The periodic magnetic Schrödinger operators: spectral gaps and tunneling effect

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References

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Notation

- M a noncompact oriented smooth manifold of dimension $n \geq 2$ such that $H^1(M, \mathbb{R}) = 0$.

- Γ a finitely generated, discrete group, which acts properly discontinuously on M so that M/Γ is a compact smooth manifold.

- g a Γ-invariant Riemannian metric on M:

$$g = \sum_{i=1}^{n} \sum_{j=1}^{n} g_{ij}(x) dx^i dx^j.$$

Magnetic field

- $\mathbf{B}$ — a real-valued Γ-invariant closed 2-form on M:

$$\mathbf{B} = \sum_{i<j} b_{ij}(x)dx^i \wedge dx^j.$$

- ASSUME: $\mathbf{A} = \sum_{i=1}^{n} a_i(x)dx^i$ a 1-form on M such that

$$d\mathbf{A} = \mathbf{B} \Longleftrightarrow b_{ij} = \frac{\partial a_j}{\partial x^i} - \frac{\partial a_i}{\partial x^j}.$$

- g_{ij} and b_{ij} Γ-periodic, a_i, in general, NOT.

The magnetic Schrödinger operator

- The Schrödinger operator with magnetic potential $\mathbf{A}$ — a self-adjoint operator in $L^2(M)$:

$$H^h = (ih\, d + \mathbf{A})^*(ih\, d + \mathbf{A}), \quad h > 0.$$

- In $\mathbb{R}^n$, a self-adjoint operator in $L^2(\mathbb{R}^n, \sqrt{g}dx)$

$$H^h = \frac{1}{\sqrt{g}} \sum_{j,k} (ih\frac{\partial}{\partial x^j} + a_j(x)) \left[g^{jk}(x)\sqrt{g}(ih\frac{\partial}{\partial x^k} + a_k(x)) \right]$$

($g = \det(g_{ij})$, g^{jk} the inverse of g_{jk})

The main problem

- A gap in the spectrum $\sigma(T)$ of a self-adjoint operator T is a maximal interval (a,b) such that

$$(a,b) \cap \sigma(T) = \emptyset$$

($\Leftrightarrow$ a component of $\mathbb{R} \setminus \sigma(T)$)

PROBLEMS:

- Are there gaps in the spectrum of H^h in the semiclassical limit (as $h \to 0$)?
- Are there arbitrarily many number of gaps in the spectrum of H^h in the semiclassical limit (as $h \to 0$)?

Some more notation

- $B(x) : T_xM \to T_xM, x \in M$ the anti-symmetric linear operator:

$$g_x(B(x)u, v) = \mathbf{B}_x(u, v), \quad u, v \in T_xM.$$

- In local coordinates

$$B^i_j = \sum_{k=1}^{n} g^{ik} b_{kj} = \sum_{k=1}^{n} g^{ik} \left(\frac{\partial a_k}{\partial x^j} - \frac{\partial a_j}{\partial x^k} \right).$$

Even more notation

- The intensity of the magnetic field

$$\mathrm{Tr}^{\,+}(B(x)) = \frac{1}{2}\mathrm{Tr}\,([B^*(x)\cdot B(x)]^{1/2}).$$

- If $\pm\lambda_j(x), j = 1, 2, \ldots, d, \lambda_j(x) > 0$, are the eigenvalues of $B(x)$, then

$$\mathrm{Tr}^{\,+}(B(x)) = \sum_{j=1}^{d} \lambda_j(x).$$

Magnetic wells

- DENOTE

$$b_0 = \min\{\mathrm{Tr}^+(B(x)) \,:\, x \in M\}.$$

- ASSUME:

  there exist a (connected) fundamental domain $\mathcal{F}$ and $\epsilon_0 > 0$ such that

$$\mathrm{Tr}^+(B(x)) \geq b_0 + \epsilon_0, \quad x \in \partial\mathcal{F}. \tag{1}$$

Magnetic wells. II

- For any $\epsilon_1 \leq \epsilon_0$, let

$$U_{\epsilon_1} = \{x \in \mathcal{F} \, : \, \mathrm{Tr}^{\,+}(B(x)) < b_0 + \epsilon_1\}.$$

  - U_{ϵ_1} an open subset of $\mathcal{F}$ such that $U_{\epsilon_1} \cap \partial\mathcal{F} = \emptyset$;
  - For $\epsilon_1 < \epsilon_0$, $\overline{U_{\epsilon_1}}$ is compact and included in the interior of $\mathcal{F}$.

- Any connected component of U_{ϵ_1} with $\epsilon_1 < \epsilon_0$ — a magnetic well (attached to the effective potential $h \cdot \mathrm{Tr}^{\,+}(B(x))$).

