Tunnel effect for Krammers-Fokker-Planck type operators: return to equilibrium and applications

F. Hérau[1] M. Hitrik[3] J. Sjöstrand[3]

[1]Laboratoire de Mathématiques
Université de Reims

[2]Départment of Mathematics
UCLA

[3]CMLS
Ecole Polytechnique

Qmath10 conference, Brasov 2007

Context

The Fokker-Planck operator :

$$P = y.h\partial_x - V'(x).h\partial_y + \gamma(-h\partial_y + y/).(h\partial_y + y/2)$$

position $x \in \mathbb{R}^d$, velocity $y \in \mathbb{R}^d$, friction coefficient γ.

Some natural 1/2 classical questions arise

- Eigenvalues, resolvent estimate
- Return to the equilibrium for the heat problem
- Tunnel effect (in the case of multiple critical points for V)
- Intrinsic structure $\rightarrow$ supersymetry

Intensive work last years : Helffer, Nier, Lebeau, Bismut...

Context

The Fokker-Planck operator :

$$P = y.h\partial_x - V'(x).h\partial_y + \gamma(-h\partial_y + y/).(h\partial_y + y/2)$$

position $x \in \mathbb{R}^d$, velocity $y \in \mathbb{R}^d$, friction coefficient γ.
Some natural 1/2 classical questions arise

- Eigenvalues, resolvent estimate
- Return to the equilibrium for the heat problem
- Tunnel effect (in the case of multiple critical points for V)
- Intrinsic structure $\rightarrow$ supersymetry

Intensive work last years : Helffer, Nier, Lebeau, Bismut...

Context

The Fokker-Planck operator :

$$P = y.h\partial_x - V'(x).h\partial_y + \gamma(-h\partial_y + y/).(h\partial_y + y/2)$$

position $x \in \mathbb{R}^d$, velocity $y \in \mathbb{R}^d$, friction coefficient γ.
Some natural 1/2 classical questions arise

- Eigenvalues, resolvent estimate
- Return to the equilibrium for the heat problem
- Tunnel effect (in the case of multiple critical points for V)
- Intrinsic structure $\rightarrow$ supersymetry

Intensive work last years : Helffer, Nier, Lebeau, Bismut...

- (Linearized) Kinetic equations $X_0 - L$,
  - Krammers-Fokker-Planck
  - Linear Boltzmann (not local)
  - Linearized Boltzmann, Landau, ...
  - Probabilistic models, other models
- Related questions and structures :
  - Hypoellipticity
  - Hypocoercivity and trend to the equilibrium
  - Supersymetry and inner structures (KFP-like)
  - Boundary, potentials, non-linear problems, perturbative study...

Villani, Mouhot, Guo, Schmeiser, Talay, Eckmann, Rey-bellet, Hairer...

- (Linearized) Kinetic equations $X_0 - L$,
  - Krammers-Fokker-Planck
  - Linear Boltzmann (not local)
  - Linearized Boltzmann, Landau, ...
  - Probabilistic models, other models
- Related questions and structures :
  - Hypoellipticity
  - Hypocoercivity and trend to the equilibrium
  - Supersymetry and inner structures (KFP-like)
  - Boundary, potentials, non-linear problems, perturbative study...

Villani, Mouhot, Guo, Schmeiser, Talay, Eckmann, Rey-bellet, Hairer...

- (Linearized) Kinetic equations $X_0 - L$,
  - Krammers-Fokker-Planck
  - Linear Boltzmann (not local)
  - Linearized Boltzmann, Landau, ...
  - Probabilistic models, other models
- Related questions and structures :
  - Hypoellipticity
  - Hypocoercivity and trend to the equilibrium
  - Supersymetry and inner structures (KFP-like)
  - Boundary, potentials, non-linear problems, perturbative study...

Villani, Mouhot, Guo, Schmeiser, Talay, Eckmann, Rey-bellet, Hairer...

Very constructive interaction

P.D.E

- 1/2 classical methods
- ϕ dO methods
- supersymetry ...

$\longleftrightarrow$

Kinetic

- hypocoercivity,
- trend to eq.
- perturbative study
- ...

Table of contents

Assumptions and main result

Back to the Fokker-Planck operator

$$P = y.h\partial_x - V'(x).h\partial_y + \gamma(-h\partial_y + y/2).(h\partial_y + y/2)$$

We impose on the potential V the following :

- $\partial^\alpha V = \mathcal{O}(1)$ for $|\alpha| \geq 2$,
- $|\nabla V| \geq 1/C$, for $|x| \geq C$ with C sufficiently large
- V has 3 critical points : 2 local minima and 1 critical point of index 1

Then P has 2 eigenvalues in the disc $D(0, C/h)$ for h sufficiently small, $\mu_0 = 0$ and μ_1, with μ_1 of the form

$$\mu_1 = h\left(a_1(h)e^{-2S_1/h} + a_{-1}(h)e^{-2S_{-1}/h}\right), \quad S_j = V(x_j) - V(x_0).$$

Assumptions and main result

Back to the Fokker-Planck operator

$$P = y.h\partial_x - V'(x).h\partial_y + \gamma(-h\partial_y + y/2).(h\partial_y + y/2)$$

We impose on the potential V the following :

- $\partial^\alpha V = \mathcal{O}(1)$ for $|\alpha| \geq 2$,
- $|\nabla V| \geq 1/C$, for $|x| \geq C$ with C sufficiently large
- V has 3 critical points : 2 local minima and 1 critical point of index 1

Then P has 2 eigenvalues in the disc $D(0, C/h)$ for h sufficiently small, $\mu_0 = 0$ and μ_1, with μ_1 of the form

$$\mu_1 = h\left(a_1(h)e^{-2S_1/h} + a_{-1}(h)e^{-2S_{-1}/h}\right), \quad S_j = V(x_j) - V(x_0).$$

Assumptions and main result

Back to the Fokker-Planck operator

$$P = y.h\partial_x - V'(x).h\partial_y + \gamma(-h\partial_y + y/2).(h\partial_y + y/2)$$

We impose on the potential V the following :

- $\partial^\alpha V = \mathcal{O}(1)$ for $|\alpha| \geq 2$,
- $|\nabla V| \geq 1/C$, for $|x| \geq C$ with C sufficiently large
- V has 3 critical points : 2 local minima and 1 critical point of index 1

Then P has 2 eigenvalues in the disc $D(0, C/h)$ for h sufficiently small, $\mu_0 = 0$ and μ_1, with μ_1 of the form

$$\mu_1 = h\left(a_1(h)e^{-2S_1/h} + a_{-1}(h)e^{-2S_{-1}/h}\right), \quad S_j = V(x_j) - V(x_0).$$

Other cases

- "Simple well" : V has precisely 1 local minimum and $V(x) \to \infty$ then from [HHS07], P has only 1 eigenvalue $\mu_0 = 0$ in the disc $D(0, C/h)$ for $h \ll 1$.
- "A well and the sea" : V has precisely 1 local minimum x_1 and 1 critical point x_0, then P has only 1 eigenvalue μ_1 in the disc $D(0, C/h)$.

$$\mu_1 = ha_1(h)e^{-2S_1/h}, \quad S_1 = V(x_1) - V(x_0).$$

- "multiple wells" : Serious hope to get similar results as in the Witten case (see recent work about linear algebra by le peutrec in the Witten case).

