Quantum Large Deviations
joint work with Wojciech De Roeck and Karel Netočný

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Quantum Mathematics 10
Moeciu, 12 September 2007
Two very related problems
how to count in quantum...

1. Establishing a microcanonical ensemble with different noncommuting constraints — what is a macro-state and how to study (joint) macroscopic fluctuations: CONFIGURATIONAL ENTROPY?
- equivalence with canonical framework...
- H-theorem...

2. LARGE DEVIATIONS and fluctuation theory
- equilibrium set-up...
- nonequilibrium fluctuations...
Two very related problems
WHY DO YOU ASK?

1. Elements of statistical mechanics:
   - relation between fluctuation functionals and thermodynamic potentials
   - counting interpretation of entropy appears relevant for quantum information theory and for microscopic understanding of the second law.

2. Fluctuations in small systems:
   - quantum transport and counting statistics
   - effects of nonlocality/entanglement

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Quantum macrostates
Commutative case

On a sequence of finite-dimensional Hilbert spaces \((\mathcal{H}^N)_N \uparrow +\infty\)
consider a uniformly bounded family of observables
\[
X^N = (X^N_1, \ldots, X^N_K), \quad N \uparrow +\infty
\]
(think of a collection of different empirical averages)
To each \(X^N_k\) assign its projection-valued measure \(Q^N_k\)

If they mutually commute then each collection \(x = (x_1, \ldots, x_K)\) is
associated with the projection
\[
Q^{N,\delta}(x) = \prod_{k=1}^K Q^{N,\delta}_k(x) = \prod_{k=1}^K \int_{x_k-\delta}^{x_k+\delta} Q^N_k(dz_k)
\]

\(\longrightarrow\) quantum microcanonical ensemble

(Boltzmann-von Neumann; microcanonical) entropy function:
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S^{N,\delta}(x) = \log \text{Tr}(Q^{N,\delta}(x))
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Von Neumann ’55:
“It is a fundamental fact with macroscopic measurements that everything which is measurable at all, is also simultaneously measurable, i.e. that all questions which can be answered separately can also be answered simultaneously.”

YET, while indeed averages
\[ A = \frac{a_1 + \ldots + a_N}{N}, \quad B = \frac{b_1 + \ldots + b_N}{N}, \]
for which all commutators \([a_i, b_j] = 0\) for \(i \neq j\), have their commutator
\([A, B] = O(1/N)\) going to zero (in the appropriate norm, corresponding to \([a_i, b_i] = O(1)\)) as \(N \uparrow +\infty\), it is not true in general that

\[
\lim_{N \uparrow +\infty} \frac{1}{N} \log \text{Tr}[e^{NA} e^{NB}] \neq \lim_{N \uparrow +\infty} \frac{1}{N} \log \text{Tr}[e^{NA+NB}]
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These generating functions are obviously important in quantum fluctuation theory...
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General case

**Idea:** Find a largest projection that “well approximates” each projection $Q_k^{N,\delta}$, $k = 1, \ldots, K$

**Def. 1.** A sequence of projections $(\mathcal{P}^N)_{N \to +\infty}$ is concentrating at $x$ whenever for all $k = 1, \ldots, K$ and $\delta > 0$,

$$\lim_{N \to +\infty} \frac{\text{Tr}(\mathcal{P}^N Q_k^{N,\delta}(x))}{\text{Tr}(\mathcal{P}^N)} = 1$$

Then write $\mathcal{P}^N \to x$.

**Def. 2.** To any macrostate $x$ assign the entropy function

$$s(x) = \limsup_{\mathcal{P}^N \to x} \frac{1}{N} \log \text{Tr}(\mathcal{P}^N)$$

**Def. 3.** A sequence $\mathcal{P}^N \to x$ is called typical sequence concentrating at $x$ iff

$$\limsup_{N \to +\infty} \frac{1}{N} \log \text{Tr}(\mathcal{P}^N) = s(x)$$
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Quantum macrostates

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\( \text{Tr}(\mathcal{P}^N) \) along a maximal concentrating sequence \( \mathcal{P}^N \) plays the role of "probability", the entropy \( s(x) \) is its well-defined rate function.