The general case

THEOREM [B. Helffer, Yu. K., 2007]

- ASSUME: there exist a (connected) fundamental domain $\mathcal{F}$ and $\epsilon_0 > 0$ such that

$$\mathrm{Tr}^{\,+}(B(x)) \geq b_0 + \epsilon_0, \quad x \in \partial\mathcal{F}.$$

($\Longleftrightarrow$ there are magnetic wells).

- THEN: for any natural N, there exists $h_0 > 0$ such that, for any $h \in (0, h_0]$,

$$\sigma(H^h) \cap [0, h(b_0 + \epsilon_0)]$$

has at least N gaps.

Potential wells with the regular point bottom

THEOREM [B. Helffer, Yu. K. 2007]

ASSUME

- H^h satisfies the condition (1) with some $\epsilon_0 > 0$:
- there exists a zero $\bar{x}_0$ of B, $B(\bar{x}_0) = 0$, such that $\exists C > 0$

$$C^{-1}|x - x_0|^k \leq \mathrm{Tr}\,^+(B(x)) \leq C|x - x_0|^k$$

for all x in some neighborhood of x_0 with some integer $k > 0$.

Potential wells with the regular point bottom

THEN

for any natural N, there exist $C > 0$ and $h_0 > 0$ such that

$$\sigma(H^h) \cap [0, Ch^{\frac{2k+2}{k+2}}]$$

has at least N gaps for any $h \in (0, h_0)$.

The spectral concentration

THEOREM [Yu. K. 2005]

ASSUME

- $b_0 = 0$.
- For some integer $k > 0$, $B(x_0) = 0 \Rightarrow \exists C > 0$

$$C^{-1}|x - x_0|^k \leq \mathrm{Tr}^{\,+}(B(x)) \leq C|x - x_0|^k$$

for all x in some neighborhood of x_0.

The spectral concentration

THEN

there exists an increasing sequence

$$\lambda_1 < \lambda_2 < \lambda_3 < \dots, \quad \lambda_m \to \infty \text{ as } m \to \infty,$$

such that for any a and b with $\lambda_m < a < b < \lambda_{m+1}$,

$$[ah^{\frac{2k+2}{k+2}}, bh^{\frac{2k+2}{k+2}}] \cap \sigma(H^h) = \emptyset.$$

Potential wells with the one-dimensional bottom

ASSUME:

- $b_0 = 0$, and the zero set of the magnetic field $\mathbf{B}$ has a connected component γ, which is a bounded smooth curve;

- there are constants $k \in \mathbb{N}$ and $C > 0$ such that for all x in some neighborhood of γ the estimates hold:

$$C^{-1} d(x, \gamma)^k \leq |B(x)| \leq C d(x, \gamma)^k \,.$$

- REMARK: For $k = 1$ the last condition means that ∇B does not vanish on γ.

More notation

- ω is the Riemannian volume form on M;
- Write the 2-form $\mathbf{B}$ as

$$\mathbf{B} = B(x)\omega, \quad x \in M;$$

More notation

- N the external unit normal vector to γ;
- $\tilde{N}$ an arbitrary extension of N to a smooth vector field on M;
- $\beta_1 \in C^\infty(M)$ given by

$$\beta_1(x) = \tilde{N}^k B(x), \quad x \in M.$$

- By assumption, we have

$$\beta_1(x) \neq 0, \quad x \in \gamma.$$

Potential wells with the one-dimensional bottom: the main result

ASSUME: the restriction of β_1 to γ (which is independent of the choice of smooth extension $\tilde{N}$) is constant along γ:

$$\beta_1(x) \equiv \beta_1 = \text{const}, \quad x \in \gamma.$$

REMARK: For $k = 1$, this means that the length of the gradient $|\nabla B|$ is constant along γ.

Potential wells with the one-dimensional bottom: the main result

THEOREM: [B. Helffer, Yu. K. 2007]

Under the given assumptions, for any natural N there exist constants $C > 0$ and $h_0 > 0$ such that

$$\sigma(H^h) \cap [0, Ch^{\frac{2k+2}{k+2}}]$$

has at least N gaps for any $h \in (0, h_0)$.

Tunneling and localization in wells

- Fix arbitrary $\epsilon_1 < \epsilon_2 < \epsilon_0$.
- H^h_D the Dirichlet realization of H^h in $D = \overline{U_{\epsilon_2}}$.