Main problem : resolvent estimates, spectral projectors not selfadjoint...

Other cases

- "Simple well" : V has precisely 1 local minimum and $V(x) \to \infty$ then from [HHS07], P has only 1 eigenvalue $\mu_0 = 0$ in the disc $D(0, C/h)$ for $h \ll 1$.
- "A well and the sea" : V has precisely 1 local minimum x_1 and 1 critical point x_0, then P has only 1 eigenvalue μ_1 in the disc $D(0, C/h)$.

$$\mu_1 = ha_1(h)e^{-2S_1/h}, \quad S_1 = V(x_1) - V(x_0).$$

- "multiple wells" : Serious hope to get similar results as in the Witten case (see recent work about linear algebra by le peutrec in the Witten case).

Main problem : resolvent estimates, spectral projectors not selfadjoint...

Other cases

- "Simple well" : V has precisely 1 local minimum and $V(x) \to \infty$ then from [HHS07], P has only 1 eigenvalue $\mu_0 = 0$ in the disc $D(0, C/h)$ for $h \ll 1$.
- "A well and the sea" : V has precisely 1 local minimum x_1 and 1 critical point x_0, then P has only 1 eigenvalue μ_1 in the disc $D(0, C/h)$.

$$\mu_1 = ha_1(h)e^{-2S_1/h}, \quad S_1 = V(x_1) - V(x_0).$$

- "multiple wells" : Serious hope to get similar results as in the Witten case (see recent work about linear algebra by le peutrec in the Witten case).

Main problem : resolvent estimates, spectral projectors not selfadjoint...

Other cases

- "Simple well" : V has precisely 1 local minimum and $V(x) \to \infty$ then from [HHS07], P has only 1 eigenvalue $\mu_0 = 0$ in the disc $D(0, C/h)$ for $h \ll 1$.
- "A well and the sea" : V has precisely 1 local minimum x_1 and 1 critical point x_0, then P has only 1 eigenvalue μ_1 in the disc $D(0, C/h)$.

$$\mu_1 = ha_1(h)e^{-2S_1/h}, \quad S_1 = V(x_1) - V(x_0).$$

- "multiple wells" : Serious hope to get similar results as in the Witten case (see recent work about linear algebra by le peutrec in the Witten case).

Main problem : resolvent estimates, spectral projectors not selfadjoint...

Heat problem

Analyse of $e^{-tP/h}$.

- In the case without "tunneling effect" (e.g. the simple well case), the return to equilibrium is of order of magnitude 1.
- What is this rate in the case of tunneling effect ("double well" and " a well and the sea").

Main result

In the double well case, Let Π_j the spectral projection associated to μ_j, then

$$\Pi_j = \mathcal{O}(1), \quad h \to 0.$$

and uniformly as $t \geq 0$, and $h \to 0$,

$$e^{-tP/h} = \Pi_0 + e^{-t\mu_1/h}\Pi_1 + \mathcal{O}(1)e^{-t/C}, \quad C > 0, \quad in \mathcal{L}(L^2, L^2).$$

Heat problem

Analyse of $e^{-tP/h}$.

- In the case without "tunneling effect" (e.g. the simple well case), the return to equilibrium is of order of magnitude 1.
- What is this rate in the case of tunneling effect ("double well" and " a well and the sea").

Main result

In the double well case, Let Π_j the spectral projection associated to μ_j, then

$$\Pi_j = \mathcal{O}(1), \quad h \to 0.$$

and uniformly as $t \geq 0$, and $h \to 0$,

$$e^{-tP/h} = \Pi_0 + e^{-t\mu_1/h}\Pi_1 + \mathcal{O}(1)e^{-t/C}, \quad C > 0, \quad in\mathcal{L}(L^2, L^2).$$

Sketch of proof

A coercive estimate

Eigenvalues [HHS07] : let p be the symbol of KFP and $p_{0,j}$ be the symbol of the quadratic approximation at ρ_j critical point.
The spectrum of P in $D(0, Bh)$ is discrete and equal to

$$\lambda_{j,k}(h) \sim h\left(\mu_{j,k} + h^{1/N_{j,k}}\mu_{j,k,1} + h^{2/N_{j,k}}\mu_{j,k,2} + \dots\right),$$

Recall that $\mu_{j,k}$ are all numbers in $D(0, B)$ of the form

$$\mu_{j,k} = \frac{1}{i}\sum_{\ell=1}^{n}\left(\nu_{j,k,\ell} + \frac{1}{2}\right)\lambda_{j,\ell}, \quad \nu_{j,k,\ell} \in \mathbf{N},$$

for some $j \in \{1, \dots N\}$. Here $\lambda_{j,\ell}$, $1 \le \ell \le n$, are the eigenvalues of the Hamilton map of the quadratic part of p at $\rho_j \in \mathcal{C}$, for which $\mathrm{Im}\,\lambda_{j,\ell} > 0$.

A coercive estimate

Eigenvalues [HHS07] : let p be the symbol of KFP and $p_{0,j}$ be the symbol of the quadratic approximation at ρ_j critical point. The spectrum of P in $D(0, Bh)$ is discrete and equal to

$$\lambda_{j,k}(h) \sim h\left(\mu_{j,k} + h^{1/N_{j,k}}\mu_{j,k,1} + h^{2/N_{j,k}}\mu_{j,k,2} + \dots\right),$$

Recall that $\mu_{j,k}$ are all numbers in $D(0, B)$ of the form

$$\mu_{j,k} = \frac{1}{i}\sum_{\ell=1}^{n}\left(\nu_{j,k,\ell} + \frac{1}{2}\right)\lambda_{j,\ell}, \quad \nu_{j,k,\ell} \in \mathbf{N},$$

for some $j \in \{1, \dots N\}$. Here $\lambda_{j,\ell}$, $1 \leq \ell \leq n$, are the eigenvalues of the Hamilton map of the quadratic part of p at $\rho_j \in \mathcal{C}$, for which $\mathrm{Im}\,\lambda_{j,\ell} > 0$.