Three immediate QUESTIONS:
1. Is there an IDENTITY WITH THE CANONICAL von Neumann ENTROPY, defined for any state \( \omega_N(\cdot) = \text{Tr}(\sigma_N \cdot) \) as
   \[
   H(\omega_N) = - \text{Tr}(\sigma_N \log \sigma_N)
   \]
2. Is there an H-THEOREM? Is it a Lyapounov function?
3. Are there nontrivial EXAMPLES?
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Quantum macrostates
Generating functions and canonical ensemble

Consider the following generating functions:

\[ p(\lambda) = \lim_{N \to +\infty} \frac{1}{N} \log \text{Tr}(\exp N \sum_k \lambda_k X_k^N) \]

\[ q_k(\kappa) = \lim_{N \to +\infty} \frac{1}{N} \log \text{Tr}(\exp N \sum_k \lambda_k X_k^N \exp \kappa N X_k^N), \quad k = 1, \ldots, K \]

Remarks:

- \( p(\lambda) \) is the “canonical pressure”

\[ \omega^N_{\lambda}(\cdot) = \text{Tr}(\sigma^N_{\lambda} \cdot) = \frac{1}{Z^N_{\lambda}} \text{Tr}(\exp N \sum_k \lambda_k X_k^N \cdot) \]

- parameterized by \( \lambda = (\lambda_1, \ldots, \lambda_K) \)

- \( q_k(\kappa) \) is the large deviation generating function for the single observable \( X_k^N \)

- In general:

\[ q_k(\kappa) \geq p(\lambda + (0, \ldots, (\kappa)_k, \ldots)) \]
Quantum macrostates
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Quantum macrostates
General case

**Theorem** (De Roeck, Maes, Netočný, ’06). Assume that

1. $p(\lambda)$ exists and has the derivative

$$\frac{dp(\kappa \lambda)}{d\kappa} \bigg|_{\kappa=1} = \sum_{k=1}^{K} \lambda_k x_k$$

2. $q_k(\kappa)$ exists and has the derivative

$$\frac{dq_k(\kappa)}{d\kappa} \bigg|_{\kappa=0} = x_k$$

for all $k = 1, \ldots, K$

Then,

$$s(x) = p(\lambda) - \sum_{k=1}^{K} \lambda_k x_k$$
Quantum macrostates

Conclusions

There are various ways how to read the above result:

1. Note that

   \[ p(\lambda) - \sum_{k=1}^{K} \lambda_k x_k = \lim_{N \uparrow +\infty} \frac{H(\omega^N_\lambda)}{N} \]

   is the von Neumann entropy of the (sequence of) canonical states \( \omega^N_\lambda \)

   \( \rightarrow \) the theorem is an equivalence of ensembles result

   \( \rightarrow \) the von Neumann entropy gets a “counting” interpretation
Quantum macrostates

Conclusions

There are various ways how to read the above result:

(2) Under slightly stronger conditions, also

\[ s(x) = \inf_{\lambda} \{ p(\lambda) - \sum_k \lambda_k x_k \} \]

and the result is a noncommutative version of the Gärtner-Ellis theorem (but only under the trace state.)
LARGE DEVIATIONS

Product states

Take a matrix algebra $M$ and consider
- the algebra of $N$ copies: $\mathcal{U}_N = \bigotimes_{i=1}^{N} M_i$
- the product state $\omega_N = \bigotimes_{i=1}^{N} \omega_i$ where $\omega_i$ are copies of a faithful state on $M$

Take a self-adjoint matrix $X = X^* \in M$ and its empirical averages

$$\bar{X}_N = \frac{1}{N} \sum_{i=1}^{N} X_i$$

Question:
What is the law of large fluctuations of $\bar{X}_N$ over the states $\omega_N$ for large $N$?

