Some mathematical remarks on the Feynman path integral for the nonrelativistic quantum electrodynamics

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A number of mathematical results on the Feynman path integral for the quantum mechanics have been obtained.

On the other hand, the speaker doesn't know any mathematical results on the Feynman path integral for the quantum electrodynamics, written as QED.

A functional integral representation for the nonrelativistic QED model on the Fock spaces with imaginary time was obtained by Hiroshima (RMP,1997) in terms of the probabilistic method.

Our aim in this talk is to give the mathematical definition of the Feynman path interal for the nonrelativstic QED, especially studied in Feynman (PR,1950) and Feynman - Hibbs (book,1965) by the time-slicing method.

In my talk the Fourier coefficient of electromagnetic potentials are quantized, which is a familiar method in physics (cf. Dirac's book,1958), and photons with large momentum are arbitrarily cut off.

In our method the Coulomb potentials between charged particles naturally appear, which is well known in physics (cf. Fermi (RMPh,1932)).

Our study is defferent from the one for the QED model on the Fock spaces.

We consider n charged nonrelativistic particles $x^{(j)} \in R^3$ ($j = 1, 2, \ldots, n$) with mass $m_j > 0$ and charge $e_j \in R$.

We take a constant $T > 0$ arbitarily.

Let $\phi(t, x) \in R$ be a scalar potential and $A(t, x) \in R^3$ a vector potential for $(t, x) \in [0, T] \times R^3$.

Then the Lagrangian function for the particles and the electromagnetic field with

$$\rho(t,x) = \sum_{j=1}^{n} e_j \delta(x - x^{(j)}),$$

$$j(t,x) = \sum_{j=1}^{n} e_j \dot{x}^{(j)} \delta(x - x^{(j)}) \in R^3$$

is given by

$$\begin{aligned} &\mathcal{L}\left(t, \overrightarrow{x}, \dot{\overrightarrow{x}}, A, \dot{A}, \frac{\partial A}{\partial x}, \phi, \frac{\partial \phi}{\partial x}\right) \\ &= \sum_{j=1}^{n} \left(\frac{m_j}{2}|\dot{x}^{(j)}|^2 - e_j\phi(t, x^{(j)}) + \frac{1}{c} e_j \dot{x}^{(j)} \cdot A(t, x^{(j)})\right) \\ &\quad + \frac{1}{8\pi}\int_{R^3}\left(|E(t,x)|^2 - |B(t,x)|^2\right) dx + C \qquad (1) \end{aligned}$$

with an indefinite constant C.

It seems that an indefinite constant C in $\mathcal{L}$ has not been used by anyone before (cf. Feynman-Hibbs (1965), Sakurai's book (1967), Spohn's book (2004)).

As in Fermi (1932) and Feynman (1950) we consider a very large box

$$V = \left[-\frac{L_1}{2}, \frac{L_1}{2}\right] \times \left[-\frac{L_2}{2}, \frac{L_2}{2}\right] \times \left[-\frac{L_3}{2}, \frac{L_3}{2}\right].$$

We consider only periodic potentials $\phi(t,x)$ and $A(t,x)$ satisfying

$$\nabla \cdot A(t,x) = 0 \text{ (the Coulomb gauge) in } [0,T] \times R^3 \tag{2}$$

and also

$$\int_V \phi(t,x)dx = 0, \quad \int_V A(t,x)dx = 0. \tag{3}$$

Let

$$k = (\frac{2\pi}{L_1}s_1, \frac{2\pi}{L_2}s_2, \frac{2\pi}{L_3}s_3) \ (s_1, s_2, s_3 \in \mathbf{Z}).$$

be wave number vectors.

We take $\overrightarrow{e_j}(k) \in R^3$ $(j = 1, 2)$ such that $\left(\overrightarrow{e_1}(k), \overrightarrow{e_2}(k), k/|k|\right)$ for $k \neq 0$ forms a set of mutually orthogonal unit vectors and

$$\overrightarrow{e_j}(-k) = -\overrightarrow{e_j}(k) \quad (j = 1, 2).$$

for all k.

