$$

The numbers μ_j located in the spectral gaps are the Dirichlet eigenvalues of H_q.

Perturbations of quasi-periodic Jacobi operators

Let H be a perturbation of H_q such that

$$\sum_{n\in\mathbb{Z}} |n|\Big(|a(n) - a_q(n)| + |b(n) - b_q(n)|\Big) < \infty.$$

Theorem 2 *(i) There exist Jost solutions $\psi_\pm(z,.)$ of*

$$H\psi_\pm(z,n) = z\psi_\pm(z,n)$$

which asymptotically look like the quasi-periodic solutions

$$\lim_{n\to\pm\infty} \Big| w(z)^{\mp n}(\psi_\pm(z,n) - \psi_{q,\pm}(z,n))\Big| = 0.$$

Spectrum of H:

(ii) $\sigma := \sigma_{ess}(H) = \sigma(H_q) = \bigcup_{j=0}^{g}[E_{2j}, E_{2j+1}]$.

(iii) $\sigma_p(H) = \{\rho_j\}_{j=1}^{q} \subset \mathbb{R}\backslash\sigma(H_q)$.

The transformation operator

We define the transformation operator by computing the Fourier expansion of $\psi_\pm(\lambda, n)$ with respect to the ONS $\psi_{q,\pm}(\lambda, n)$

$$K_\pm(n,m) = \oint_\sigma \psi_\pm(\lambda,n)\psi_{q,\mp}(\lambda,m) d\omega.$$

By construction we have $HK_\pm = K_\pm H_q$. By the Cauchy theorem,

$$K_\pm(n,m) = 0 \quad \text{for} \quad \pm m < \pm n,$$

therefore

$$\psi_\pm(z,n) = \sum_{m=n}^{\pm\infty} K_\pm(n,m)\psi_{q,\pm}(z,m).$$

[Boutet de Monvel-Egorova '04]: Transformation operator for periodic bg.

Properties of the transformation operator

$HK_\pm = K_\pm H_q$ implies

Theorem 3

$$
\begin{aligned}
\frac{a(n)}{a_q(n)} &= \frac{K_+(n+1,n+1)}{K_+(n,n)} = \frac{K_-(n,n)}{K_-(n+1,n+1)},\\
b(n)-b_q(n) &= a_q(n)\frac{K_+(n,n+1)}{K_+(n,n)} - a_q(n-1)\frac{K_+(n-1,n)}{K_+(n-1,n-1)}\\
&= a_q(n-1)\frac{K_-(n,n-1)}{K_-(n,n)} - a_q(n)\frac{K_-(n+1,n)}{K_-(n+1,n+1)},\\
\prod_{m=-\infty}^{\infty}\frac{a_q(m)}{a(m)} &= K_+(n,n)K_-(n,n).
\end{aligned}
$$

$K_\pm$ satisfy the crucial technical estimate

$$
|K_\pm(n,m)| \le C \sum_{j=[\frac{m+n}{2}]\pm 1}^{\pm\infty} \Big(|a(j)-a_q(j)| + |b(j)-b_q(j)|\Big), \quad \pm m > \pm n.
$$

Scattering data

Define the transmission T and reflection $R_\pm$ coefficients via the scattering relations

$$T(\lambda)\psi_\mp(\lambda,n) = \overline{\psi_\pm(\lambda,n)} + R_\pm(\lambda)\psi_\pm(\lambda,n), \qquad \lambda \in \sigma(H_q),$$

and for each eigenvalue ρ_j the norming constants $\gamma_{\pm,j}$ via

$$\gamma_{\pm,j}^{-1} = \sum_{n\in\mathbb{Z}} \hat{\psi}_\pm(\rho_j,n)^2, \qquad \hat{\psi}_\pm(z,n) = \Big(\prod_{\mu_\ell\in M_\pm}(z-\mu_\ell)\Big)\psi_\pm(z,n).$$

Then the left/right scattering data are the sets

$$S_\pm(H) = \{R_\pm(\lambda), \lambda\in\sigma;\ (\rho_j,\gamma_{\pm,j}), 1\le j\le q\}.$$

Relation between the left/right scattering data

There is a meromorphic continuation of T to Π_+ with simple poles at the eigenvalues ρ_j. The residua are given by

$$(\mathrm{res}_{\rho_j} T)^2 = \gamma_{+,j}\gamma_{-,j} \prod_{k=0}^{2g+1} (\rho_j - E_k).$$

By direct computation,

$$|T(z)|^2 + |R_\pm(z)|^2 = 1,$$
$$T(z)\overline{R_+(z)} + \overline{T(z)} R_-(z) = 0.$$

Thus if the transmission coefficient T is known, we can compute $S_-(H)$ from $S_+(H)$ and vice versa.