Various answers, depending on the precise formulation!
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Large Deviations for Product states

Answer 1:
Spectral theorem: there is a probability measure $\mu_N$ on $\text{sp}(\tilde{X}_N) \subset [-\|X\|, \|X\|]$ such that

$$\omega_N(F(\tilde{X}_N)) = \int \mu_N(d\tilde{x}) F(\tilde{x}), \quad F \in C([-\|X\|, \|X\|])$$

- $\mu_N$ is physically the distribution of outcomes when measuring $\tilde{X}_N$ (von Neumann measurement)
- Explicitly: for any $D \subset \mathbb{R}$ a Borel set

$$\mu_N(D) = \omega_N(\tilde{Q}_N(D))$$

$$= \sum_{x_1, \ldots, x_N \in \text{sp}(X)} \omega(Q(x_1)) \ldots \omega(Q(x_N)) \chi\left(\frac{1}{N} \sum_{i=1}^{N} x_i \in D\right)$$

where $Q$ is the projection-valued measure for $X$ and $\tilde{Q}_N$ the projection-valued measure for $\tilde{X}_N$
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Large Deviations for Product states

Since $\mu_N$ satisfy large deviations, we have that

$$\limsup_{N \to +\infty} \frac{1}{N} \log \omega_N(\bar{Q}_N(D)) \leq -\inf_{\bar{x} \in D} I(\bar{x})$$

for $D$ closed

$$\liminf_{N \to +\infty} \frac{1}{N} \log \omega_N(\bar{Q}_N(D)) \geq -\sup_{\bar{x} \in D} I(\bar{x})$$

for $D$ open

or, equivalently, for any $-\|X\| < \bar{x} < \|X\|$,

$$\lim_{\delta \downarrow 0} \lim_{N \to +\infty} \frac{1}{N} \log \omega_N(\bar{Q}_N(\bar{x} - \delta, \bar{x} + \delta)) = -I(\bar{x})$$

with the rate function

$$I(\bar{x}) = \sup_t \{t\bar{x} - q(t)\}$$

$$q(t) = \lim_{N \to +\infty} \frac{1}{N} \log \mu_N(e^{tN\bar{x}}) = \log \omega(e^{tX})$$
Large Deviations for Product states

Remarks:

- Note that the spectral theorem was essential here
- A *naive* attempt to repeat Cramer’s lifting on the level of quantum states $\omega_N$ fails:

One might be tempted to look for a modification of the state $\omega(\cdot) = \text{Tr}(e^A \cdot)$ to

$$\omega^t(\cdot) = \frac{\text{Tr}(e^{A+ t X} \cdot)}{\text{Tr}(e^{A+ t X})}$$

which makes a fixed $\bar{x}$ “typical”, i.e., $\omega^t(X) = \bar{x}$

However, the heuristics

$$\omega_N(\bar{Q}(\bar{x} - \delta, \bar{x} + \delta)) \sim (\text{Tr}(e^{A+ t X}))^N e^{-t \bar{x} N} \frac{\omega^t_N(\bar{Q}(\bar{x} - \delta, \bar{x} + \delta))}{\sim 1}$$

only works when $A$ and $X$ commute!
Hence, the *candidate* generating function

\[ \hat{q}(t) = \log \text{Tr}(e^{A+tX}) \]

does generally not determine the statistics of large fluctuations! By the Golden-Thompson inequality, \( \hat{q}(t) \leq q(t) \).
Large deviations for Product states

Answer 2:
Study the asymptotics of Laplace-Varadhan type:

$$\lim_{N \to +\infty} \frac{1}{N} \log \text{Tr}(\exp \frac{1}{K} \sum_{i=1}^{N} A_i \exp \frac{N}{K} G(\bar{X}_N))^K$$

for various $K > 0$ and $G \in C(\mathbb{R})$

$K = 1$ corresponds to the Varadhan formula over the measures $\mu_N$:

$$\lim_{N \to +\infty} \frac{1}{N} \log \omega_N(e^{NG(\bar{X}_N)})$$

$$= \lim_{N \to +\infty} \frac{1}{N} \log \int \mu_N(d\bar{x}) e^{NG(\bar{x})}$$

$$= \sup_{\bar{x}} \{G(\bar{x}) - I(\bar{x})\}$$
Large Deviations for Product states

