Entanglement of formation
for some special two-mode Gaussian states

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Outline

- Undisplaced two-mode Gaussian states (TMGS’s)
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- Uncertainty relations
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- Gaussian Entanglement of Formation (GEoF)
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- Manipulating the GEoF equations
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- Explicit evaluations
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Manipulating the GEoF equations
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Conclusions: is GEoF=EoF?
Two-mode Gaussian states (TMGS’s)

- Density operator $\rho_G$.
- Characteristic function (CF):
  \[
  \chi_G(x) = \exp \left( -\frac{1}{2} x^T \mathcal{V} x \right),
  \]
  with $x^T$ denoting a real row vector $(x_1 \ x_2 \ x_3 \ x_4)$ and $\mathcal{V}$ the $4 \times 4$ covariance matrix (CM).
- A TMGS is fully described by its CM:
  \[
  \rho_G \longleftrightarrow \chi_G(x) \longleftrightarrow \mathcal{V}.
  \]
Scaled standard forms

- Equivalence class of locally similar TMGS’s:
  \[ S = S_1 \oplus S_2, \quad S \in \text{Sp}(2, \mathbb{R}) \times \text{Sp}(2, \mathbb{R}) \]
  \[ \rightarrow U(S) = U_1(S_1) \otimes U_2(S_2). \]

- Consider two independent one-mode squeeze factors
  \[ u_1 = \exp(2r_1), \quad u_2 = \exp(2r_2). \]

- CM of a scaled standard state \( \rho(u_1, u_2) \):
  \[
  \mathcal{V}(u_1, u_2) = \begin{pmatrix}
  b_1 u_1 & 0 & c \sqrt{u_1 u_2} & 0 \\
  0 & b_1/u_1 & 0 & d/\sqrt{u_1 u_2} \\
  c \sqrt{u_1 u_2} & 0 & b_2 u_2 & 0 \\
  0 & d/\sqrt{u_1 u_2} & 0 & b_2/u_2
  \end{pmatrix}.
  \]
Uncertainty relations

- Standard form I (unscaled): \( \mathcal{V}_I := \mathcal{V}(1, 1) \).
- Robertson-Schrödinger uncertainty relations:

\[
\mathcal{V} + \frac{i}{2} \Omega \geq 0, \quad \Omega := i \left( \bigoplus_{j=1}^{2} \sigma_2 \right) \text{ equivalent to:}
\]

\[
b_1 \geq 1/2, \quad b_2 (b_1 b_2 - c^2) - \frac{b_1}{4} \geq 0, \quad b_2 \geq 1/2, \quad b_1 (b_1 b_2 - c^2) - \frac{b_2}{4} \geq 0,
\]

\[
(\kappa_-^2 - 1/4)(\kappa_+^2 - 1/4) \geq 0.
\]

(\( \kappa_- \), \( \kappa_+ \) are the symplectic eigenvalues).
Non-classicality

• Classicality of a scaled standard state:

\[ \mathcal{V}(u_1, u_2) \geq \frac{1}{2} I_4 \] equivalent to

\[ u_1 \leq 2b_1, \quad u_2 \leq 2b_2, \]

\[ (b_1 u_1 - \frac{1}{2})(b_2 u_2 - \frac{1}{2}) \geq c^2 u_1 u_2, \]

\[ (b_1/u_1 - \frac{1}{2})(b_2/u_2 - \frac{1}{2}) \geq d^2 / (u_1 u_2). \]

• Non-classical state \(\iff\) Matrix condition not fulfilled.
Inseparability

R. Simon’s separability criterion (2000):

- TMGS’s with $d \geq 0$ are separable.
- for $d < 0$, one has to check the sign of the invariant

$$S(\rho_G) = (b_1b_2 - c^2)(b_1b_2 - d^2) - \frac{1}{4}(b_1^2 + b_2^2 + 2c|d|) + \frac{1}{16}$$

that can be written as

$$S(\rho_G) = (\tilde{\kappa}^- - 1/4)(\tilde{\kappa}^+ - 1/4).$$

Entangled TMGS’s fulfil the condition

$$S(\rho_G) < 0.$$
Duan et al. (2000) introduced a scaled standard state for which the separability and classicality conditions coincide.

Standard form II of the CM (separability=classicality):

$$V_{II} := V(v_1, v_2)$$

with $v_1, v_2$ satisfying the algebraic system

$$\frac{b_1(v_1^2 - 1)}{2b_1 - v_1} = \frac{b_2(v_2^2 - 1)}{2b_2 - v_2},$$

$$b_1 b_2 (v_1^2 - 1)(v_2^2 - 1) = (cv_1 v_2 - |d|)^2.$$
Squeezed vacuum states (SVS’s)

The standard-form CM $\mathcal{V}_I$ of an entangled pure TMGS has the property

$$\det(\mathcal{V}_I + \frac{i}{2} \Omega) = 0$$

as a product of two vanishing factors

$$b_1 = b_2 = b, \quad d = -c < 0, \quad b^2 - c^2 = 1/4.$$ 

This state is a SVS and has minimal symplectic eigenvalues:

$$\kappa_- = \kappa_+ = 1/2.$$ 

The smallest symplectic eigenvalue of $\tilde{\mathcal{V}} \leftrightarrow \rho_{PT}^{G}$ is

$$\tilde{\kappa}_- = b - c < \frac{1}{2}.$$
Entanglement of formation (EoF)