Reconstructing T from its boundary values

Since we know the absolute value of T on the boundary, $|T(z)|^2 = 1 - |R_\pm(z)|^2$, and since T is meromorphic in Π_+, we can reconstruct T from S_+ (this is nontrivial, since Π_+ is *not* simply connected):

$$T(z) = \exp\Big(- \sum_j g(z, \rho_j)\Big) \exp\Big(\frac{1}{4\pi} \int_{\partial \Pi_+} \ln(1 - |R_\pm(z)|^2) \mu(z, x) dx \Big)$$

Here $g(z, z_0)$ is the Green function of the domain Π_+ and is given by $g(z, z_0) = \int_{E_0}^{z} \omega_{z_0, \tilde{z}_0}$, where $\tilde{z}_0$ is the complex conjugate of z_0 on the lower sheet Π_-. Moreover, $\mu(z, x_0)$ is the harmonic measure

$$\frac{\partial}{\partial y_0} g(z, x_0 + \mathrm{i} y_0)\Big|_{y_0 = 0}.$$

Note that T is not single-valued in general, since both the Blaschke produkt and the outer function are not single-valued (it depends on the path of integration).

Gel'fand-Levitan-Marchenko equation

Taking the Fourier transform of $T(\lambda)\psi_{\mp}(\lambda,n) = \overline{\psi_{\pm}(\lambda,n)} + R_{\pm}(\lambda)\psi_{\pm}(\lambda,n)$, $\lambda \in \sigma(H_q)$, gives the Gel'fand-Levitan-Marchenko equation

$$K_{\pm}(n,m) + \sum_{l=n}^{\pm\infty} K_{\pm}(n,l) F^{\pm}(l,m) = \frac{\delta(n,m)}{K_{\pm}(n,n)}, \qquad \pm m \geq \pm n,$$

where

$$\begin{aligned} F^{\pm}(l,m) &= \tilde{F}^{\pm}(l,m) + \sum_{j=1}^{q} \gamma_{\pm,j} \hat{\psi}_{q,\pm}(\rho_j,l) \hat{\psi}_{q,\pm}(\rho_j,m), \\ \tilde{F}^{\pm}(l,m) &= \oint_{\sigma} R_{\pm}(\lambda) \psi_{q,\pm}(\lambda,l) \psi_{q,\pm}(\lambda,m) d\omega. \end{aligned}$$

Theorem 4 *The GLM equation has a unique solution, so the scattering data $S_+(H)$ determine H uniquely and H can be reconstructed from $S_+(H)$ solving the GLM equation.*

Inverse scattering theory

We can reconstruct the operator H from given scattering data

$$S_\pm = \{R_\pm(\lambda), \lambda \in \sigma; (\rho_j, \gamma_{\pm,j}), 1 \le j \le q\}$$

and a given quasi-periodic Jacobi operator H_q.

The remaining question is when given scattering data $S_+(H)$ give rise to a Jacobi operator H?

Conditions: **I** $R_\pm(\lambda)$ are continuous except possibly at E_j,

$$R_\pm(\lambda^{\mathrm{u}}) = \overline{R_\pm(\lambda^{\mathrm{l}})}, \qquad |R_\pm(\lambda)| < 1 \quad \text{for} \quad \lambda \neq E_j.$$

II The eigenvalues ρ_j must be such that the transmission coefficient $T(z)$ extends to a single valued function on Π_+.

Note that **II** is void in the constant background case!