- $K = +\infty$ corresponds to the problem

$$\lim_{N \uparrow +\infty} \frac{1}{N} \log \text{Tr}(\exp \sum_{i=1}^{N} A_i + N G(\tilde{X}_N))$$

(by Trotter product formula)

**Theorem** (Petz, Raggio, Verbeure ‘89).

$$\lim_{N \uparrow +\infty} \frac{1}{N} \log \text{Tr}(\exp \sum_{i=1}^{N} A_i + N G(\tilde{X}_N)) = \sup_{\tilde{x}} \{G(\tilde{x}) - \hat{l}(\tilde{x})\}$$

where

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The case $K = +\infty$ is very different from $K = 1$ since the former cannot be rephrased as a classical Varadhan formula upon a classical probability model!
Large Deviations for Product states

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OUR NEW RESULTS

generalizations beyond product states

Large deviation and central limit results can be generalized in the following ways:

▶ Instead of product states one can consider KMS (= equilibrium) states of quantum spin lattice systems in the regimes of
  ▶ high temperature
  ▶ low temperature with a unique ground state and “unbiased” observables

▶ The method uses that the projected state on the observable’s subspace is a classical GIBBS measure. That is related to finite entanglement length.
Classical lattice spin models

Set up

- **Space.** $\mathbb{Z}^d$ a regular lattice, $d = 1, 2, \ldots$
- **Configurations.** $\Omega = \times_{i \in \mathcal{L}} \Omega_i$ where $\Omega_i$ is a finite set of "spins" at site $i$
- **Potential.** $\Phi = (\Phi(A))_{A \subseteq \mathbb{Z}^d}$ where $\Phi(A, \cdot) : \Omega_A \mapsto \mathbb{R}$ are interactions
  - Summability condition:
    \[
    \sup_i \sum_{A \ni i} \| \Phi(A) \| < +\infty
    \]
- **Local Hamiltonians**
  \[
  H_\Lambda(\eta) = \sum_{A \subseteq \Lambda} \Phi(A, \eta_A)
  \]
- **Local Gibbs states**
  \[
  \mu_\Lambda^\beta(\eta) = \frac{1}{Z_\Lambda^\beta} \exp\left( -\beta H_\Lambda(\eta) \right)
  \]
  \[
  Z_\Lambda^\beta = \sum_{\eta \in \Omega_\Lambda} \exp\left( -\beta H_\Lambda(\eta) \right)
  \]
Classical lattice spin models

DLR theory

- **Thermodynamic limit.** Define $\mu^\beta$ as the weak limit (provided it exists) $\mu^\beta = \lim_{\Lambda} \mu^\beta_{\Lambda}$
  - often sufficient but not always!
- **general definition of Gibbs states:**
  - **Relative Hamiltonians:**
    
    $$H^\beta_{\Lambda}(x_{\Lambda} \mid x_{\Lambda^c}) = \sum_{A \cap \Lambda \neq 0} \Phi(A, x_A)$$

- **DLR equations:** for all finite $\Lambda$ and $\mu^\beta$—almost surely
  
  $$\mu^\beta(x_{\Lambda} \mid x_{\Lambda^c}) = \frac{1}{\mathcal{Z}^\beta_{\Lambda}(x_{\Lambda^c})} e^{-\beta H^\beta_{\Lambda}(x_{\Lambda} \mid x_{\Lambda^c})}$$
  
  $$\mathcal{Z}^\beta_{\Lambda}(x_{\Lambda^c}) = \sum_{x_{\Lambda} \in \Omega_{\Lambda}} e^{-\beta H^\beta_{\Lambda}(x_{\Lambda} \mid x_{\Lambda^c})}$$

- turn the logic around:
  a measure $\mu$ on $\Omega_{\mathbb{Z}^d}$ is Gibbs whenever it satisfies the DLR equations with some summable potential $\Phi$
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  $$\mu^\beta(x_\Lambda \mid x_{\Lambda^c}) = \frac{1}{Z_\Lambda^\beta(x_{\Lambda^c})} e^{-\beta H_\Lambda(x_\Lambda \mid x_{\Lambda^c})}$$
  $$Z_\Lambda^\beta(x_{\Lambda^c}) = \sum_{x_\Lambda \in \Omega_\Lambda} e^{-\beta H_\Lambda(x_\Lambda \mid x_{\Lambda^c})}$$
- turn the logic around:
  a measure $\mu$ on $\Omega_{\mathbb{Z}^d}$ is **Gibbs** whenever it satisfies the DLR equations with some summable potential $\Phi$
Gibbs $\implies$ LD $\implies$ CLT