Pure-state decompositions of a mixed state $\rho$:

$$\rho = \sum_k p_k |\Psi_k\rangle\langle \Psi_k|, \quad \sum_k p_k = 1.$$ 

EoF of a mixed bipartite state (Bennett et al., 1996):

$$EoF(\rho) := \inf_{\{p_k\}} \sum_k p_k E_0(|\Psi_k\rangle\langle \Psi_k|),$$

where $E_0(|\Psi_k\rangle\langle \Psi_k|)$ is any acceptable measure of pure-state entanglement.
Two-field superpositions

- Glauber (1963): superposition $\rho_S$ of two fields

$$\rho_S = \int d^2 \beta P_2(\beta) D(\beta) \rho_1 D^\dagger(\beta)$$

where $D(\alpha) := \exp (\alpha a^\dagger - \alpha^* a)$ is a Weyl displacement operator,

$a$ is a photon annihilation operator;

$\rho_1$ is a one-mode field state and $P_2(\beta)$ denotes the $P$ representation of a classical one-mode field state $\rho_2$.

- Equivalent formulation (Marian & Marian, 1996):

$$\chi_S^{(N)}(\lambda) = \chi_1^{(N)}(\lambda)\chi_2^{(N)}(\lambda);$$

$\chi^{(N)}$ denotes the normally-ordered CF.
Gaussian EoF

- A pure-state decomposition of a mixed TMGS is
  \[ \rho_G = \int d^2\beta_1 d^2\beta_2 P(\beta_1, \beta_2) D_1(\beta_1) D_2(\beta_2) \rho_0 D_2^\dagger(\beta_2) D_1^\dagger(\beta_1), \]
  
  where \( \rho_0 \) is a pure TMGS.

  The most general \( \rho_0 \), which is a scaled SVS, was employed by Wolf et al. (2003) → Gaussian EoF (GEoF):
  \[ GEoF(\rho_G) = E(\rho_0^{\text{optimal}}). \]

- Main problem: find the optimal decomposition (=determine \( \rho_0 \) having the minimal entanglement).
- Giedke et al. (2003) evaluated the exact EoF for symmetric TMGS’s (\( b_1 = b_2 \)).

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Our work

- exploit the factorization formula of the CF’s

\[ \chi_G(\lambda_1, \lambda_2) = \chi_0(\lambda_1, \lambda_2)\chi_{cl}(\lambda_1, \lambda_2) \exp \left( -\frac{|\lambda_1|^2}{2} - \frac{|\lambda_2|^2}{2} \right). \]

- choose \( \chi_0(\lambda_1, \lambda_2) \) to be a SVS with the CM

\[ V_0 = \begin{pmatrix} x & 0 & y & 0 \\ 0 & x & 0 & -y \\ y & 0 & x & 0 \\ 0 & -y & 0 & x \end{pmatrix}, \quad x^2 - y^2 = 1/4. \]
Relation between CM’s

Reason for $\mathcal{V}_0$: A SVS is the pure state with minimal entanglement at a given EPR correlation (Giedke et al., 2003).

\[ \Delta_{EPR} = 2(x - y) \]

- Gaussian $\chi_0(\lambda_1, \lambda_2) \leftrightarrow$ Gaussian $\chi_{cl}(\lambda_1, \lambda_2)$.

- For any pure-state decomposition of the TMGS $\rho$

\[ \mathcal{V} = \mathcal{V}_0 + \mathcal{V}_{cl} - \frac{1}{2}I_4. \]
Variables

For any entangled TMGS \( d = -|d| < 0 \).

\[ \mathcal{V} = \begin{pmatrix} b_1 u_1 & 0 & c \sqrt{u_1 u_2} & 0 \\ 0 & b_1/u_1 & 0 & -|d|/\sqrt{u_1 u_2} \\ c \sqrt{u_1 u_2} & 0 & b_2 u_2 & 0 \\ 0 & -|d|/\sqrt{u_1 u_2} & 0 & b_2/u_2 \end{pmatrix}. \]

- Given parameters: \( b_1, b_2, c, |d|, (|d| \leq c) \).
- Any measure of pure-state entanglement is a monotonous function of \( x \rightarrow \) we have to find the minimal value of \( x \) as a function of the variables \( u_1, u_2 \).
Analytical method

- concentrate on the added classical state $\mathcal{V}_{cl}$.