A coercive estimate

Eigenvalues [HHS07] : let p be the symbol of KFP and $p_{0,j}$ be the symbol of the quadratic approximation at ρ_j critical point. The spectrum of P in $D(0, Bh)$ is discrete and equal to

$$\lambda_{j,k}(h) \sim h\left(\mu_{j,k} + h^{1/N_{j,k}}\mu_{j,k,1} + h^{2/N_{j,k}}\mu_{j,k,2} + \dots\right),$$

Recall that $\mu_{j,k}$ are all numbers in $D(0, B)$ of the form

$$\mu_{j,k} = \frac{1}{i}\sum_{\ell=1}^{n}\left(\nu_{j,k,\ell} + \frac{1}{2}\right)\lambda_{j,\ell}, \quad \nu_{j,k,\ell} \in \mathbf{N},$$

for some $j \in \{1, \dots N\}$. Here $\lambda_{j,\ell}$, $1 \leq \ell \leq n$, are the eigenvalues of the Hamilton map of the quadratic part of p at $\rho_j \in \mathcal{C}$, for which $\mathrm{Im}\,\lambda_{j,\ell} > 0$.

Crucial estimate :

Lemma

Let $B \geq 0$ and Π_B the corresponding spectral projector, then for all $u \in \mathrm{Ran}(1 - \Pi_B)$,

$$\left\| e^{-tP/h} u \right\| = \mathcal{O}(1) e^{-t/C}, \quad C = C(B)$$

difficulties :

- Π_B not selfadjoint,
- $\mathrm{Re}\,(Pu, u) \geq Ch \|u\|^2$ not true
- $\mathrm{Re}\,(Pu, u)_\varepsilon \geq Ch \|u\|_\varepsilon^2$ true with a modified norm !

Crucial estimate :

Lemma

Let $B \geq 0$ and Π_B the corresponding spectral projector, then for all $u \in \mathrm{Ran}(1 - \Pi_B)$,

$$\left\| e^{-tP/h} u \right\| = \mathcal{O}(1) e^{-t/C}, \quad C = C(B)$$

difficulties :

- Π_B not selfadjoint,
- $\mathrm{Re}\,(Pu, u) \geq Ch \|u\|^2$ not true
- $\mathrm{Re}\,(Pu, u)_\varepsilon \geq Ch \|u\|_\varepsilon^2$ true with a modified norm !

Exists Global FIO A_ε exploiting the hypoelliptic properties of P such that $\|u\|_\varepsilon \stackrel{\text{def}}{=} \|A_\varepsilon u\| \sim \|u\|$.

Study of $P_\varepsilon = A_\varepsilon^{-1} P_\varepsilon A_\varepsilon$ in L^2 :

We already know [HHS07] that for all u

$$\mathrm{Re}\,((P_\varepsilon + K_\varepsilon)u, u) \geq ch\,\|u\|^2$$

where K_ε is (micro-)localized near the critical points. Sufficient to prove $\|K_\varepsilon u\| \ll h\,\|u\|$.

- Building a selfadjoint operator Q (an harmonic oscillator) adapted to the evs $\leq B$.
- Posing $K_\varepsilon = \chi(Q/B)$.
- This gives the semi-group property.

Exists Global FIO A_ε exploiting the hypoelliptic properties of P such that $\|u\|_\varepsilon \stackrel{\text{def}}{=} \|A_\varepsilon u\| \sim \|u\|$.

$$\text{Study of } P_\varepsilon = A_\varepsilon^{-1} P_\varepsilon A_\varepsilon \text{ in } L^2 :$$

We already know [HHS07] that for all u

$$\mathrm{Re}\,((P_\varepsilon + K_\varepsilon)u, u) \geq ch\,\|u\|^2$$

where K_ε is (micro-)localized near the critical points. Sufficient to prove $\|K_\varepsilon u\| \ll h\,\|u\|$.

- Building a selfadjoint operator Q (an harmonic oscillator) adapted to the evs $\leq B$.
- Posing $K_\varepsilon = \chi(Q/B)$.
- This gives the semi-group property.

Exists Global FIO A_ε exploiting the hypoelliptic properties of P such that $\|u\|_\varepsilon \overset{\text{def}}{=} \|A_\varepsilon u\| \sim \|u\|$.

$$\text{Study of } P_\varepsilon = A_\varepsilon^{-1} P_\varepsilon A_\varepsilon \text{ in } L^2 :$$

We already know [HHS07] that for all u

$$\mathrm{Re}\,((P_\varepsilon + K_\varepsilon)u, u) \geq ch \|u\|^2$$

where K_ε is (micro-)localized near the critical points. Sufficient to prove $\|K_\varepsilon u\| \ll h \|u\|$.

- Building a selfadjoint operator Q (an harmonic oscillator) adapted to the evs $\leq B$.
- Posing $K_\varepsilon = \chi(Q/B)$.
- This gives the semi-group property.

General form

Previous results valid for a large class of 2nd order differential operators :

$$p = p_2 + ip_1 + p_0$$

where

$$p_2 = \sum b_{j,k}\xi_j\xi_k, \quad p_1 = \sum c_j(x)\xi_j, \quad p_0 = p_0(x)$$

with the following assumptions :

- positivity p_2 and $p_0 \geq 0$,
- growth $|\partial^\alpha b| + |\partial^{\alpha+1} c| + |\partial^{\alpha+2} p_0| = \mathcal{O}(1), \quad |\alpha| \geq 0$,
- finite critical set $\{(x_l, 0) \text{ with } p_0(x_l) = 0,\ c(x_l) = 0\}$,

and if $< p >= \frac{1}{T}\int_{[-T,T]}(p_0 + p_2/ < \xi >^2)dt$,

- local dynamic $< p >\sim |\rho - \rho_j|^2$ near ρ_j
- global dynamic $< p >\geq C$ away.

General form

Previous results valid for a large class of 2nd order differential operators :

$$p = p_2 + ip_1 + p_0$$

where

$$p_2 = \sum b_{j,k}\xi_j\xi_k, \quad p_1 = \sum c_j(x)\xi_j, \quad p_0 = p_0(x)$$

with the following assumptions :

- positivity p_2 and $p_0 \geq 0$,
- growth $|\partial^\alpha b| + |\partial^{\alpha+1} c| + |\partial^{\alpha+2} p_0| = \mathcal{O}(1), \quad |\alpha| \geq 0$,
- finite critical set $\{(x_l, 0) \text{ with } p_0(x_l) = 0, \; c(x_l) = 0\}$,

and if $< p >= \frac{1}{T}\int_{[-T,T]}(p_0 + p_2/ < \xi >^2)dt$,

- local dynamic $< p >\sim |\rho - \rho_j|^2$ near ρ_j
- global dynamic $< p >\geq C$ away.