Let $|V| = L_1 L_2 L_3$.
We can expand $\phi(t,x)$ and $A(t,x)$ into Fourier series

$$A(x, \{a_{|k}\}) = \frac{\sqrt{4\pi}}{|V|} c \sum_{k \neq 0} \{ a_{1k} \mathrm{e}^{ik\cdot x} \overrightarrow{e_1}(k) + a_{2k} \mathrm{e}^{ik\cdot x} \overrightarrow{e_2}(k) \}, \tag{4}$$

$$\phi(x, \{\phi_k\}) = \frac{1}{|V|} \sum_{k \neq 0} \phi_k \mathrm{e}^{ik\cdot x} \tag{5}$$

from the Coulomb gauge (2) and $a_{10} = a_{20} = \phi_0 = 0$ (3).

We write

$$a_{\mathsf{l}k} =: \frac{a_{\mathsf{l}k}^{(1)} - i a_{\mathsf{l}k}^{(2)}}{\sqrt{2}} \ (\mathsf{l} = 1, 2), \tag{6}$$

$$\phi_k =: \phi_k^{(1)} - i\phi_k^{(2)}. \tag{7}$$

Take acccount of the constraint condition, roughly $\nabla \cdot E = 4\pi\rho$, in $\mathcal{L}$ (1) and determine an indefinite constant C formally in $\mathcal{L}$ as

$$\frac{2\pi}{|V|}\left(\sum_{j=1}^{n} e_j\right)\sum_{k\neq 0}\frac{1}{|k|^2} + \frac{1}{2}\sum_{k\neq 0}\frac{\hbar c|k|}{2}. \tag{8}$$

Then, $\mathcal{L}$ (1) can be written as

$$\mathcal{L}_c(\overrightarrow{x}, \dot{\overrightarrow{x}}, \{a_{\mathsf{l}k}\}, \{\dot{a_{\mathsf{l}k}}\}) = \sum_{j=1}^{n} \frac{m_j}{2} |\dot{x}^{(j)}|^2$$

$$- \frac{2\pi}{|V|} \sum_{j,l=1, j\neq l}^{n} \sum_{k\neq 0} \frac{e_j e_l \cos k \cdot (x^{(j)} - x^{(l)})}{|k|^2}$$

$$+ \frac{1}{c} \sum_{j=1}^{n} e_j \dot{x}^{(j)} \cdot A(x^{(j)}, \{a_{\mathsf{l}k}\})$$

$$+ \frac{1}{2} \sum_{k\neq 0, \mathsf{i}, \mathsf{l}} \left(\frac{|\dot{a}_{\mathsf{l}k}^{(\mathsf{i})}|^2}{2|V|} - \frac{(c|k|)^2 |a_{\mathsf{l}k}^{(\mathsf{i})}|^2}{2|V|} + \frac{\hbar c|k|}{2} \right). \tag{9}$$

We shall arbitrarily cut off the terms with the large wave number k in $\mathcal{L}_c$ (9).

Let M_1 be an arbitrary positive number and consider

$$\Lambda := \{k = \left(\frac{2\pi}{L_1}s_1, \frac{2\pi}{L_2}s_2, \frac{2\pi}{L_3}s_3\right) \neq 0; |k| \leq M_1\}.$$

We can write

$$\Lambda = \Lambda' \cup -\Lambda',\ \Lambda' \cap -\Lambda' = \text{empty set}.$$

Since

$$a^{(1)}_{\mathsf{l}-k} = -a^{(1)}_{\mathsf{l}k},\ a^{(2)}_{\mathsf{l}-k} = a^{(2)}_{\mathsf{l}k}$$

from $A \in R^3$, so $a_{\Lambda'} := \{a^{(\mathsf{i})}_{\mathsf{l}k}\}_{\mathsf{i},\mathsf{l},k\in\Lambda'} \in R^{4N}$ ($N := \sharp\Lambda'$) are independent variables. In addition, replace A in $\mathcal{L}_c$ (4) with

$$
\begin{aligned}
\tilde{A}(x, a_{\Lambda'}) = \frac{\sqrt{4\pi c}}{|V|} g(x) \sum_{k \in \Lambda} \Big\{ \Big(\psi(a_{1k}^{(1)}/\sqrt{2}) \cos k \cdot x \\
+ \psi(a_{1k}^{(2)}/\sqrt{2}) \sin k \cdot x \Big) \overrightarrow{e_1}(k) + \Big(\psi(a_{2k}^{(1)}/\sqrt{2}) \cos k \cdot x \\
+ \psi(a_{2k}^{(2)}/\sqrt{2}) \sin k \cdot x \Big) \overrightarrow{e_2}(k) \Big\}. \qquad (10)
\end{aligned}
$$

