Entanglement of formation

for some special two-mode Gaussian states

Paulina Marian & Tudor A. Marian

Centre for Advanced Quantum Physics

University of Bucharest, Romania

Outline

- Undisplaced two-mode Gaussian states (TMGS's)

Outline

- Undisplaced two-mode Gaussian states (TMGS's)
- Uncertainty relations

Outline

- Undisplaced two-mode Gaussian states (TMGS's)
- Uncertainty relations
- Non-classicality & inseparability

Outline

- Undisplaced two-mode Gaussian states (TMGS’s)
- Uncertainty relations
- Non-classicality & inseparability
- Entanglement of Formation (EoF)

Outline

- Undisplaced two-mode Gaussian states (TMGS's)
- Uncertainty relations
- Non-classicality & inseparability
- Entanglement of Formation (EoF)
- Continuous-variable pure-state decompositions

Outline

- Undisplaced two-mode Gaussian states (TMGS's)
- Uncertainty relations
- Non-classicality & inseparability
- Entanglement of Formation (EoF)
- Continuous-variable pure-state decompositions
- Gaussian Entanglement of Formation (GEoF)

Outline

- Undisplaced two-mode Gaussian states (TMGS's)
- Uncertainty relations
- Non-classicality & inseparability
- Entanglement of Formation (EoF)
- Continuous-variable pure-state decompositions
- Gaussian Entanglement of Formation (GEoF)
- Factorization of the characteristic functions (CF's)

Outline

- Undisplaced two-mode Gaussian states (TMGS’s)
- Uncertainty relations
- Non-classicality & inseparability
- Entanglement of Formation (EoF)
- Continuous-variable pure-state decompositions
- Gaussian Entanglement of Formation (GEoF)
- Factorization of the characteristic functions (CF’s)
- Properties of the added classical state

Outline

- Undisplaced two-mode Gaussian states (TMGS's)
- Uncertainty relations
- Non-classicality & inseparability
- Entanglement of Formation (EoF)
- Continuous-variable pure-state decompositions
- Gaussian Entanglement of Formation (GEoF)
- Factorization of the characteristic functions (CF's)
- Properties of the added classical state
- Manipulating the GEoF equations

Outline

- Undisplaced two-mode Gaussian states (TMGS's)
- Uncertainty relations
- Non-classicality & inseparability
- Entanglement of Formation (EoF)
- Continuous-variable pure-state decompositions
- Gaussian Entanglement of Formation (GEoF)
- Factorization of the characteristic functions (CF's)
- Properties of the added classical state
- Manipulating the GEoF equations
- Explicit evaluations

Outline

- Undisplaced two-mode Gaussian states (TMGS's)
- Uncertainty relations
- Non-classicality & inseparability
- Entanglement of Formation (EoF)
- Continuous-variable pure-state decompositions
- Gaussian Entanglement of Formation (GEoF)
- Factorization of the characteristic functions (CF's)
- Properties of the added classical state
- Manipulating the GEoF equations
- Explicit evaluations
- Conclusions: is GEoF=EoF?

Two-mode Gaussian states (TMGS's)

- Density operator ρ_G.
- Characteristic function (CF):

$$\chi_G(x) = \exp\left(-\frac{1}{2}x^T \mathcal{V} x\right),$$

with x^T denoting a real row vector $(x_1\ x_2\ x_3\ x_4)$ and $\mathcal{V}$ the 4×4 covariance matrix (CM).

- A TMGS is fully described by its CM:

$$\rho_G \longleftrightarrow \chi_G(x) \longleftrightarrow \mathcal{V}.$$

Scaled standard forms

- Equivalence class of locally similar TMGS's:

$$\mathcal{S} = \mathcal{S}_1 \oplus \mathcal{S}_2, \quad \mathcal{S} \in \mathrm{Sp}(2,\mathbb{R}) \times \mathrm{Sp}(2,\mathbb{R})$$

$$\rightarrow U(\mathcal{S}) = U_1(\mathcal{S}_1) \otimes U_2(\mathcal{S}_2).$$

- Consider two independent one-mode squeeze factors

$$u_1 = \exp(2r_1), \quad u_2 = \exp(2r_2).$$

- CM of a scaled standard state $\rho(u_1, u_2)$:

$$\mathcal{V}(u_1, u_2) = \begin{pmatrix} b_1 u_1 & 0 & c\sqrt{u_1 u_2} & 0 \\ 0 & b_1/u_1 & 0 & d/\sqrt{u_1 u_2} \\ c\sqrt{u_1 u_2} & 0 & b_2 u_2 & 0 \\ 0 & d/\sqrt{u_1 u_2} & 0 & b_2/u_2 \end{pmatrix}.$$