General form

Previous results valid for a large class of 2nd order differential operators :

$$p = p_2 + ip_1 + p_0$$

where

$$p_2 = \sum b_{j,k}\xi_j\xi_k, \quad p_1 = \sum c_j(x)\xi_j, \quad p_0 = p_0(x)$$

with the following assumptions :

- positivity p_2 and $p_0 \geq 0$,
- growth $|\partial^\alpha b| + |\partial^{\alpha+1} c| + |\partial^{\alpha+2} p_0| = \mathcal{O}(1), \quad |\alpha| \geq 0$,
- finite critical set $\{(x_l, 0) \text{ with } p_0(x_l) = 0,\ c(x_l) = 0\}$,

and if $< p >= \frac{1}{T}\int_{[-T,T]}(p_0 + p_2/ < \xi >^2)dt$,

- local dynamic $< p >\sim |\rho - \rho_j|^2$ near ρ_j
- global dynamic $< p >\geq C$ away.

Supersymetry

They are defined through

- An invertible real $d \times d$ matrix $A = B + C$, B sym., C skew.
- A morse function ϕ where $\partial^\alpha \phi$ and $\partial^\alpha < B\nabla\phi, \nabla\phi >$ are $\mathcal{O}(1)$.

The Witten Hodge Laplacian is

$$\sum -h^2 \sum \partial_j B_{j,k} \partial_k + \sum \partial_j \phi B_{j,k} \partial_k \phi - htr(B\phi'') + \sum \partial_j \phi C_{j,k} \partial_k + \sum \partial_j C_{j,k} \partial_k \phi$$

Principal symbol : $p = \langle B\xi, \xi \rangle + 2i \langle C\nabla\phi, \xi \rangle + \langle B\nabla\phi, \nabla\phi \rangle$, Of Witten Hodge Laplacian type : $-\Delta_A = d_\phi^{A,*} d_\phi$ on $k-$forms.

Supersymetry

They are defined through

- An invertible real $d \times d$ matrix $A = B + C$, B sym., C skew.
- A morse function ϕ where $\partial^\alpha \phi$ and $\partial^\alpha < B\nabla\phi, \nabla\phi >$ are $\mathcal{O}(1)$.

The Witten Hodge Laplacian is

$$\sum -h^2 \sum \partial_j B_{j,k} \partial_k + \sum \partial_j \phi B_{j,k} \partial_k \phi - htr(B\phi'')$$
$$+ \sum \partial_j \phi C_{j,k} \partial_k + \sum \partial_j C_{j,k} \partial_k \phi$$

Principal symbol : $p = \langle B\xi, \xi\rangle + 2i \langle C\nabla\phi, \xi\rangle + \langle B\nabla\phi, \nabla\phi\rangle$, Of Witten Hodge Laplacian type : $-\Delta_A = d_\phi^{A,*} d_\phi$ on $k-$forms.

Supersymetry

They are defined through

- An invertible real $d \times d$ matrix $A = B + C$, B sym., C skew.
- A morse function ϕ where $\partial^\alpha \phi$ and $\partial^\alpha < B\nabla\phi, \nabla\phi >$ are $\mathcal{O}(1)$.

The Witten Hodge Laplacian is

$$\sum -h^2 \sum \partial_j B_{j,k} \partial_k + \sum \partial_j \phi B_{j,k} \partial_k \phi - htr(B\phi'')$$
$$+ \sum \partial_j \phi C_{j,k} \partial_k + \sum \partial_j C_{j,k} \partial_k \phi$$

Principal symbol : $p = \langle B\xi, \xi \rangle + 2i \langle C\nabla\phi, \xi \rangle + \langle B\nabla\phi, \nabla\phi \rangle$, Of Witten Hodge Laplacian type : $-\Delta_A = d_\phi^{A,*} d_\phi$ on $k-$forms.

Return to equilibrium

With the supersymetric structure : reduce the problem of exp. small evs to a finite dimensional problem.
In particular build the corresponding eigenfunctions for $-\Delta_A^{(0)}$ and $-\Delta_A^{(1)}$, as in the treatment by Helffer-Sjostrand [HS80'].

$$e.g. \quad e_j(x) = h^{-n/4} c_j(h) e^{\frac{1}{h}(\phi(x)-\phi(x_j))}$$

Back to the double well case : 2 minima, therefore 2 exp. small evs (0 and μ_1).

- Express explicitly the projectors Π_k for each exp. small eigenvalue.
- Write $e^{-tP/h} = e^{-tP/h}(\Pi_0 + \Pi_1 + \Pi_{(2-B)} + (1 - \Pi_B))$
- Use the former result for the last term, and the 2 first (the third one is easy with resolvent estimates from [HHS07]).

Return to equilibrium

With the supersymetric structure : reduce the problem of exp. small evs to a finite dimensional problem.
In particular build the corresponding eigenfunctions for $-\Delta_A^{(0)}$ and $-\Delta_A^{(1)}$, as in the treatment by Helffer-Sjostrand [HS80'].

$$e.g. \quad e_j(x) = h^{-n/4} c_j(h) e^{\frac{1}{h}(\phi(x)-\phi(x_j))}$$

Back to the double well case : 2 minima, therefore 2 exp. small evs (0 and μ_1).

- Express explicitly the projectors Π_k for each exp. small eigenvalue.
- Write $e^{-tP/h} = e^{-tP/h}(\Pi_0 + \Pi_1 + \Pi_{(2-B)} + (1 - \Pi_B))$
- Use the former result for the last term, and the 2 first (the third one is easy with resolvent estimates from [HHS07]).

Return to equilibrium

With the supersymetric structure : reduce the problem of exp. small evs to a finite dimensional problem.
In particular build the corresponding eigenfunctions for $-\Delta_A^{(0)}$ and $-\Delta_A^{(1)}$, as in the treatment by Helffer-Sjostrand [HS80'].

$$e.g. \quad e_j(x) = h^{-n/4} c_j(h) e^{\frac{1}{h}(\phi(x)-\phi(x_j))}$$

Back to the double well case : 2 minima, therefore 2 exp. small evs (0 and μ_1).

- Express explicitly the projectors Π_k for each exp. small eigenvalue.
- Write $e^{-tP/h} = e^{-tP/h}(\Pi_0 + \Pi_1 + \Pi_{(2-B)} + (1 - \Pi_B))$
- Use the former result for the last term, and the 2 first (the third one is easy with resolvent estimates from [HHS07]).

Return to equilibrium

With the supersymetric structure : reduce the problem of exp. small evs to a finite dimensional problem.
In particular build the corresponding eigenfunctions for $-\Delta_A^{(0)}$ and $-\Delta_A^{(1)}$, as in the treatment by Helffer-Sjostrand [HS80'].

$$e.g. \quad e_j(x) = h^{-n/4} c_j(h) e^{\frac{1}{h}(\phi(x)-\phi(x_j))}$$

Back to the double well case : 2 minima, therefore 2 exp. small evs (0 and μ_1).