Conditions for the scattering data

III $\tilde{F}^{\pm}(l,m)$ *must have the proper decay rate:*

$$|\tilde{F}^{\pm}(n,m)| \leq \sum_{j=n+m}^{\pm\infty} q(j), \qquad q(j) \geq 0, \qquad |j|q(j) \in \ell^1(\mathbb{Z}),$$

$$\sum_{n=n_0}^{\pm\infty} |n| \Big| \tilde{F}^{\pm}(n,n) - \tilde{F}^{\pm}(n \pm 1, n \pm 1) \Big| < \infty,$$

$$\sum_{n=n_0}^{\pm\infty} |n| \Big| a_q(n) \tilde{F}^{\pm}(n,n+1) - a_q(n-1) \tilde{F}^{\pm}(n-1,n) \Big| < \infty.$$

IV *Transmission and reflection coefficients satisfy*

$$\lim_{z \to E} R_{2g+2}^{1/2}(z) \frac{R_{\pm}(z)+1}{T(z)} = 0, \qquad E \neq \mu_\ell,$$
$$\lim_{z \to E} R_{2g+2}^{1/2}(z) \frac{R_{\pm}(z)-1}{T(z)} = 0, \qquad E = \mu_\ell,$$

and the consistency conditions

$$\frac{\overline{R_-(\lambda)}}{\overline{R_+(\lambda)}} = -\frac{T(\lambda)}{\overline{T(\lambda)}}, \qquad \gamma_{+,j}\,\gamma_{-,j} = \frac{\Big(\mathit{res}_{\rho_j} T(\lambda)\Big)^2}{\prod_{l=0}^{2g+1}(\rho_j - E_l)}.$$

Main Theorem

Theorem 5 Conditions I–IV *are necessary and sufficient for* $S_{\pm}$ *to be the left/right scattering data of a unique Jacobi operator* H*. The associated coefficients* a*,* b *satisfy the short range assumption.*

Perturbations with step-like quasi-periodic background

Consider two quasi-periodic Jacobi operators $H_q^{\pm}$ with Dirichlet divisors $(\mu_j^{\pm}, \sigma_j^{\pm})$ and spectra $\sigma_{\pm}$. Let H be a perturbation of $H_q^{\pm}$ such that

$$\sum_{n=0}^{\pm\infty} |n| \Big(|a(n) - a_q^{\pm}(n)| + |b(n) - b_q^{\pm}(n)| \Big) < \infty.$$

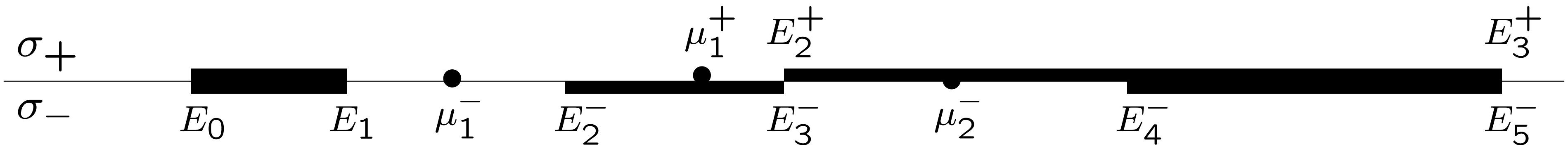
There exist Jost solutions $\psi_{\pm}(z,.)$ satisfying $H\psi = z\psi$ and

$$\psi_{\pm}(z,n) = \sum_{m=n}^{\pm\infty} K_{\pm}(n,m) \psi_q^{\pm}(z,m),$$

where $\psi_q^{\pm}(z,n)$ are the Weyl solutions of $H_q^{\pm}\psi = z\psi$ decaying for $z \in \mathbb{C} \setminus \sigma_{\pm}$ as $n \to \pm\infty$ and $K_{\pm}(n,m)$ satisfy a similar estimate as before.

Spectrum of H

Typical mutual locations of σ_- and σ_+:



$$\sigma = \sigma_{ess}(H) = \sigma_+ \cup \sigma_-, \qquad \sigma_p(H) = \{\rho_j\}_{j=1}^q \subset \mathbb{R}\backslash\sigma$$

No restriction on the mutual location of σ_+ and σ_-,
no restriction on the location of the Dirichlet eigenvalues!