Then, $\mathcal{L}_c$ (9) is changed to

$$\tilde{\mathcal{L}}_c(\overrightarrow{x}, \dot{\overrightarrow{x}}, a_{\Lambda'}, \dot{a}_{\Lambda'}) = \sum_{j=1}^{n} \frac{m_j}{2}|\dot{x}^{(j)}|^2$$

$$- \frac{2\pi}{|V|} \sum_{j,l=1, j\neq l}^{n} \sum_{k\in\Lambda} \frac{e_j e_l \cos k\cdot(x^{(j)} - x^{(l)})}{|k|^2}$$

$$+ \frac{1}{c}\sum_{j=1}^{n} e_j \dot{x}^{(j)} \cdot \tilde{A}(x^{(j)}, a_{\Lambda'})$$

$$+ \frac{1}{2}\sum_{k\in\Lambda, \mathrm{i}, \mathrm{l}} \left(\frac{|\dot{a}^{(\mathrm{i})}_{\mathrm{l}k}|^2}{2|V|} - \frac{(c|k|)^2|a^{(\mathrm{i})}_{\mathrm{l}k}|^2}{2|V|} + \frac{\hbar c|k|}{2} \right). \qquad (11)$$

Let

$$\Delta : 0 = \tau_0 < \tau_1 < \ldots < \tau_\nu = T, \;\; |\Delta| := \max_{1 \le l \le \nu} (\tau_l - \tau_{l-1}).$$

Let $\overrightarrow{x} := (x^{(1)}, \ldots, x^{(n)}) \in R^{3n}$ and $a_{\Lambda'} \in R^{4N}$ be fixed.
We take arbitrarily

$$\overrightarrow{x}^{(0)}, \ldots, \overrightarrow{x}^{(\nu-1)} \in R^{3n}$$

and

$$a_{\Lambda'}^{(0)}, \ldots, a_{\Lambda'}^{(\nu-1)} \in R^{4N}.$$

Then, we write the broken line paths on $[0,T]$ connecting $\overrightarrow{x}^{(l)}$ at $\theta = \tau_l$ $(l = 0, 1, \ldots, \nu,\ \overrightarrow{x}^{(\nu)} = \overrightarrow{x})$ as $\overrightarrow{q}_{\triangle}(\theta) \in R^{3n}$.

In the same way we define the broken line paths $a_{\Lambda'\triangle}(\theta) \in R^{4N}$ on $[0,T]$ for $a_{\Lambda'}^{(0)}, \ldots, a_{\Lambda'}^{(\nu-1)}, a_{\Lambda'}$.

We write the classical action

$$S_c(T,0;\overrightarrow{q}_{\triangle}, a_{\Lambda'\triangle}) = \int_0^T \tilde{\mathcal{L}}_c\big(\overrightarrow{q}_{\triangle}(\theta), \dot{\overrightarrow{q}}_{\triangle}(\theta), a_{\Lambda'\triangle}(\theta), \dot{a}_{\Lambda'\triangle}(\theta)\big)d\theta,$$

which is a function with respect to $\overrightarrow{x} = (x^{(1)}, \ldots, x^{(n)})$ and $a_{\Lambda'} = \{a_{\mathsf{l}k}^{(\mathsf{i})}\}_{\mathsf{i},\mathsf{l},k\in\Lambda'}$.