- Express explicitly the projectors Π_k for each exp. small eigenvalue.
- Write $e^{-tP/h} = e^{-tP/h}(\Pi_0 + \Pi_1 + \Pi_{(2-B)} + (1 - \Pi_B))$
- Use the former result for the last term, and the 2 first (the third one is easy with resolvent estimates from [HHS07]).

Return to equilibrium

With the supersymetric structure : reduce the problem of exp. small evs to a finite dimensional problem.
In particular build the corresponding eigenfunctions for $-\Delta_A^{(0)}$ and $-\Delta_A^{(1)}$, as in the treatment by Helffer-Sjostrand [HS80'].

$$e.g. \quad e_j(x) = h^{-n/4} c_j(h) e^{\frac{1}{h}(\phi(x)-\phi(x_j))}$$

Back to the double well case : 2 minima, therefore 2 exp. small evs (0 and μ_1).

- Express explicitly the projectors Π_k for each exp. small eigenvalue.
- Write $e^{-tP/h} = e^{-tP/h}(\Pi_0 + \Pi_1 + \Pi_{(2-B)} + (1 - \Pi_B))$
- Use the former result for the last term, and the 2 first (the third one is easy with resolvent estimates from [HHS07]).

Examples of KFP type operators

Probability

Some problem may come from Probability. Let

$$dx(t) = b(x(t))dt + \sigma dw$$

where w is a d-dimentionnal process, σ a constant matrix and $\partial^\alpha b = \mathcal{O}(1)$. Then there exists a unique solution (in an L^2 adapted space) for $x_0 \coprod w$.

Define the associated semi group by

$$\mathbb{E}\left(\phi(x(t))\right) = T^t \phi(x_0)$$

This is a strongly semi-group (on $\mathcal{C}_\infty$, we can work also on L^2) whose infinitesimal generator is

$$L = \nabla.D\nabla + b(x).\nabla \qquad \text{with } D = \frac{1}{2}\sigma\sigma^t$$

Probability

Some problem may come from Probability. Let

$$dx(t) = b(x(t))dt + \sigma dw$$

where w is a d-dimentionnal process, σ a constant matrix and $\partial^\alpha b = \mathcal{O}(1)$. Then there exists a unique solution (in an L^2 adapted space) for $x_0 \coprod w$.
Define the associated semi group by

$$\mathbb{E}\left(\phi(x(t))\right) = T^t \phi(x_0)$$

This is a strongly semi-group (on $\mathcal{C}_\infty$, we can work also on L^2) whose infinitesimal generator is

$$L = \nabla.D\nabla + b(x).\nabla \qquad \text{with } D = \frac{1}{2}\sigma\sigma^t$$

Probability

Some problem may come from Probability. Let

$$dx(t) = b(x(t))dt + \sigma dw$$

where w is a d-dimentionnal process, σ a constant matrix and $\partial^\alpha b = \mathcal{O}(1)$. Then there exists a unique solution (in an L^2 adapted space) for $x_0 \coprod w$.
Define the associated semi group by

$$\mathbb{E}\left(\phi(x(t))\right) = T^t \phi(x_0)$$

This is a strongly semi-group (on $\mathcal{C}_\infty$, we can work also on L^2) whose infinitesimal generator is

$$L = \nabla.D\nabla + b(x).\nabla \quad \text{with } D = \frac{1}{2}\sigma\sigma^t$$

Its formal adjoint is

$$L^* = \nabla.D\nabla - \nabla.b(x)$$

and is the infinitesimal generator of $(T^t)^*)$ (at least its closure in an L^2 setting).

This means that if $\mu_0 = f_0 dx$ is the a.c. measure of probability of x_0, then $\mu_t = f(t,.)dx$ is the one of $x(t)$ and $\partial_t f - L^* f = 0$, ie

$$\begin{cases} \partial_t f + (-\nabla.D.\nabla + \nabla.b)f = 0 \\ f|_{t=0} = f_0 \end{cases}$$

An invariant measure will be associated to a time-independant function $\mathcal{M}$. What remains in particular cases is

- exhibit the Maxwellian $\mathcal{M}$,
- do a conjugation, a 1/2 classical scaling
- recognize a supersymetric structure

Its formal adjoint is

$$L^* = \nabla.D\nabla - \nabla.b(x)$$

and is the infinitesimal generator of $(T^t)^*)$ (at least its closure in an L^2 setting).
This means that if $\mu_0 = f_0 dx$ is the a.c. measure of probability of x_0, then $\mu_t = f(t,.)dx$ is the one of $x(t)$ and $\partial_t f - L^* f = 0$, ie

$$\begin{cases} \partial_t f + (-\nabla.D.\nabla + \nabla.b)f = 0 \\ f|_{t=0} = f_0 \end{cases}$$

An invariant measure will be associated to a time-independant function $\mathcal{M}$. What remains in particular cases is

- exhibit the Maxwellian $\mathcal{M}$,
- do a conjugation, a 1/2 classical scaling
- recognize a supersymetric structure

Its formal adjoint is

$$L^* = \nabla.D\nabla - \nabla.b(x)$$

and is the infinitesimal generator of $(T^t)^*)$ (at least its closure in an L^2 setting).
This means that if $\mu_0 = f_0 dx$ is the a.c. measure of probability of x_0, then $\mu_t = f(t,.)dx$ is the one of $x(t)$ and $\partial_t f - L^* f = 0$, ie

$$\begin{cases} \partial_t f + (-\nabla.D.\nabla + \nabla.b)f = 0 \\ f|_{t=0} = f_0 \end{cases}$$

An invariant measure will be associated to a time-independant function $\mathcal{M}$. What remains in particular cases is

- exhibit the Maxwellian $\mathcal{M}$,
- do a conjugation, a 1/2 classical scaling
- recognize a supersymetric structure

Its formal adjoint is

$$L^* = \nabla . D \nabla - \nabla . b(x)$$

and is the infinitesimal generator of $(T^t)^*$) (at least its closure in an L^2 setting).
This means that if $\mu_0 = f_0 dx$ is the a.c. measure of probability of x_0, then $\mu_t = f(t, .)dx$ is the one of $x(t)$ and $\partial_t f - L^* f = 0$, ie

$$\begin{cases} \partial_t f + (-\nabla . D . \nabla + \nabla . b) f = 0 \\ f|_{t=0} = f_0 \end{cases}$$

An invariant measure will be associated to a time-independant function $\mathcal{M}$. What remains in particular cases is

- exhibit the Maxwellian $\mathcal{M}$,
- do a conjugation, a 1/2 classical scaling
- recognize a supersymetric structure

Witten case

$$W : dx = -\gamma \partial_x V dt + \sqrt{2\gamma T} dw$$

- Parameters : $D = \sigma^* \sigma / 2 = \gamma T$ and $b(x) = -\gamma \partial_x V$.
- Density : $\partial_t f - \gamma \partial_x . (T \partial_x + \partial_x V) f = 0$.
- Let $t = h$ and $\times h$: $h \partial_t f - \gamma h \partial_x . (h \partial_x + \partial_x V) f = 0$.
- Maxwellian : $\mathcal{M} = e^{-V/h}$.
- Conjugation $f = \mathcal{M}^{1/2} h$:
  $h \partial_t u + \gamma(-h \partial_x + \partial_x V/2).(h \partial_x + \partial_x V/2) u = 0$.
- Supersymetry : $A = \gamma I$, $\phi(x) = V(x)/2$.
- Potential : V Morse and
  $\partial^\alpha V(x) = \mathcal{O}(1)$ when $|\alpha| = 2$ and $\mathcal{O}(< x >^{-1})$ when $|\alpha| \geq 3$.