[Bazargan-Egorova '03]: Two operators of period 2 and a special choice for the respective spectra

Scattering data for step-like case

Now we have two transmission $T_\pm$ and reflection $R_\pm$ coefficients defined via the scattering relations

$$T_\mp(\lambda)\psi_\pm(\lambda,n) = \overline{\psi_\mp(\lambda,n)} + R_\mp(\lambda)\psi_\mp(\lambda,x), \qquad \lambda\in\sigma_\mp^{\mathrm{u,l}},$$

and again for each eigenvalue ρ_j the norming constant $\gamma_{\pm,j}$,

$$\gamma_{\pm,j}^{-1} = \sum_{n\in\mathbb{Z}} \hat{\psi}_\pm(\rho_j,n)^2, \qquad \hat{\psi}_\pm(z,n) = \Big(\prod_{\mu_\ell\in M_\pm}(z-\mu_\ell^\pm)\Big)\psi_\pm(z,n).$$

Then the scattering data is the set

$$\mathcal{S} = \Big\{ R_+(\lambda),\, T_+(\lambda),\, \lambda\in\sigma_+^{\mathrm{u,l}};\, R_-(\lambda),\, T_-(\lambda),\, \lambda\in\sigma_-^{\mathrm{u,l}};$$
$$\rho_1,\dots,\rho_q\in\mathbb{R}\setminus\sigma,\, \gamma_{\pm,1},\dots,\gamma_{\pm,q}\in\mathbb{R}_+\Big\}.$$

Minimal scattering data?

Kernel of the GLM equation

The Gel'fand-Levitan-Marchenko equation has the form

$$K_{\pm}(n,m) + \sum_{l=n}^{\pm\infty} K_{\pm}(n,l)F^{\pm}(l,m) = \frac{\delta(n,m)}{K_{\pm}(n,n)}, \qquad \pm m \geq \pm n,$$

where

$$\begin{aligned} F_{\pm}(m,n) &= \oint_{\sigma_{\pm}} R_{\pm}(\lambda)\psi_q^{\pm}(\lambda,m)\psi_q^{\pm}(\lambda,n)d\omega_{\pm} \\ &+ \int_{\sigma_{\mp}^{(1),u}} |T_{\mp}(\lambda)|^2 \psi_q^{\pm}(\lambda,m)\psi_q^{\pm}(\lambda,n)d\omega_{\mp} \\ &+ \sum_{k=1}^{q} \gamma_{\pm,k}\widehat{\psi}_q^{\pm}(\rho_k,n)\widehat{\psi}_q^{\pm}(\rho_k,m). \end{aligned}$$

Main result in the step-like case

Theorem 6 *The GLM equation has a unique solution* $K_\pm(n,m)$ *and* $a_\pm, b_\pm$ *satisfy the short range assumption* $n\left\{|a_\pm(n) - a_q^\pm(n)| + |b_\pm - b_q^\pm(n)|\right\} \in \ell^1(\mathbb{Z}_\pm)$.
If in addition,
(i) $\overline{R_\pm(\lambda)} T_\pm(\lambda) + R_\mp(\lambda) \overline{T_\pm(\lambda)} = 0$ *for* $\lambda \in \sigma^{(2)}$,
(ii) for $E \in \partial\sigma_+ \cap \partial\sigma_-$ *and* $\hat{W}(E) \neq 0$,

$$R_\pm(E) = \begin{cases} -1 & \text{for} \quad E \neq \mu_\ell^\pm, \\ 1 & \text{for} \quad E = \mu_\ell^\pm, \end{cases}$$

then $a_- = a_+ = a$, $b_- = b_+ = b$ *and the data* $\mathcal{S}$ *is the scattering data for the Jacobi operator* H *associated with* a, b.

[Boutet de Monvel-Egorova-Teschl '07]: 1-dim Schrödinger operator with step-like periodic bg

PART II

Initial value problem of the Toda Hierarchy

The Toda equation

Assume

$a(n,t), b(n,t) \in \ell^\infty(\mathbb{Z},\mathbb{R}), \quad t \mapsto (a(t),b(t))$ differentiable in $\ell^\infty(\mathbb{Z})\oplus\ell^\infty(\mathbb{Z})$.