**Theorem** (Lanford, Olla,... ; simplified to level−1). If $\mu$ is a translation-invariant Gibbs measure and $F : \Omega_A \mapsto \mathbb{R}$, $A \subset \subset \mathbb{Z}^d$ a local observable, then the empirical average

$$F_\Lambda(\cdot) = \frac{1}{|\Lambda|} \sum_{i : i + A \subset \Lambda} F(\tau_i \cdot)$$

satisfies the large deviation principle with rate function

$$I^F(\bar{x}) = \sup_t \{t\bar{x} - q^F(t)\}$$

$$q^F(t) = \lim_{\Lambda \uparrow \mathbb{Z}^d} \frac{1}{|\Lambda|} \log \int d\mu(x_\Lambda) e^{t|\Lambda|F_\Lambda(x_\Lambda)}$$

Note 1: This goes beyond the Gärtner-Ellis theorem because $q^F$ does not have to be differentiable (in phase transitions)!
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$$F_\Lambda(\cdot) = \frac{1}{|\Lambda|} \sum_{i : i+A \subset \Lambda} F(\tau_i \cdot)$$

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$$I^F(\bar{x}) = \sup_t \{t\bar{x} - q^F(t)\}$$

$$q^F(t) = \lim_{\Lambda \uparrow \mathbb{Z}^d} \frac{1}{|\Lambda|} \log \int d\mu(x_\Lambda) e^{t|\Lambda|F_\Lambda(x_\Lambda)}$$

**Note 1:**
This goes beyond the Gärtner-Ellis theorem because $q^F$ does not have to be differentiable (in phase transitions)!
Gibbs $\rightarrow$ LD $\rightarrow$ CLT

Note 1:
This goes beyond the Gärtner-Ellis theorem because $q^F$ does not have to be differentiable (in phase transitions)!

Note 2:
The generating function is (set $\beta = 1$)

$$q^F(t) = \lim_{\Lambda \uparrow \mathbb{Z}^d} \frac{1}{|\Lambda|} \log \frac{1}{\mathcal{Z}_{\Lambda}} \sum_{x_{\Lambda} \in \Omega_{\Lambda}} \exp - H_{\Lambda}(x_{\Lambda}) + t \sum_{i: i + A \subset \Lambda} F(x_{\Lambda})$$

$$= P^\Phi(t) - P^\Phi$$

$P^\Phi(t)$ is the “pressure” of a modified potential!
$\implies$ the link between LD and thermodynamic potentials
Gibbs $\implies$ LD $\implies$ CLT

**Theorem** (Bryc, ‘93). If the generating function $q^F(t)$ is analytic on a neighborhood of $t = 0$ then

$$W_\Lambda(x_\Lambda) = \frac{1}{\sqrt{|\Lambda|}} \sum_{i: i+\Lambda \subset \Lambda} [F(\tau_i x_\Lambda) - \int d\mu(x_\Lambda) F(x_\Lambda)]$$

has the limit

$$\lim_{\Lambda \uparrow \mathbb{Z}^d} \int d\mu(x_\Lambda) e^{itW_\Lambda(x_\Lambda)} = e^{-\frac{t^2\sigma^2}{2}}, \quad \sigma^2 \geq 0$$
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Quantum lattice spin models

- **Observables.** $M$ a finite-dimensional matrix algebra $\rightarrow$ local algebras $\mathcal{U}_\Lambda = \bigotimes_{i \in \Lambda} M_i$, $\Lambda \subset \mathbb{Z}^d$

- **Potential.** $\Phi = (\Phi_A)_{A \subset \mathbb{Z}^d}$ a family of self-adjoint elements $\Phi(A) = \Phi(A)^* \in \mathcal{U}_A$
  - Local Hamiltonians
    \[ H_\Lambda = \sum_{A \subset \Lambda} \Phi(A) \]