Uncertainty relations

- Standard form I (unscaled): $\mathcal{V}_I := \mathcal{V}(1,1)$.
- Robertson-Schrödinger uncertainty relations:

$$\mathcal{V} + \frac{i}{2}\Omega \geq 0, \quad \Omega := i\left(\bigoplus_{j=1}^{2} \sigma_2\right) \text{ equivalent to :}$$

$$b_1 \geq 1/2, \quad b_2(b_1 b_2 - c^2) - \frac{b_1}{4} \geq 0,$$

$$b_2 \geq 1/2, \quad b_1(b_1 b_2 - c^2) - \frac{b_2}{4} \geq 0,$$

$$(\kappa_-^2 - 1/4)(\kappa_+^2 - 1/4) \geq 0.$$

($\kappa_-, \ \kappa_+$ are the symplectic eigenvalues).

Non-classicality

- Classicality of a scaled standard state:

$$\mathcal{V}(u_1, u_2) \geq \frac{1}{2} I_4 \quad \text{equivalent to}$$

$$u_1 \leq 2b_1, \quad u_2 \leq 2b_2,$$

$$(b_1 u_1 - \frac{1}{2})(b_2 u_2 - \frac{1}{2}) \geq c^2 u_1 u_2,$$

$$(b_1/u_1 - \frac{1}{2})(b_2/u_2 - \frac{1}{2}) \geq d^2/(u_1 u_2).$$

- Non-classical state $\longleftrightarrow$ Matrix condition not fulfilled.

Inseparability

R. Simon's separability criterion (2000):

- TMGS's with $d \geq 0$ are separable.
- for $d < 0$, one has to check the sign of the invariant

$$S(\rho_G) = (b_1 b_2 - c^2)(b_1 b_2 - d^2) - \frac{1}{4}(b_1^2 + b_2^2 + 2c|d|) + \frac{1}{16}$$

that can be written as

$$S(\rho_G) = (\tilde{\kappa}_-^2 - 1/4)(\tilde{\kappa}_+^2 - 1/4).$$

Entangled TMGS's fulfil the condition

$$S(\rho_G) < 0.$$

EPR approach

Duan *et al.*(2000) introduced a scaled standard state for which the separability and classicality conditions coincide.
Standard form II of the CM (separability=classicality):

$$\mathcal{V}_{II} := \mathcal{V}(v_1, v_2)$$

with v_1, v_2 satisfying the algebraic system

$$\frac{b_1(v_1^2 - 1)}{2b_1 - v_1} = \frac{b_2(v_2^2 - 1)}{2b_2 - v_2},$$

$$b_1 b_2 (v_1^2 - 1)(v_2^2 - 1) = (c v_1 v_2 - |d|)^2.$$

Squeezed vacuum states (SVS's)

The standard-form CM $\mathcal{V}_I$ of an entangled pure TMGS has the property

$$\det(\mathcal{V}_I + \frac{i}{2}\Omega) = 0$$

as a product of two vanishing factors $\longrightarrow$

$$b_1 = b_2 = b, \quad d = -c < 0, \quad b^2 - c^2 = 1/4.$$

This state is a SVS and has minimal symplectic eigenvalues:

$$\kappa_- = \kappa_+ = 1/2.$$

The smallest symplectic eigenvalue of $\tilde{\mathcal{V}} \longleftrightarrow \rho_G^{PT2}$ is

$$\tilde{\kappa}_- = b - c < \frac{1}{2}.$$

Entanglement of formation (EoF)

Pure-state decompositions of a mixed state ρ:

$$\rho = \sum_k p_k |\Psi_k\rangle\langle\Psi_k|, \qquad \sum_k p_k = 1.$$

EoF of a mixed bipartite state (Bennett *et al.*, 1996):

$$EoF(\rho) := \inf_{\{p_k\}} \sum_k p_k E_0(|\Psi_k\rangle\langle\Psi_k|),$$

where $E_0(|\Psi_k\rangle\langle\Psi_k|)$ is any acceptable measure of pure-state entanglement.