The Toda lattice [1967] is a simple model for a nonlinear one-dimensional crystal. The Toda equation (in Flaschka's variables [1974])

$$\begin{aligned}\dot a(n,t) &= a(n,t)\Big(b(n+1,t) - b(n,t)\Big),\\ \dot b(n,t) &= 2\Big(a(n,t)^2 - a(n-1,t)^2\Big)\end{aligned}$$

is equivalent to the Lax equation

$$\frac{d}{dt}H(t) = P_2(t)H(t) - H(t)P_2(t),$$

where $H(t)$ is our Jacobi operator and $P_2(t) = a(t)S^+ - a^-(t)S^-$. Here $(S^\pm f)(n) = f^\pm(n) = f(n\pm 1)$.

The Toda Hierarchy

Replacing P_2 with more general operators P_{2r+2} of order $2r+2$ yields the Toda hierarchy

$$\frac{d}{dt}H(t) = P_{2r+2}(t)H(t) - H(t)P_{2r+2}(t) \quad \Leftrightarrow \quad \mathrm{TL}_r(a(t), b(t)) = 0.$$

The r-th Toda equation is given by

$$\begin{aligned} \dot{a}(n,t) &= a(t)\Big(g_{r+1}(n+1,t) - g_{r+1}(n,t)\Big), \\ \dot{b}(n,t) &= \Big(h_{r+1}(n,t) - h_{r+1}(n-1,t)\Big), \end{aligned}$$

where

$$\begin{aligned} g_j(n,t) &= \sum_{l=0}^{j} c_{j-l}\langle\delta_n, H(t)^l\delta_n\rangle, \\ h_j(n,t) &= 2a(n,t)\sum_{l=0}^{j} c_{j-l}\langle\delta_{n+1}, H(t)^l\delta_n\rangle + c_{j+1} \end{aligned}$$

for some arbitrarily chosen constants $\{c_j\}_{j=0}^{r}$ with $c_0 = 1$.

The Toda Hierarchy

The operator P_{2r+2} is given by

$$P_{2r+2}(t) = -H(t)^{r+1} + \sum_{j=0}^{r} (2a(t)g_j(t)S^+ - h_j(t))H(t)^{r-j} + g_{r+1}(t).$$

One obtains for the first few equations of the Toda hierarchy

$$\mathrm{TL}_0(a,b) = \begin{pmatrix} \dot{a} - a(b^+ - b) \\ \dot{b} - 2(a^2 - (a^-)^2) \end{pmatrix},$$

$$\mathrm{TL}_1(a,b) = \begin{pmatrix} \dot{a} - a((a^+)^2 - (a^-)^2 + (b^+)^2 - b^2) \\ \dot{b} - 2a^2(b^+ + b) + 2(a^-)^2(b + b^-) \end{pmatrix} - c_1 \begin{pmatrix} a(b^+ - b) \\ 2(a^2 - (a^-)^2) \end{pmatrix}.$$

The Lax equation $\dot{H} = [P_{2r+2}, H]$ implies existence of a unitary propagator $U_r(t,s)$ such that the family of operators $H(t)$, $t \in \mathbb{R}$, are unitarily equivalent, $H(t) = U_r(t,s)H(s)U_r(s,t)$, that is,

$$\sigma(H) \equiv \sigma(H(t)) = \sigma(H(0)), \quad t \in \mathbb{R}.$$

Finite-gap solutions of the Toda Hierarchy

Existence and uniqueness of global solutions of the initial value problem

$$\mathrm{TL}_r(a,b) = 0, \qquad (a(0), b(0)) = (a_0, b_0), \qquad a_0, b_0 \in \ell^\infty(\mathbb{Z}, \mathbb{R}),$$

is well known.

Starting with quasi-periodic initial conditions $(a_{q,0}, b_{q,0})$ one can explicitely solve $\mathrm{TL}_r(a_q(t), b_q(t)) = 0$: [Bulla-Gesztesy-Holden-Teschl 1998]

$$a_q(n,t)^2 = \tilde{a}^2 \frac{\theta(\underline{z}(n+1,t))\theta(\underline{z}(n-1,t))}{\theta(\underline{z}(n,t))^2},$$

$$b_q(n,t) = \tilde{b} + \sum_{j=1}^{g} c_j(g) \frac{\partial}{\partial w_j} \ln\Big(\frac{\theta(\underline{w} + \underline{z}(n,t))}{\theta(\underline{w} + \underline{z}(n-1,t))}\Big)\Big|_{\underline{w}=0}.$$

The constants $\tilde{a}$, $\tilde{b}$, $c_j(g)$ depend only on the Riemann surface.