Witten case

W : $dx = -\gamma\partial_x V dt + \sqrt{2\gamma T} dw$

- Parameters : $D = \sigma^*\sigma/2 = \gamma T$ and $b(x) = -\gamma\partial_x V$.
- Density : $\partial_t f - \gamma\partial_x.(T\partial_x + \partial_x V)f = 0$.
- Let $t = h$ and $\times h$: $h\partial_t f - \gamma h\partial_x.(h\partial_x + \partial_x V)f = 0$.
- Maxwellian : $\mathcal{M} = e^{-V/h}$.
- Conjugation $f = \mathcal{M}^{1/2}h$:
  $h\partial_t u + \gamma(-h\partial_x + \partial_x V/2).(h\partial_x + \partial_x V/2)u = 0$.
- Supersymetry : $A = \gamma I$, $\phi(x) = V(x)/2$.
- Potential : V Morse and
  $\partial^\alpha V(x) = \mathcal{O}(1)$ when $|\alpha| = 2$ and $\mathcal{O}(< x >^{-1})$ when $|\alpha| \geq 3$.

Witten case

W : $dx = -\gamma \partial_x V dt + \sqrt{2\gamma T} dw$

- Parameters : $D = \sigma^* \sigma / 2 = \gamma T$ and $b(x) = -\gamma \partial_x V$.
- Density : $\partial_t f - \gamma \partial_x . (T \partial_x + \partial_x V) f = 0$.
- Let $t = h$ and $\times h$: $h \partial_t f - \gamma h \partial_x . (h \partial_x + \partial_x V) f = 0$.
- Maxwellian : $\mathcal{M} = e^{-V/h}$.
- Conjugation $f = \mathcal{M}^{1/2} h$:
  $h \partial_t u + \gamma(-h \partial_x + \partial_x V/2).(h \partial_x + \partial_x V/2) u = 0$.
- Supersymetry : $A = \gamma I$, $\phi(x) = V(x)/2$.
- Potential : V Morse and
  $\partial^\alpha V(x) = \mathcal{O}(1)$ when $|\alpha| = 2$ and $\mathcal{O}(< x >^{-1})$ when $|\alpha| \geq 3$.

Witten case

W : $dx = -\gamma\partial_x V dt + \sqrt{2\gamma T} dw$

- Parameters : $D = \sigma^*\sigma/2 = \gamma T$ and $b(x) = -\gamma\partial_x V$.
- Density : $\partial_t f - \gamma\partial_x.(T\partial_x + \partial_x V)f = 0$.
- Let $t = h$ and $\times h$: $h\partial_t f - \gamma h\partial_x.(h\partial_x + \partial_x V)f = 0$.
- Maxwellian : $\mathcal{M} = e^{-V/h}$.
- Conjugation $f = \mathcal{M}^{1/2} h$:
  $h\partial_t u + \gamma(-h\partial_x + \partial_x V/2).(h\partial_x + \partial_x V/2)u = 0$.
- Supersymetry : $A = \gamma I$, $\phi(x) = V(x)/2$.
- Potential : V Morse and
  $\partial^\alpha V(x) = \mathcal{O}(1)$ when $|\alpha| = 2$ and $\mathcal{O}(< x >^{-1})$ when $|\alpha| \geq 3$.

Witten case

W : $dx = -\gamma \partial_x V dt + \sqrt{2\gamma T} dw$

- Parameters : $D = \sigma^* \sigma / 2 = \gamma T$ and $b(x) = -\gamma \partial_x V$.
- Density : $\partial_t f - \gamma \partial_x . (T \partial_x + \partial_x V) f = 0$.
- Let $t = h$ and $\times h$: $h \partial_t f - \gamma h \partial_x . (h \partial_x + \partial_x V) f = 0$.
- Maxwellian : $\mathcal{M} = e^{-V/h}$.
- Conjugation $f = \mathcal{M}^{1/2} h$:
  $h \partial_t u + \gamma(-h\partial_x + \partial_x V/2).(h\partial_x + \partial_x V/2)u = 0$.
- Supersymetry : $A = \gamma I$, $\phi(x) = V(x)/2$.
- Potential : V Morse and
  $\partial^\alpha V(x) = \mathcal{O}(1)$ when $|\alpha| = 2$ and $\mathcal{O}(< x >^{-1})$ when $|\alpha| \geq 3$.

Witten case

W : $dx = -\gamma \partial_x V dt + \sqrt{2\gamma T} dw$

- Parameters : $D = \sigma^* \sigma/2 = \gamma T$ and $b(x) = -\gamma \partial_x V$.
- Density : $\partial_t f - \gamma \partial_x.(T\partial_x + \partial_x V)f = 0$.
- Let $t = h$ and $\times h$: $h\partial_t f - \gamma h \partial_x.(h\partial_x + \partial_x V)f = 0$.
- Maxwellian : $\mathcal{M} = e^{-V/h}$.
- Conjugation $f = \mathcal{M}^{1/2} h$:
  $h\partial_t u + \gamma(-h\partial_x + \partial_x V/2).(h\partial_x + \partial_x V/2)u = 0$.
- Supersymetry : $A = \gamma I$, $\phi(x) = V(x)/2$.
- Potential : V Morse and
  $\partial^\alpha V(x) = \mathcal{O}(1)$ when $|\alpha| = 2$ and $\mathcal{O}(< x >^{-1})$ when $|\alpha| \geq 3$.