Theorem 1. *We assume for* $g(x)$ *and* $\psi(\theta)$ *in* $\tilde{A}$ (10) *that* $\forall l \geq 1, \exists \delta_l > 0$ *satisfying*

$$|\partial_\theta^l \psi(\theta)| \leq C_l (1 + |\theta|)^{-(1+\delta_l)}, \ \theta \in R$$

and that $\forall \alpha, \exists \delta_\alpha > 0$ *satisfying*

$$|\partial_x^\alpha g(x)| \leq C_\alpha (1 + |x|)^{-(1+\delta_\alpha)}, \ x \in R^3.$$

Let $f(\overrightarrow{x}, a_{\Lambda'}) \in L^2(R^{3n+4N})$.
Then, the function

$$\left(\prod_{j=1}^{n}\prod_{l=1}^{\nu}\sqrt{\frac{m_j}{2\pi i\hbar(\tau_l-\tau_{l-1})}}^{3}\right)\prod_{l=1}^{\nu}\sqrt{\frac{1}{2|V|\pi i\hbar(\tau_l-\tau_{l-1})}}^{4N}$$

$$\times\ Os-\int\cdots\int\left(\exp i\hbar^{-1}S_c(T,0;\overrightarrow{q}_{\triangle},a_{\Lambda'\triangle})\right)f\Big(\overrightarrow{q}_{\triangle}(0),$$

$$a_{\Lambda'\triangle}(0)\Big)d\overrightarrow{x}^{(0)}\cdots d\overrightarrow{x}^{(\nu-1)}da_{\Lambda'}^{(0)}\cdots da_{\Lambda'}^{(\nu-1)} \tag{12}$$

converges to the so-called Feynman path integral

$$\iint\left(\exp i\hbar^{-1}S_c(T,0;\overrightarrow{q},a_{\Lambda'})\right)f(\overrightarrow{q}(0),a_{\Lambda'}(0))\mathcal{D}\overrightarrow{q}\mathcal{D}a_{\Lambda'}$$

in $L^2(R^{3n+4N})$ *as* $|\triangle| \to 0$.

In addition, this limit satisfies the Schrödinger type equation

$$i\hbar\frac{\partial}{\partial t}u(t) = H(t)u(t),\ u(0) = f,$$

where

$$H(t) = \sum_{j=1}^{n} \frac{1}{2m_j}\left|\frac{\hbar}{i}\frac{\partial}{\partial x^{(j)}} - \frac{e_j}{c}\tilde{A}(x^{(j)}, a_{\Lambda'})\right|^2$$
$$+ \frac{2\pi}{|V|}\sum_{j,l=1, j\neq l}^{n}\sum_{k\in\Lambda}\frac{e_j e_l \cos k\cdot(x^{(j)} - x^{(l)})}{|k|^2}$$
$$+ \sum_{k\in\Lambda', \mathrm{i}, \mathrm{l}}\left\{\frac{|V|}{2}\left(\frac{\hbar}{i}\frac{\partial}{\partial a^{(\mathrm{i})}_{\mathrm{l}k}}\right)^2 + \frac{(c|k|)^2}{2|V|}\left|a^{(\mathrm{i})}_{\mathrm{l}k}\right|^2 - \frac{\hbar c|k|}{2}\right\}. \qquad (13)$$

Remark 1. *We note about the term in* $\mathcal{L}_c$ (11) *and* $H(t)$ *that we have*

$$\lim_{L_1,L_2,L_3\to\infty}\lim_{M_1\to\infty}\frac{2\pi}{|V|}\sum_{j,l=1,j\neq l}^{n}\sum_{k\in\Lambda}\frac{e_je_l\cos k\cdot(x^{(j)}-x^{(l)})}{|k|^2}$$

$$=\frac{1}{2}\sum_{j,l=1,j\neq l}^{n}\frac{e_je_l}{|x^{(j)}-x^{(l)}|}\quad \textit{in } \mathcal{S}'(R^{3n}) \tag{14}$$

(the Coulomb potentials) as in Fermi (1932) by means of

$$\frac{1}{(2\pi)^2}\int \mathrm{e}^{ik\cdot x}/|k|^2dk=\frac{1}{2}\frac{1}{|x|}=\frac{1}{(2\pi)^2}\int\cos k\cdot x/|k|^2dk$$

in $\mathcal{S}'(R^3)$.