Inverse scattering transform

The IST is one of the main tools for solving completely integrable wave equations.

Korteweg-de Vries equation:
constant bg: Gardner-Green-Kruskal-Miura 1967
non-constant bg: Kuznetsov-A.V. Mikhaĭlov 1975
(They used the Weierstraß elliptic function as stationary bg solution.)
periodic bg: Firsova 1988

Toda equation:
constant bg: Flaschka 1974
Toda hierarchy with constant bg: Teschl 1999
step-like constant bg: Boutet de Monvel-Egorova '00

Short range assumption

Theorem 7 *Suppose a, b and $\tilde{a}$, $\tilde{b}$ are two arbitrary bounded solutions of the Toda system. If*

$$\sum_{n\in\mathbb{Z}} w(n)\Big(|a(n,t)-\tilde{a}(n,t)| + |b(n,t)-\tilde{b}(n,t)|\Big) < \infty$$

holds for one $t_0 \in \mathbb{R}$, then it holds for all $t \in \mathbb{R}$. Here $w(n) > 0$ is an arbitrary function.

In particular, a short range perturbation of a quasi-periodic finite-gap solution will stay short range for all times!

Time dependent scattering theory

Suppose a, b is a solution of the Toda system satisfying

$$\sum_{n\in\mathbb{Z}} |n|\Big(|a(n,t)-a_q(n,t)| + |b(n,t)-b_q(n,t)|\Big) < \infty.$$

We can define Jost solutions, transmission coefficients, etc. as before, now they depend on an additional parameter $t \in \mathbb{R}$. The scattering data are given by

$$S_\pm(H(t)) = \{R_\pm(\lambda,t), \lambda\in\sigma; (\rho_j, \gamma_{\pm,j}(t)), 1\le j\le q\}.$$

How do $S_\pm(H(t))$ evolve with t?