Local Gibbs states

\[ \omega^\beta_\Lambda(\cdot) = \frac{1}{Z^\beta_\Lambda} \text{Tr}_\Lambda(e^{-\beta H_\Lambda} \cdot), \quad Z^\beta_\Lambda = \text{Tr}_\Lambda(e^{-\beta H_\Lambda}) \]

- **Thermodynamic limit.** The weak limit $\omega^\beta = \lim_\Lambda \omega^\beta_\Lambda$
  (general formalism of quantum Gibbs and KMS states)

http://itf.fys.kuleuven.be/~christ/
Quantum high temperature $\Longrightarrow$ classical Gibbs

Given $X = X^* \in M$ with $Q$ its projection-valued measure

- the family $(X_i)_{i \in \mathbb{Z}^d}$ of copies over $\mathbb{Z}^d$ has the common projection-valued measure

$$Q_\Lambda(x_\Lambda) = \bigotimes_{i \in \mathbb{Z}^d} Q_i(x_i), \quad x_\Lambda \in \Omega_\Lambda \equiv (\text{sp}(X))^\Lambda$$

- there is the probability measure $\mu^{\beta,X}$ given on cylindric sets by

$$\mu^{\beta,X}(x_\Lambda) = \omega^\beta(Q_\Lambda(x_\Lambda))$$

**Theorem** (De Roeck, Maes, Netočný). Let for some $\epsilon > 0$

$$\|\Phi\|_\epsilon \equiv \sup_i \sum_{A \ni i} e^{\epsilon |A|} \|\Phi(A)\| < +\infty$$

There there exists $\beta_0 > 0$ such that for all $|\beta| \leq \beta_0$ and for all $X = X^* \in M$, the measure $\mu^{\beta,X}$ is Gibbs.

Moreover, there is an effective potential $\Psi^{\beta,X}$ for $\mu^{\beta,X}$ such that any $\Psi^{\beta,X}(A, \cdot)$ is analytic in $|\beta| < \beta_0$. 
Quantum high temperature \( \Longrightarrow \) classical Gibbs

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Moreover, there is an effective potential \( \psi^{\beta,X} \) for \( \mu^{\beta,X} \) such that any \( \psi^{\beta,X}(A, \cdot) \) is analytic in \( |\beta| < \beta_0 \).
Quantum high temperature $\rightarrow$ classical Gibbs

In particular, one obtains the large deviation property for the empirical average

$$\bar{X}_\Lambda = \frac{1}{|\Lambda|} \sum_{i \in \Lambda} X_i$$

with the real analytic generating function

$$q(t) = \lim_{\Lambda \uparrow \mathbb{Z}^d} \frac{1}{|\Lambda|} \log \omega^\beta (\exp t \sum_{i \in \Lambda} X_i)$$

Remember:

- That $q(t)$ is not a pressure of a modified quantum systems $\rightarrow$ it is generally difficult to prove both its existence and its smoothness!
- It is however a pressure of the modified effective classical system with the Hamiltonian

$$H^{\beta, X, t}_{\Lambda}(x_\Lambda) = \sum_{A \subset \Lambda} \psi^{\beta, X}(A, x_A) + t \sum_{i \in \Lambda} X_i$$
Some related results

- **Goderis, Verbeure, Vets ‘89**: Fluctuation algebra (= CLT formalism for several observables)
- **Lebowitz, Lenci, Spohn ‘99**: LD for particle density in noninteracting quantum gases
- **Gallavotti, Lebowitz, Mastropietro ‘02**: extension for weakly interacting quantum gases
- **Netočný, Redig ‘04**: existence and real analyticity of $q(t)$ at high temperatures
- **Lenci, Rey Bellet ‘04**: existence of $q(t)$ at high temperatures
- **Abou Salem ‘07**: extension to nonequilibrium steady states
- **Hiai, Mosonyi, Ohno, Petz ‘07**: free energy density for mean field perturbations of one-dimensional spin chains
- **I. Bjelakovic, J.-D. Deuschel, T. Krueger, R. Seiler, Ra. Siegmund-Schultze, A. Szkola ‘07**: Sanov large deviations
Some related results