Witten case

W : $dx = -\gamma \partial_x V dt + \sqrt{2\gamma T} dw$

- Parameters : $D = \sigma^* \sigma / 2 = \gamma T$ and $b(x) = -\gamma \partial_x V$.
- Density : $\partial_t f - \gamma \partial_x . (T \partial_x + \partial_x V) f = 0$.
- Let $t = h$ and $\times h$: $h \partial_t f - \gamma h \partial_x . (h \partial_x + \partial_x V) f = 0$.
- Maxwellian : $\mathcal{M} = e^{-V/h}$.
- Conjugation $f = \mathcal{M}^{1/2} h$:
  $h \partial_t u + \gamma(-h\partial_x + \partial_x V/2).(h\partial_x + \partial_x V/2) u = 0$.
- Supersymetry : $A = \gamma I$, $\phi(x) = V(x)/2$.
- Potential : V Morse and
  $\partial^\alpha V(x) = \mathcal{O}(1) \text{ when } |\alpha| = 2 \text{ and } \mathcal{O}(< x >^{-1}) \text{ when } |\alpha| \geq 3$.

Fokker-Planck case

$$\text{KFP} : \begin{cases} dx = ydt \\ dy = -\gamma y dt - \partial_x V dt + \sqrt{2\gamma T} dw \end{cases} .$$

- Parameters : $D = \begin{bmatrix} 0 & 0 \\ 0 & \gamma T \end{bmatrix}$ and $b(x) = \begin{bmatrix} y \\ -\gamma y - \partial_x V \end{bmatrix}$.
- Density, scaling : $h\partial_t f - \gamma h\partial_y.(h\partial_y + y)f + yh\partial_x f - \partial_x V h\partial_y f = 0$.
- Maxwellian : $\mathcal{M} = C^{-1} e^{(-V(x)+y^2/2)/h}$.
- Conjugation : $h\partial_t u + \gamma(-h\partial_y + y/2).(h\partial_y + y/2)u + \gamma yh\partial_x u - \partial_x V h\partial_y u = 0$.
- Supersymetry : $A = \begin{bmatrix} 0 & -I \\ I & \gamma \end{bmatrix}$ and $\phi(x, v) = V(x)/2 + y^2/4$.
- potential : V Morse and $\partial^\alpha V(x) = \mathcal{O}(1)$.

Fokker-Planck case

$$\text{KFP}: \begin{cases} dx = ydt \\ dy = -\gamma ydt - \partial_x Vdt + \sqrt{2\gamma T}dw \end{cases} .$$

- Parameters : $D = \begin{bmatrix} 0 & 0 \\ 0 & \gamma T \end{bmatrix}$ and $b(x) = \begin{bmatrix} y \\ -\gamma y - \partial_x V \end{bmatrix}$.
- Density, scaling : $h\partial_t f - \gamma h\partial_y.(h\partial_y + y)f + yh\partial_x f - \partial_x Vh\partial_y f = 0$.
- Maxwellian : $\mathcal{M} = C^{-1}e^{(-V(x)+y^2/2)/h}$.
- Conjugation : $h\partial_t u + \gamma(-h\partial_y + y/2).(h\partial_y + y/2)u + \gamma yh\partial_x u - \partial_x Vh\partial_y u = 0$.
- Supersymetry : $A = \begin{bmatrix} 0 & -I \\ I & \gamma \end{bmatrix}$ and $\phi(x, v) = V(x)/2 + y^2/4$.
- potential : V Morse and $\partial^\alpha V(x) = \mathcal{O}(1)$.

Fokker-Planck case

$$\text{KFP}: \begin{cases} dx = ydt \\ dy = -\gamma ydt - \partial_x V dt + \sqrt{2\gamma T}\, dw \end{cases}.$$

- Parameters : $D = \begin{bmatrix} 0 & 0 \\ 0 & \gamma T \end{bmatrix}$ and $b(x) = \begin{bmatrix} y \\ -\gamma y - \partial_x V \end{bmatrix}$.
- Density, scaling :
  $h\partial_t f - \gamma h\partial_y.(h\partial_y + y)f + yh\partial_x f - \partial_x V h\partial_y f = 0.$
- Maxwellian : $\mathcal{M} = C^{-1} e^{(-V(x)+y^2/2)/h}$.
- Conjugation :
  $h\partial_t u + \gamma(-h\partial_y + y/2).(h\partial_y + y/2)u + \gamma yh\partial_x u - \partial_x V h\partial_y u = 0.$
- Supersymetry : $A = \begin{bmatrix} 0 & -I \\ I & \gamma \end{bmatrix}$ and $\phi(x, v) = V(x)/2 + y^2/4$.
- potential : V Morse and $\partial^\alpha V(x) = \mathcal{O}(1)$.

Fokker-Planck case

$$\text{KFP}: \begin{cases} dx = ydt \\ dy = -\gamma ydt - \partial_x Vdt + \sqrt{2\gamma T}dw \end{cases} .$$

- Parameters : $D = \begin{bmatrix} 0 & 0 \\ 0 & \gamma T \end{bmatrix}$ and $b(x) = \begin{bmatrix} y \\ -\gamma y - \partial_x V \end{bmatrix}$.
- Density, scaling :
  $h\partial_t f - \gamma h\partial_y.(h\partial_y + y)f + yh\partial_x f - \partial_x Vh\partial_y f = 0.$
- Maxwellian : $\mathcal{M} = C^{-1}e^{(-V(x)+y^2/2)/h}$.
- Conjugation :
  $h\partial_t u+\gamma(-h\partial_y+y/2).(h\partial_y+y/2)u+\gamma yh\partial_x u-\partial_x Vh\partial_y u = 0.$
- Supersymetry : $A = \begin{bmatrix} 0 & -I \\ I & \gamma \end{bmatrix}$ and $\phi(x, v) = V(x)/2 + y^2/4$.
- potential : V Morse and $\partial^\alpha V(x) = \mathcal{O}(1)$.

Fokker-Planck case

$$\text{KFP}: \begin{cases} dx = ydt \\ dy = -\gamma ydt - \partial_x Vdt + \sqrt{2\gamma T}dw \end{cases}.$$

- Parameters : $D = \begin{bmatrix} 0 & 0 \\ 0 & \gamma T \end{bmatrix}$ and $b(x) = \begin{bmatrix} y \\ -\gamma y - \partial_x V \end{bmatrix}$.
- Density, scaling :
  $h\partial_t f - \gamma h\partial_y.(h\partial_y + y)f + yh\partial_x f - \partial_x Vh\partial_y f = 0.$
- Maxwellian : $\mathcal{M} = C^{-1}e^{(-V(x)+y^2/2)/h}$.
- Conjugation :
  $h\partial_t u+\gamma(-h\partial_y+y/2).(h\partial_y+y/2)u+\gamma yh\partial_x u-\partial_x Vh\partial_y u = 0.$
- Supersymetry : $A = \begin{bmatrix} 0 & -I \\ I & \gamma \end{bmatrix}$ and $\phi(x, v) = V(x)/2 + y^2/4$.
- potential : V Morse and $\partial^\alpha V(x) = \mathcal{O}(1)$.