Two-field superpositions

- Glauber (1963): superposition ρ_S of two fields

$$\rho_S = \int d^2\beta P_2(\beta) D(\beta) \rho_1 D^\dagger(\beta)$$

where $D(\alpha) := \exp\left(\alpha a^\dagger - \alpha^* a\right)$ is a Weyl displacement operator,
a is a photon annihilation operator;
ρ_1 is a one-mode field state and $P_2(\beta)$ denotes the P representation of a classical one-mode field state ρ_2.

- Equivalent formulation (Marian & Marian, 1996):

$$\chi_S^{(N)}(\lambda) = \chi_1^{(N)}(\lambda)\chi_2^{(N)}(\lambda);$$

$\chi^{(N)}$ denotes the normally-ordered CF.

Gaussian EoF

- A pure-state decomposition of a mixed TMGS is

$$\rho_G = \int d^2\beta_1 d^2\beta_2 P(\beta_1, \beta_2) D_1(\beta_1) D_2(\beta_2) \rho_0 D_2^\dagger(\beta_2) D_1^\dagger(\beta_1),$$

where ρ_0 is a pure TMGS.
The most general ρ_0, which is a scaled SVS, was employed by Wolf *et al.* (2003)⟶Gaussian EoF (GEoF):

$$GEoF(\rho_G) = E(\rho_0^{optimal}).$$

- Main problem: find the optimal decomposition (=determine ρ_0 having the minimal entanglement).
- Giedke *et al.* (2003) evaluated the exact EoF for symmetric TMGS's ($b_1 = b_2$).

Our work

- exploit the factorization formula of the CF's

$$\chi_G(\lambda_1, \lambda_2) = \chi_0(\lambda_1, \lambda_2)\chi_{cl}(\lambda_1, \lambda_2) \exp\left(-\frac{|\lambda_1|^2}{2} - \frac{|\lambda_2|^2}{2}\right).$$

- choose $\chi_0(\lambda_1, \lambda_2)$ to be a SVS with the CM

$$\mathcal{V}_0 = \begin{pmatrix} x & 0 & y & 0 \\ 0 & x & 0 & -y \\ y & 0 & x & 0 \\ 0 & -y & 0 & x \end{pmatrix}, \quad x^2 - y^2 = 1/4.$$

Relation between CM's

Reason for $\mathcal{V}_0$: A SVS is the pure state with minimal entanglement at a given EPR correlation (Giedke *et al.*, 2003).

$$\Delta_{EPR} = 2(x - y)$$

- Gaussian $\chi_0(\lambda_1, \lambda_2) \longleftrightarrow$ Gaussian $\chi_{cl}(\lambda_1, \lambda_2)$.
- For any pure-state decomposition of the TMGS ρ

$$\mathcal{V} = \mathcal{V}_0 + \mathcal{V}_{cl} - \frac{1}{2} I_4.$$

Variables

For any entangled TMGS $d = -|d| < 0.$

$$\mathcal{V} = \begin{pmatrix} b_1 u_1 & 0 & c\sqrt{u_1 u_2} & 0 \\ 0 & b_1/u_1 & 0 & -|d|/\sqrt{u_1 u_2} \\ c\sqrt{u_1 u_2} & 0 & b_2 u_2 & 0 \\ 0 & -|d|/\sqrt{u_1 u_2} & 0 & b_2/u_2 \end{pmatrix}.$$

- Given parameters: $b_1,\ b_2,\ c,\ |d|,\ (|d| \leq c).$
- Any measure of pure-state entanglement is a monotonous function of $x \longrightarrow$ we have to find the minimal value of x as a function of the variables $u_1,\ u_2.$

Analytical method

- concentrate on the added classical state $\mathcal{V}_{cl}$.

First step: Towards the optimal pure-state decomposition, $\mathcal{V}_{cl}$ should reach *the classicality threshold*

$$\det(\mathcal{V}_{cl} - \frac{1}{2} I_4) = 0$$

as a product of two vanishing factors:

$$(b_1 u_1 - x)(b_2 u_2 - x) = (c\sqrt{u_1 u_2} - y)^2,$$

$$(b_1/u_1 - x)(b_2/u_2 - x) = (|d|/\sqrt{u_1 u_2} - y)^2.$$

(derived by Wolf *et al.* (2003) on different grounds).