THEOREM [B. Helffer, Yu. K., 2006] $\exists C, c, h_0 > 0 \ \forall h \in (0, h_0]$

$$\sigma(H^h) \cap [0, h(b_0 + \epsilon_1)] \subset \{\lambda \in [0, h(b_0 + \epsilon_2)] : \operatorname{dist}(\lambda, \sigma(H^h_D)) < Ce^{-c/\sqrt{h}}\},$$

$$\sigma(H^h_D) \cap [0, h(b_0 + \epsilon_1)] \subset \{\lambda \in [0, h(b_0 + \epsilon_2)] : \operatorname{dist}(\lambda, \sigma(H^h)) < Ce^{-c/\sqrt{h}}\}.$$

Quasimodes and spectral gaps

THEOREM: Let $N \geq 1$.

SUPPOSE $\mu_0^h < \mu_1^h < \ldots < \mu_N^h$ a subset of an interval $I(h) \subset [0, h(b_0 + \epsilon_1))$:

1. There exist constants $c > 0$ and $M \geq 1$ such that for any $h > 0$ small enough

$$\mu_j^h - \mu_{j-1}^h > ch^M, \quad j = 1, \ldots, N,$$
$$\mathrm{dist}(\mu_0^h, \partial I(h)) > ch^M, \quad \mathrm{dist}(\mu_N^h, \partial I(h)) > ch^M;$$

Quasimodes and spectral gaps

2. Each μ_j^h is an approximate eigenvalue of H_D^h:

$$\|H_D^h v_j^h - \mu_j^h v_j^h\| = \alpha_j(h)\|v_j^h\|,$$

where $v_j^h \in C_c^\infty(D)$ and $\alpha_j(h) = o(h^M)$ as $h \to 0$.

THEN

$\sigma(H^h) \cap I(h)$ has at least N gaps for any sufficiently small $h > 0$.

Quasimodes and spectral gaps: sketch of the proof

There exists $\lambda_j^h \in \sigma(H^h) \cap I(h), j = 0, 1, \ldots, N$

$$\lambda_j^h - \mu_j^h = o(h^M), \quad h \to 0.$$

For any $h > 0$ small enough, we have

$$\lambda_j^h - \lambda_{j-1}^h > ch^M, \quad j = 1, \ldots, N.$$

Quasimodes and spectral gaps: sketch of the proof

DENOTE

$N_h(\alpha, \beta)$ — the number of eigenvalues of H_D^h on an arbitrary interval $(h\alpha, h\beta)$.

LEMMA: For some C and h_0

$$N_h(\alpha, \beta) \leq Ch^{-n}, \quad \forall h \in (0, h_0] \ .$$

Quasimodes and spectral gaps: sketch of the proof

- **LEMMA:** Let $M > 0$ and $c > 0$. There exist $C > 0$ and $h_1 > 0$ such that

  IF α^h and β^h are two points in the spectrum of H^h on the interval $I(h)$ with $\beta^h - \alpha^h > ch^M$,

  THEN for any $h \in (0, h_1]$, $\sigma(H^h) \cap (\alpha^h, \beta^h)$ has at least one gap of length $\geq Ch^{M+n}$.

- By this lemma, each interval $(\lambda_j^h, \lambda_{j+1}^h)$ contains at least one gap in the spectrum of H^h of length $\geq Ch^{M+n}$

- $\Longrightarrow$ The spectrum of H^h on the interval $I(h)$ has at least N gaps of length $\geq Ch^{M+n}$ for any h small enough.

The general case: sketch of the proof

- Fix some natural N. Choose some

$$b_0 < \mu_0 < \mu_1 < \ldots < \mu_N < b_0 + \epsilon_1.$$

- For any $j = 0, 1, \ldots, N$, take any $x_j \in D$ such that

$$\mathrm{Tr}^{\,+}(B(x_j)) = \mu_j.$$

The general case: sketch of the proof

- **PROPOSITION** (Helffer-Mohamed, JFA, 1996):

  For any μ in the image of $\mathrm{Tr}\,^{+}B$ and for any $h \in (0,1]$, there exists $C > 0$ such that
  $$(-h^{4/3}C + h\mu, h\mu + h^{4/3}C) \cap \sigma(H_D^h) \neq \emptyset\,.$$

- So Theorem follows from the abstract result with

$$\mu_j^h = h\mu_j, \quad M = 1.$$

The general case: a construction of quasimodes

- Choose a local chart $f_j : U_j \to \mathbb{R}^n$ defined in a neighborhood U_j of x_j with local coordinates $X = (X_1, X_2, \dots, X_n) \in \mathbb{R}^n$.

- Suppose that
  - $f_j(U_j)$ is a ball $B = B(0, r)$ in $\mathbb{R}^n$, $f_j(x_j) = 0$,
  - the Riemannian metric at x_j becomes the standard Euclidean metric on $\mathbb{R}^n$,
  - $\mathbf{B}(x_j) = \sum_{k=1}^{d_j} \mu_{jk} dX_{2k-1} \wedge dX_{2k}$.