Remark 2. *We can write the last term of* $H(t)$ *defined by* (13) *as*

$$H_{rad} := \sum_{k\in\Lambda',\mathsf{l}} \sum_{\mathsf{i}=1}^{2} \left\{ \frac{|V|}{2} \left(\frac{\hbar}{i} \frac{\partial}{\partial a_{\mathsf{l}k}^{(\mathsf{i})}} \right)^2 + \frac{(c|k|)^2}{2|V|} \left| a_{\mathsf{l}k}^{(\mathsf{i})} \right|^2 - \frac{\hbar c|k|}{2} \right\}$$

$$=: \sum_{k\in\Lambda',\mathsf{l}} \hbar c|k| \hat{a}_{\mathsf{l}k}^{\dagger} \hat{a}_{\mathsf{l}k}.$$

We see that

$$\Psi_0(a_{\Lambda'}) := \prod_{k\in\Lambda',\mathsf{l}} \sqrt{\frac{c|k|}{\hbar|V|}} \exp\left\{ -\frac{c|k|}{2\hbar|V|} \left(a_{\mathsf{l}k}^{(1)^2} + a_{\mathsf{l}k}^{(2)^2} \right) \right\}$$

is the ground state of H_{rad}*, called vacuum, whose energy is* 0 *and that the state of* n' *photons of momentum* $\hbar k$ $(k \in \Lambda')$ *and polarization* l *are given by* $\Psi_{n'\mathsf{l}k}(a_{\Lambda'}) := (\hat{a}_{\mathsf{l}k}^{\dagger})^{n'} \Psi_0(a_{\Lambda'})$ *written concretely.*

Remark 3. *We determined an indefinite constant* C *in* $\mathcal{L}$ *by* (8). *This gives the disapperance of* $\infty = \sum_{j=1}^{n} \sum_{j=l} \sum_{k \neq 0} e_j e_l$ $\times \cos k \cdot (x^{(j)} - x^{(l)})/|k|^2$ *in* $\tilde{\mathcal{L}}_c$ (11) *and* $H(t)$ (13), *and also the disappearance of the ground state energy* $\sum_{k \in \Lambda} \hbar c|k| \big(\to \infty$ *as* $M_1 \to \infty\big)$.

Let's not consider the constraint condition.

Let $\overrightarrow{q}_{\Delta}(\theta) \in R^{3n}$ and $a_{\Lambda'\Delta}(\theta) \in R^{4N}$ be the broken line paths defined before.

Let $\overrightarrow{\xi}_k := \left\{\xi_k^{(\mathrm{i})}\right\}_{\mathrm{i}=1,2} \in R^2$. Take $\overrightarrow{\xi}_k^{(0)}, \overrightarrow{\xi}_k^{(1)}, \ldots$ and $\overrightarrow{\xi}_k^{(\nu-1)}$ in R^2 arbitrarily. Then, we define the path

$$\phi_{k\Delta}(\theta) := \overrightarrow{\xi}_k^{(l)} + \frac{4\pi\rho_k(\overrightarrow{q}_{\Delta}(\theta))}{|k|^2} \in R^2, \ \tau_{l-1} < \theta \leq \tau_l, \tag{15}$$

where $\rho_k(\overrightarrow{x}) \in R^2$ is the vector consisting of the real part and the imaginary part of the Fourier coefficient of ρ.

Let S be the classical action for $\tilde{\mathcal{L}}(\overrightarrow{x}, \dot{\overrightarrow{x}}, a_{\Lambda'}, \dot{a}_{\Lambda'}, \{\phi_k\}_{k\in\Lambda'})$ where A is replaced with $\tilde{A}$ (10) in $\mathcal{L}$ (1).

Theorem 2. *Let $f(\overrightarrow{x}, a_{\Lambda'}) \in L^2(R^{3n+4N})$. Then, under the assumptions of Theorem 1 the function*

$$\left(\prod_{j=1}^{n}\prod_{l=1}^{\nu}\sqrt{\frac{m_j}{2\pi i\hbar(\tau_l-\tau_{l-1})}}^{3}\right)\prod_{l=1}^{\nu}\left\{\sqrt{\frac{1}{2|V|\pi i\hbar(\tau_l-\tau_{l-1})}}^{4N}\right.$$
$$\left.\times\prod_{k\in\Lambda'}\left(-\frac{i|k|^2(\tau_l-\tau_{l-1})}{4\pi^2|V|\hbar}\right)\right\}Os-\int\cdots\int\Big(\exp i\hbar^{-1}S(T,0;$$
$$\overrightarrow{q}_{\triangle}, a_{\Lambda'\triangle}, \{\phi_{k\triangle}\}_{k\in\Lambda'})\Big) f\left(\overrightarrow{q}_{\triangle}(0), a_{\Lambda'\triangle}(0)\right) d\overrightarrow{x}^{(0)}\cdots d\overrightarrow{x}^{(\nu-1)}$$
$$\cdot\, da_{\Lambda'}^{(0)}\cdots da_{\Lambda'}^{(\nu-1)}\prod_{k\in\Lambda'} d\overrightarrow{\xi}_k^{(0)} d\overrightarrow{\xi}_k^{(1)}\cdots d\overrightarrow{\xi}_k^{(\nu-1)} \tag{16}$$