Moreover:
For general empirical averages with some $F \in \mathcal{U}_A$, $A \subset \subset \mathbb{Z}^d$,

$$F_{\Lambda} = \frac{1}{|\Lambda|} \sum_{i : i + A \subset \Lambda} \tau_i(F)$$

no associate (classical) Gibbs measure exists unless $[\tau_i(F), \tau_j(F)] = 0$, yet

- one can prove $q^F(t)$ to exist and to be analytic in a neighborhood of $t = 0$
- but no proof of existence of $q^F(t)$ for all $t \in \mathbb{R}$ is known!
Quantum Laplace-Varadhan formula

**Theorem** (De Roeck, Maes, Netočný). Let for some $\epsilon > 0$

$$\|\Phi\|_\epsilon \equiv \sup_i \sum_{A \ni i} e^{\epsilon |A|} \|\Phi(A)\| < +\infty$$

There there exists $\beta_0 > 0$ such that for all $|\beta| \leq \beta_0$, for all $X = X^* \in M$, and for all $G \in C([-\|X\|, \|X\|])$ concave,

$$\lim_{\Lambda \uparrow \mathbb{Z}^d} \frac{1}{|\Lambda|} \log \frac{1}{Z_\beta} \text{Tr}(e^{-\beta H_\Lambda + |\Lambda| G(\bar{X}_\Lambda)}) = \sup_{-\|X\| < u < \|X\|} \{G(u) - \hat{l}(u)\}$$

where

$$\hat{l}(u) = \sup_t \{tu - \hat{q}(t)\}$$

$$\hat{q}(t) = \lim_{\Lambda \uparrow \mathbb{Z}^d} \frac{1}{|\Lambda|} \log \frac{1}{Z_\beta} \text{Tr}(\exp(-\beta H_\Lambda + t \sum_{i \in \Lambda} X_i))$$
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Strategy of the equivalence-proof and related results

The proof consists of the next two steps:

**Step 1.**

\( \rho(\kappa \lambda) \) differentiable at \( \kappa = 1 \)

\[ \Longrightarrow \] the projection-valued measure \( \bar{Q}^N \) for \( \sum_k \lambda_k X_k^N \) satisfies

\[
\int_{\lambda^k x_k - \delta}^{\lambda^k x_k + \delta} \omega^N_\lambda (\bar{Q}^N(dz)) \geq 1 - e^{-\tilde{C}(\delta)N}, \quad \tilde{C}(\delta) > 0
\]

\[ \Longrightarrow \] there is sequence \( \delta_N \downarrow 0 \) such that

\[
P^N := \int_{\lambda^k x_k - \delta_N}^{\lambda^k x_k + \delta_N} \bar{Q}^N(dz)
\]

satisfies

- \( \lim_{N \uparrow +\infty} \omega^N_\lambda (P^N) = 1 \)
- \( \lim_{N \uparrow +\infty} \frac{1}{N} \log \text{Tr}_N(P^N) = \lim_{N \uparrow +\infty} \frac{1}{N} H(\omega^N_\lambda) \)
Strategy of the proof and related results

This is a special case of quantum Shannon-McMillan theorem

For a more general result of this type, see:

- *Bjelaković, Krüger, Siegmund-Schultze, Szkoła ‘04*

See also

- *Bjelaković, Deuschel, Krüger, Seiler, Siegmund-Schultze, Szkoła ‘05: quantum Sanov theorem*
Strategy of the proof and related results

The proof consists of the next two steps:

**Step 2.**

$q_k(\kappa)$ differentiable at $\kappa = 0$

\[ \implies \] the projection-valued measure $Q_k^N$ for $X_k^N$ satisfies

\[ \omega_N^{\chi}(Q_k^N, \delta(x_k)) \geq 1 - e^{-C_k(\delta)N}, \quad C_k(\delta) > 0 \]

\[ \implies \]

\[ \frac{\text{Tr}_N(\mathcal{P}_k^N Q_k^N, \delta(x_k))}{\text{Tr}_N(\mathcal{P}^N)} \geq 1 - e^{-C_k(\delta)N}, \quad C_k(\delta) > C_k'(\delta) > 0 \]

\[ \implies \mathcal{P}_k^N \text{ is exponentially concentrating at } x \]
What is a current fluctuation?