The general case: a construction of quasimodes

- Let φ_j be a smooth function on B such that

$$|\mathbf{A}(X) - d\varphi_j(X) - A_j^q(X)| \leq C|X|^2,$$

where $A_j^q(X) = \frac{1}{2}\sum_{k=1}^{d_j} \mu_{jk}\,(X_{2k-1}dX_{2k} - X_{2k}dX_{2k-1})$.

- Write $X'' = (X_{2d_j+1}, \ldots, X_n)$.

- Let $\chi_j \in C_c^\infty(D)$ supported in a neighborhood of x_j, and $\chi_j(x) \equiv 1$ near x_j.

The general case: a construction of quasimodes

- $v_j^h \in C_c^\infty(D)$ defined as

$$v_j^h(x) = \chi_j(x) \exp\left(-i\frac{\varphi_j(x)}{h}\right) \times$$

$$\times \exp\left(-\frac{1}{4h}\sum_{k=1}^{d_j} \mu_{jk}(X_{2k-1}^2 + X_{2k}^2)\right) \exp\left(-\frac{|X''|^2}{h^{2/3}}\right).$$

- THEN

$$\|(H_D^h - h\mu_j)v_j^h\| \leq Ch^{4/3}\|v_j^h\|.$$

The regular point bottom: the model operator

- ASSUME: $\bar{x}_0$ a zero of $\mathbf{B}$ such that, for all x in some neighborhood of x_0,

$$C^{-1}|x - x_0|^k \leq \mathrm{Tr}^{\,+}(B(x)) \leq C|x - x_0|^k.$$

- Write the 2-form $\mathbf{B}$ in the local coordinates

$$f : U(\bar{x}_0) \to f(U(\bar{x}_0)) = B \subset \mathbb{R}^n, \quad f(\bar{x}_0) = 0,$$

as

$$\mathbf{B}(X) = \sum_{1 \leq l < m \leq n} b_{lm}(X)\, dX_l \wedge dX_m, \quad X = (X_1, \ldots, X_n) \in B.$$

The regular point bottom: the model operator

- $\mathbf{B}^0$ the 2-form in $\mathbb{R}^n$ with polynomial components

$$\mathbf{B}^0(X) = \sum_{1 \le l < m \le n} \sum_{|\alpha| = k} \frac{X^\alpha}{\alpha!} \frac{\partial^\alpha b_{lm}}{\partial X^\alpha}(0)\, dX_l \wedge dX_m,$$

- $\exists \mathbf{A}^0$ a 1-form on $\mathbb{R}^n$ with polynomial components:

$$d\mathbf{A}^0(X) = \mathbf{B}^0(X), \quad X \in \mathbb{R}^n.$$

The regular point bottom: the model operator

- $K^h_{\bar{x}_0}$ a self-adjoint differential operator in $L^2(\mathbb{R}^n)$:

$$K^h_{\bar{x}_0} = (ih\, d + \mathbf{A}^0)^*(ih\, d + \mathbf{A}^0),$$

  where the adjoints are taken with respect to the Hilbert structure in $L^2(\mathbb{R}^n)$ given by the flat Riemannian metric $(g_{lm}(0))$ in $\mathbb{R}^n$:

$$K^h_{\bar{x}_0} = \sum_{j,k} g^{jk}(0) \left(ih \frac{\partial}{\partial x^j} + a^0_j(x) \right) \left(ih \frac{\partial}{\partial x^k} + a^0_k(x) \right).$$

The regular point bottom: a construction of quasimodes

- For any $j \in \mathbb{N}$, let

$$K_{\bar{x}_0}^h w_j^h = h^{\frac{2k+2}{k+2}} \lambda_j w_j^h, \quad w_j^h \in L^2(\mathbb{R}^n).$$

- Let $\chi \in C_c^\infty(U(\bar{x}_0))$ equal 1 in a neighborhood of $\bar{x}_0$.

- Define

$$v_j^h(x) = \chi(x) w_j^h(x).$$

The regular point bottom: a construction of quasimodes

- We have

$$\| \left(H_D^h - h^{\frac{2k+2}{k+2}} \lambda_j \right) v_j^h \| \leq C_j h^{\frac{2k+3}{k+2}} \| v_j^h \|.$$

- For a given natural N, choose any $C > \lambda_{N+1}$.

- Then the result follows from the abstract theorem with

$$\mu_j^h = h^{\frac{2k+2}{k+2}} \lambda_j.$$