Nature of $\mathcal{V}_{cl}$

Second step: By minimization of the function $x(u_1, u_2)$, we proved that in the optimal pure-state decomposition, $\mathcal{V}_{cl}$ is also at *the separability threshold*:

$$S(\rho_{cl}) = 0,$$

i.e., $\mathcal{V}_{cl}$ has the standard form II:

$$\frac{b_1 u_1 - x}{b_1/u_1 - x} = \frac{b_2 u_2 - x}{b_2/u_2 - x},$$

$$b_1 b_2 (u_1^2 - 1)(u_2^2 - 1) = (c u_1 u_2 - |d|)^2.$$

EoF equations

System of algebraic equations with four unknowns:

$$(b_1 w_1 - x)(b_2 w_2 - x) = (c\sqrt{w_1 w_2} - y)^2,$$

$$(b_1/w_1 - x)(b_2/w_2 - x) = (|d|/\sqrt{w_1 w_2} - y)^2,$$

$$\frac{b_1 w_1 - x}{b_1/w_1 - x} = \frac{b_2 w_2 - x}{b_2/w_2 - x},$$

$$b_1 b_2 (w_1^2 - 1)(w_2^2 - 1) = (c w_1 w_2 - |d|)^2,$$

$$x^2 - y^2 = 1/4.$$

Solution only in some particular cases.

Symmetric TMGS's

$$b_1 = b_2 = b \longrightarrow \tilde{\kappa}_- = \sqrt{(b-c)(b-|d|)}.$$

Results

$$w_1 = w_2 = \sqrt{\frac{b-|d|}{b-c}},$$

$$x = \frac{\tilde{\kappa}_-^2 + 1/4}{2\tilde{\kappa}_-};$$

$$x - y = \tilde{\kappa}_-.$$

x is a function of $\tilde{\kappa}_-$ only that coincides with its expression for the exact EoF (Giedke *et al.*, 2003).

STS's

$$c = |d| \longrightarrow \tilde{\kappa}_- = \frac{1}{2}[b_1 + b_2 - \sqrt{(b_1 - b_2)^2 + 4c^2}].$$

- important mixed states used as two-mode resource in quantum teleportation of one-mode states.
- proved to have the maximal negativity at fixed local purities: Adesso *et al.* (2004,2005).

Results

$$w_1 = w_2 = 1,$$

$$x = \frac{(b_1 + b_2)(b_1 b_2 - c^2 + 1/4) - 2c\sqrt{\det(\mathcal{V} + \frac{i}{2}\Omega)}}{(b_1 + b_2)^2 - 4c^2}:$$

x not depending on $\tilde{\kappa}_-$ only.

States with $\kappa_- = 1/2$

- Mixed TMGS's with

$$\det(\mathcal{V} + \frac{i}{2}\Omega) = 0 \longleftrightarrow \kappa_- = 1/2;$$

- proved to have minimal negativity at fixed local and global purities: Adesso *et al.* (2004,2005).

Results depending on a parameter inequality, as follows.

States with $\kappa_- = 1/2$

I. $b_1 > b_2, \quad c > |d|, \quad b_2 c \leq b_1 |d|$:

$$w_1 = \left[\frac{b_2(b_1 b_2 - d^2) - b_1/4}{b_2(b_1 b_2 - c^2) - b_1/4} \right]^{1/2},$$

$$w_2 = \left[\frac{b_1(b_1 b_2 - d^2) - b_2/4}{b_1(b_1 b_2 - c^2) - b_2/4} \right]^{1/2},$$

$$x = \frac{b_1^2 - b_2^2}{8(\det(\mathcal{V}) - 1/16)}.$$

States with $\kappa_- = 1/2$

II. $b_1 > b_2, \quad c > |d|, \quad b_2 c > b_1 |d|$:

$$w_1 = 2\sqrt{\frac{b_1}{b_2}(b_1 b_2 - d^2)},$$

$$w_2 = 2\sqrt{\frac{b_2}{b_1}(b_1 b_2 - d^2)},$$

$$x = \frac{1}{2}\sqrt{\frac{b_1 b_2}{b_1 b_2 - d^2}}.$$

Conclusions I

- We have reformulated the problem of GEoF in terms of CF's and CM's.
- The added classical state is at the classicality and separability threshold as well: its CM has the standard form II.
- We have retrieved in a unitary way previous results for some important classes of entangled TMGS's.
- General case hard to be exploited analytically. Work in progress.

Conclusions II

- The GEoF built with the Bures metric is proved to coincide with the Bures entanglement for symmetric TMGS's, as well as for STS's.

- Main question: Is GEoF=EoF?

  Answer: Yes.

  This is based on the above-mentioned theorem of Giedke *et al.* (2003): A SVS is the pure state with minimal entanglement at a given EPR correlation

$$\Delta_{EPR} = 2(x - y).$$