**First step:** Towards the optimal pure-state decomposition, $\mathcal{V}_{cl}$ should reach the **classicality threshold**

$$\det(\mathcal{V}_{cl} - \frac{1}{2}I_4) = 0$$

as a product of two vanishing factors:

$$(b_1u_1 - x)(b_2u_2 - x) = (c\sqrt{u_1u_2} - y)^2,$$

$$(b_1/u_1 - x)(b_2/u_2 - x) = (|d|/\sqrt{u_1u_2} - y)^2.$$  

(derived by Wolf et al. (2003) on different grounds).
Nature of $\mathcal{V}_{cl}$

**Second step:** By minimization of the function $x(u_1, u_2)$, we proved that in the optimal pure-state decomposition, $\mathcal{V}_{cl}$ is also at the *separability threshold*:

$$S(\rho_{cl}) = 0,$$

i.e., $\mathcal{V}_{cl}$ has the standard form II:

$$\frac{b_1 u_1 - x}{b_1/u_1 - x} = \frac{b_2 u_2 - x}{b_2/u_2 - x},$$

$$b_1 b_2 (u_1^2 - 1)(u_2^2 - 1) = (cu_1 u_2 - |d|)^2.$$
EoF equations

System of algebraic equations with four unknowns:

\[(b_1 w_1 - x)(b_2 w_2 - x) = (c \sqrt{w_1 w_2} - y)^2,\]

\[(b_1/w_1 - x)(b_2/w_2 - x) = (|d|/\sqrt{w_1 w_2} - y)^2,\]

\[\frac{b_1 w_1 - x}{b_1/w_1 - x} = \frac{b_2 w_2 - x}{b_2/w_2 - x},\]

\[b_1 b_2 (w_1^2 - 1)(w_2^2 - 1) = (c w_1 w_2 - |d|)^2,\]

\[x^2 - y^2 = 1/4.\]

Solution only in some particular cases.
Symmetric TMGS’s

\[ b_1 = b_2 = b \quad \rightarrow \quad \tilde{\kappa}_- = \sqrt{(b - c)(b - |d|)}. \]

Results

\[ w_1 = w_2 = \sqrt{\frac{b - |d|}{b - c}}, \]

\[ x = \frac{\tilde{\kappa}_-^2 + 1/4}{2\tilde{\kappa}_-}, \]

\[ x - y = \tilde{\kappa}_-. \]

\( x \) is a function of \( \tilde{\kappa}_- \) only that coincides with its expression for the exact EoF (Giedke et al., 2003).
STS’s

\[ c = |d| \quad \longrightarrow \quad \tilde{\kappa}_- = \frac{1}{2}[b_1 + b_2 - \sqrt{(b_1 - b_2)^2 + 4c^2}] . \]

- important mixed states used as two-mode resource in quantum teleportation of one-mode states.
- proved to have the maximal negativity at fixed local purities: Adesso et al. (2004, 2005).

Results

\[ w_1 = w_2 = 1 , \]

\[ x = \frac{(b_1 + b_2)(b_1 b_2 - c^2 + 1/4) - 2c \sqrt{\det(\mathcal{V} + \frac{i}{2} \Omega)}}{(b_1 + b_2)^2 - 4c^2} : \]

\( x \) not depending on \( \tilde{\kappa}_- \) only.
States with $\kappa_- = 1/2$

- Mixed TMGS’s with

$$\text{det}(\mathcal{V} + \frac{i}{2}\Omega) = 0 \iff \kappa_- = 1/2;$$

- proved to have minimal negativity at fixed local and global purities: Adesso et al. (2004, 2005).

**Results** depending on a parameter inequality, as follows.
States with $\kappa_- = 1/2$

I. $b_1 > b_2$, $c > |d|$, $b_2c \leq b_1|d|$

$$w_1 = \left[ \frac{b_2(b_1b_2 - d^2) - b_1/4}{b_2(b_1b_2 - c^2) - b_1/4} \right]^{1/2},$$

$$w_2 = \left[ \frac{b_1(b_1b_2 - d^2) - b_2/4}{b_1(b_1b_2 - c^2) - b_2/4} \right]^{1/2},$$

$$x = \frac{b_1^2 - b_2^2}{8(\det(\mathcal{V}) - 1/16)}.$$
States with $\kappa_- = 1/2$

II. $b_1 > b_2$, $c > |d|$, $b_2 c > b_1 |d|$:

$$w_1 = 2 \sqrt{\frac{b_1}{b_2} (b_1 b_2 - d^2)},$$

$$w_2 = 2 \sqrt{\frac{b_2}{b_1} (b_1 b_2 - d^2)},$$

$$x = \frac{1}{2} \sqrt{\frac{b_1 b_2}{b_1 b_2 - d^2}}.$$
Conclusions I

- We have reformulated the problem of GEoF in terms of CF’s and CM’s.
- The added classical state is at the classicality and separability threshold as well: its CM has the standard form II.
- We have retrieved in a unitary way previous results for some important classes of entangled TMGS’s.
- General case hard to be exploited analytically. Work in progress.
The GEoF built with the Bures metric is proved to coincide with the Bures entanglement for symmetric TMGS’s, as well as for STS’s.

Main question: Is GEoF=EoF?

Answer: Yes.

This is based on the above-mentioned theorem of Giedke et al. (2003): A SVS is the pure state with minimal entanglement at a given EPR correlation

\[ \Delta_{EPR} = 2(x - y). \]