Time evolution of the scattering data

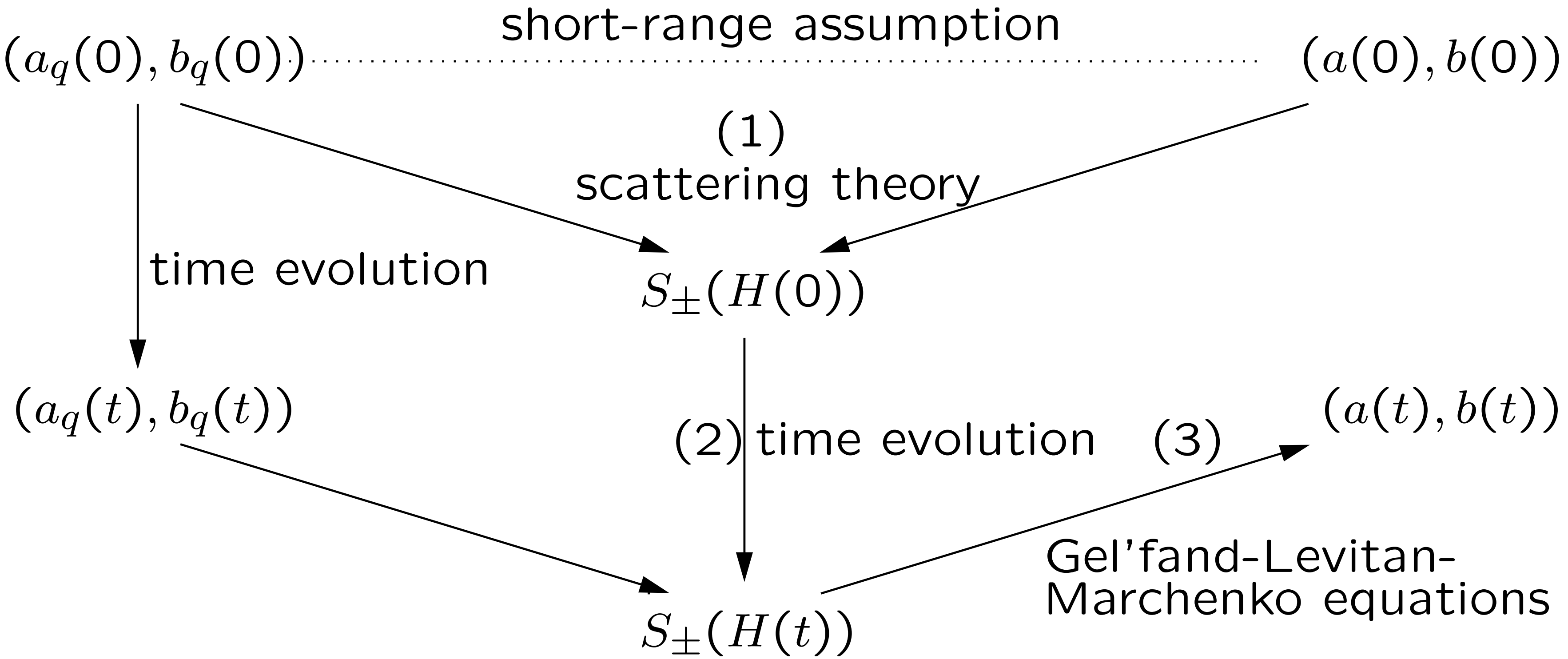
Theorem 8

$$\begin{aligned} T(\lambda,t) &= T(\lambda,0), \\ R_{\pm}(\lambda,t) &= R_{\pm}(\lambda,0)\exp(\pm(\alpha_{s,+}(z,t)-\alpha_{s,-}(z,t))), \\ \gamma_{\pm,j}(t) &= \gamma_{\pm,j}(0)\exp(2\alpha_{s,\pm}(\rho_j,t)), \qquad 1\le j\le q, \end{aligned}$$

where

$$\begin{aligned} \exp\left(\alpha_{s,\pm}(z,t)\right) &= \sqrt{\frac{G_g(z,0,t)}{G_g(z,0,0)}}\exp\left(\pm R^{1/2}_{2g+2}(z)\int_0^t \frac{\widehat{G}_s(z,0,x)}{G_g(z,0,x)}dx\right), \\ G_g(z,n,t) &= \prod_{j=1}^{g}(z-\mu_j(n,t)). \end{aligned}$$

Inverse scattering transform



Time dependent normalisation

Before we have multiplied $\psi_{q,\pm}(\lambda, n, t)$ by a factor to cancel the poles at the Dirichlet eigenvalues. However, since the Dirichlet eigenvalues change sheets during time evolution, this normalization renders $\widehat{\psi}_{q,\pm}(\lambda, n, t)$ discontinuous with respect to t and complicates the time evolution of the corresponding norming constants:

$$\begin{aligned}\widehat{\gamma}_{\pm,j}(t) \;&= \widehat{\gamma}_{\pm,j}(0)\exp\Bigg(\pm 2R_{2g+2}^{1/2}(\rho_j)\int_0^t \tfrac{\widehat{G}_s(\rho_j,0,x)}{G_g(\rho_j,0,x)}dx \\ &\qquad \pm \sum_{l=1}^{g}\int_0^t \tfrac{2\widehat{G}_s(\mu_l(x),0,x)R_{2g+2}^{1/2}(\mu_l(x))}{(\rho_j-\mu_l(x))\prod_{k\neq l}(\mu_l(x)-\mu_k(x))}dx\Bigg).\end{aligned}$$

To avoid the poles of the Baker-Akhiezer function, we assume that none of the eigenvalues ρ_j coincides with a Dirichlet eigenvalue $\mu_k(0,0)$ (w.l.o.g. shift initial time $t_0 = 0$ if necessary).

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1. *Scattering theory for Jacobi operators with quasi-periodic background*, Comm. Math. Phys. **264-3**, 811-842 (2006).
2. *Inverse scattering transform for the Toda hierarchy with quasi-periodic background*, Proc. Amer. Math. Soc. **135**, 1817-1827 (2007).
3. *Soliton solutions of the Toda hierarchy on quasi-periodic background revisited*, Math. Nach. (to appear).
4. *Scattering theory for Jacobi operators with general step-like quasi-periodic background*, in preparation.

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