Imagine reservoirs in thermal equilibrium at inverse temperatures $\beta_k$ with composite system described by

$$H_\lambda = H + \sum_{k \in K} H_k + \lambda \sum_{k \in K} H_{S_k},$$

Initial state represented by density matrix $\rho_0$ of the form

$$\rho_0 = \rho \otimes \left[ \bigotimes_{k \in K} \rho_{k,\beta_k} \right]$$

where the states $\rho_{k,\beta_k}$ are equilibrium states at $\beta_k$ on the $k$’th reservoir, and $\rho$ is an arbitrary density matrix on the system.

**QUESTION:** how much energy has flown out of/into the different reservoirs after some time $t$. 
What is a current fluctuation?

Answer 1 Introduce “current operator”

\[ l_k(t) := -i U_{-t}^\lambda [H_\lambda, H_k] U_t^\lambda \]

where \( U_t^\lambda := \exp(-itH_\lambda) \) generated by the total Hamiltonian \( H_\lambda \) and \( H_k \) is the free Hamiltonian of the \( k \)’th reservoir only.

Obviously,

\[ U_{-t}^\lambda H_k U_t^\lambda - H_k = \int_0^t ds \ l_k(s) \]

Therefore heat fluctuations

\[ \rho_0 \left[ \exp -i \sum_k \kappa_k (U_{-t}^\lambda H_k U_t^\lambda - H_k) \right] \]

but not at all clear whether the operator associated to the “change of energy in the \( k \)–th reservoir” should really be given by that: \( U_{-t}^\lambda H_k U_t^\lambda \) does not in general commute with \( H_k \), hence their difference does not have a clear physical meaning.
What is a current fluctuation?

Answer 2 Assume for simplicity that \((H_k)_{k \in \mathcal{K}}\) have discrete spectrum, indicating that we have not taken the thermodynamic limit and let \(x \in X\) label a complete set of eigenvectors \(|x\rangle\) of \((H_k)_{k \in \mathcal{K}}\) with nondegenerate eigenvalues \((H_k)_{k \in \mathcal{K}} (x)\). The corresponding spectral projections are denoted \(P_x := |x\rangle \langle x|\).

\[
\chi(\kappa, t, \lambda, \rho_0) := \sum_{x, y \in X} \text{Tr} \left[ P_y U^\lambda_t P_x \rho_0 P_x U^\lambda_{-t} P_y \right] \exp \left( -i \sum_{k \in \mathcal{K}} \kappa_k (H_k(y) - H_k(x)) \right)
\]

Measure (thereby projecting the reservoirs on the eigenstates \(x\)), then switch on the time evolution \(U^\lambda_t\), finally measure again (projecting on the eigenstates \(y\)). Use that the initial state \(\rho_0\) is diagonal in the basis \(|x\rangle\) to rewrite

\[
\chi(\kappa, t, \lambda, \rho_0) = \rho_0 \left[ \exp \left( -i \sum_{k \in \mathcal{K}} \kappa_k H_k U^\lambda_t \left( \exp \left( i \sum_{k \in \mathcal{K}} \kappa_k H_k \right) \right) U_{-t}^\lambda \right] \right].
\]
What is a current fluctuation?

References:
Conclusions

- For a high-temperature quantum lattice system we have constructed an **effective Gibbs measure** for a classical subsystem, which yields **large deviations** and **CLT** for a **single observable**.

- As an alternative approach to quantum large deviations, we have extended the **quantum Laplace-Varadhan** formula of Petz, Raggio, and Verbeure to the same high-temperature regime.

- Similar results can be obtained for other **perturbative regimes**, e.g., at low temperatures with a unique ground state.

- We have discussed one approach to the problem of joint **large deviations for noncommuting observables** based on the construction of a **generalized microcanonical** ensemble; an extension from **trace state** to a **general** state remains to be understood.