is equal to (12) *in Theorem 1.*

So it follows from Theorem 1 that as $|\Delta| \to 0$, then (16) just above converges to the Feynman path integral

$$\iiint \left(\exp i\hbar^{-1} S(T, 0; \overrightarrow{q}, a_{\Lambda'}, \phi_{\Lambda'})\right) f\Big(\overrightarrow{q}(0),$$

$$a_{\Lambda'}(0)\Big)\mathcal{D}\overrightarrow{q}\,\mathcal{D}a_{\Lambda'}\mathcal{D}\phi_{\Lambda'}.$$

This form is given in §9-8 in Feynman-Hibbs (1965) without any comments.

The outline of the proof of Theorem 1. The proof is mainly done by studying the oscillatory integral operators and by applying the abstract Ascoli-Arzelà theorem on the weighted Sobolev spaces.

For $a = 1, 2, \ldots$ we consider

$$B^a := \{ f(\overrightarrow{x}, a_{\Lambda'}) \in L^2(R^{3n+4N});\ \|f\|_{B^a} := \|f\| + \sum_{|\alpha|=a} (\|\boldsymbol{x}^\alpha f\| + \|(\hbar \partial_{\boldsymbol{x}})^\alpha f\|) < \infty \},\ \boldsymbol{x} := (\overrightarrow{x}, a_{\Lambda'}).$$

We set $B^0 = L^2$.

We can prove : (1) $\exists \rho^* > 0$ and $\exists K_a \geq 0$ $(a = 0, 1, 2, \ldots)$ such that we have a somewhat delicate estimate

$$\|\mathcal{C}_\Delta(t, 0) f\|_{B^a} \leq \mathrm{e}^{K_a t} \|f\|_{B^a},\ 0 \leq |\Delta| \leq \rho^*, \tag{17}$$

where $(\mathcal{C}_\Delta(t, 0) f)(\overrightarrow{x}, a_{\Lambda'})$ is defined by (12) in Theorem 1.

(2) $\exists M \geq 2$ such that for $\leq t, t' \leq T$ we have

$$\left\| i\hbar \left(\mathcal{C}_{\Delta}(t,0)f - \mathcal{C}_{\Delta}(t',0)f \right) - \int_{t'}^{t} H(\theta)\mathcal{C}_{\Delta}(\theta,0) f d\theta \right\|_{B^a}$$

$$\leq C_a \sqrt{|\Delta|}|t - t'|\|f\|_{B^{a+M}}, \ \ 0 \leq |\Delta| \leq \rho^* \tag{18}$$

and so

$$\left\| \mathcal{C}_{\Delta}(t,0)f - \mathcal{C}_{\Delta}(t',0)f \right\|_{B^a} \leq \text{Const.} |t - t'|\|f\|_{B^{a+M}}.$$

The embedding map from $B^{a+2M} \to B^{a+M}$ is compact.

Let $f \in B^{a+2M}$. Then we can apply the abstract Ascoli-Arzerà theorem to $\{\mathcal{C}_\Delta(t,0)f\}_\Delta$ in $C^0([0,T]; B^{a+M})$ from the compactness and the equicontinuity.

Consequently we can prove

$$\exists \lim_{|\Delta_\mu|\to 0} \mathcal{C}_{\Delta_\mu}(t,0)f$$

uniformly in $C^0([0,T]; B^{a+M})$.

This limit satisfies the Schrödinger type equation from (18).

We can prove Theorem 1 from the above together with the uniqueness of solutions to the Schrödinger type equation, and (17) just above.

Theorem 2 is proved by means of

$$\int_{-\infty}^{\infty} \mathrm{e}^{ia\theta^2} d\theta = \sqrt{\frac{i\pi}{a}} \ (a > 0).$$