Fokker-Planck case

$$\text{KFP}: \begin{cases} dx = ydt \\ dy = -\gamma ydt - \partial_x Vdt + \sqrt{2\gamma T}dw \end{cases}.$$

- Parameters : $D = \begin{bmatrix} 0 & 0 \\ 0 & \gamma T \end{bmatrix}$ and $b(x) = \begin{bmatrix} y \\ -\gamma y - \partial_x V \end{bmatrix}$.
- Density, scaling :
  $h\partial_t f - \gamma h\partial_y.(h\partial_y + y)f + yh\partial_x f - \partial_x Vh\partial_y f = 0.$
- Maxwellian : $\mathcal{M} = C^{-1}e^{(-V(x)+y^2/2)/h}$.
- Conjugation :
  $h\partial_t u + \gamma(-h\partial_y + y/2).(h\partial_y + y/2)u + \gamma yh\partial_x u - \partial_x Vh\partial_y u = 0.$
- Supersymetry : $A = \begin{bmatrix} 0 & -I \\ I & \gamma \end{bmatrix}$ and $\phi(x, v) = V(x)/2 + y^2/4$.
- potential : V Morse and $\partial^\alpha V(x) = \mathcal{O}(1)$.

Fokker-Planck case

$$\text{KFP}: \begin{cases} dx = ydt \\ dy = -\gamma ydt - \partial_x Vdt + \sqrt{2\gamma T}dw \end{cases}.$$

- Parameters : $D = \begin{bmatrix} 0 & 0 \\ 0 & \gamma T \end{bmatrix}$ and $b(x) = \begin{bmatrix} y \\ -\gamma y - \partial_x V \end{bmatrix}$.
- Density, scaling :
  $h\partial_t f - \gamma h\partial_y.(h\partial_y + y)f + yh\partial_x f - \partial_x Vh\partial_y f = 0.$
- Maxwellian : $\mathcal{M} = C^{-1}e^{(-V(x)+y^2/2)/h}$.
- Conjugation :
  $h\partial_t u+\gamma(-h\partial_y+y/2).(h\partial_y+y/2)u+\gamma yh\partial_x u-\partial_x Vh\partial_y u = 0.$
- Supersymetry : $A = \begin{bmatrix} 0 & -I \\ I & \gamma \end{bmatrix}$ and $\phi(x, v) = V(x)/2 + y^2/4$.
- potential : V Morse and $\partial^\alpha V(x) = \mathcal{O}(1)$.

Chains case

$$\mathrm{Ch}: \begin{cases} dx_1 = y_1 dt \\ dy_1 = -\partial_{x_1} V dt + z_1 dt \\ dz_1 = -\gamma z_1 dt + \gamma x_1 dt - \sqrt{2\gamma T_1} dw_1 \\ dz_2 = -\gamma z_1 dt + \gamma x_2 dt - \sqrt{2\gamma T_2} dw_2 \\ dy_2 = -\partial_{x_2} V dt + z_2 dt \\ dx_2 = y_2 dt. \end{cases}$$

where $V(x_1, x_2) = V_p(x_1) + V_p(x_2) + V_c(x_1 - x_2)$.

- Parameters : $D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \gamma T \end{bmatrix}$ and $b(x) = \begin{bmatrix} y \\ -\partial_x V + z \\ \gamma(x - z) \end{bmatrix}$.
- Scaling $T_1 = \alpha_1 h$, $T_2 = \alpha_2 h$, and $\times h$:

$$\begin{aligned} h\partial_t f &+ \gamma\alpha_1(-h\partial_{z_1}).(h\partial_{z_1} + (z_1 - x_1)/\alpha_1) f \\ &+ \gamma\alpha_2(-h\partial_{z_2}).(h\partial_{z_2} + (z_2 - x_2)/\alpha_2) f \\ &+ (y\partial_x f - (\partial_x V - z)\partial_y) f = 0. \end{aligned}$$

Chains case

$$\mathrm{Ch}: \begin{cases} dx_1 = y_1 dt \\ dy_1 = -\partial_{x_1} V dt + z_1 dt \\ dz_1 = -\gamma z_1 dt + \gamma x_1 dt - \sqrt{2\gamma T_1} dw_1 \\ dz_2 = -\gamma z_1 dt + \gamma x_2 dt - \sqrt{2\gamma T_2} dw_2 \\ dy_2 = -\partial_{x_2} V dt + z_2 dt \\ dx_2 = y_2 dt. \end{cases}$$

where $V(x_1, x_2) = V_p(x_1) + V_p(x_2) + V_c(x_1 - x_2)$.

- Parameters : $D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \gamma T \end{bmatrix}$ and $b(x) = \begin{bmatrix} y \\ -\partial_x V + z \\ \gamma(x - z) \end{bmatrix}$.
- Scaling $T_1 = \alpha_1 h$, $T_2 = \alpha_2 h$, and $\times h$:

$$\begin{aligned} h\partial_t f &+ \gamma\alpha_1(-h\partial_{z_1}).(h\partial_{z_1} + (z_1 - x_1)/\alpha_1)f \\ &+ \gamma\alpha_2(-h\partial_{z_2}).(h\partial_{z_2} + (z_2 - x_2)/\alpha_2)f \\ &+ (y\partial_x f - (\partial_x V - z)\partial_y)f = 0. \end{aligned}$$

Chains case

$$
\mathrm{Ch}: \begin{cases} dx_1 = y_1 dt \\ dy_1 = -\partial_{x_1} V dt + z_1 dt \\ dz_1 = -\gamma z_1 dt + \gamma x_1 dt - \sqrt{2\gamma T_1} dw_1 \\ dz_2 = -\gamma z_1 dt + \gamma x_2 dt - \sqrt{2\gamma T_2} dw_2 \\ dy_2 = -\partial_{x_2} V dt + z_2 dt \\ dx_2 = y_2 dt. \end{cases}
$$

where $V(x_1, x_2) = V_p(x_1) + V_p(x_2) + V_c(x_1 - x_2)$.

- Parameters : $D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \gamma T \end{bmatrix}$ and $b(x) = \begin{bmatrix} y \\ -\partial_x V + z \\ \gamma(x - z) \end{bmatrix}$.
- Scaling $T_1 = \alpha_1 h$, $T_2 = \alpha_2 h$, and $\times h$:

$$
\begin{aligned} h\partial_t f &+ \gamma\alpha_1(-h\partial_{z_1}).(h\partial_{z_1} + (z_1 - x_1)/\alpha_1)f \\ &+ \gamma\alpha_2(-h\partial_{z_2}).(h\partial_{z_2} + (z_2 - x_2)/\alpha_2)f \\ &+ (y\partial_x f - (\partial_x V - z)\partial_y)f = 0. \end{aligned}
$$

Chains case

$$\mathrm{Ch}: \begin{cases} dx_1 = y_1 dt \\ dy_1 = -\partial_{x_1} V dt + z_1 dt \\ dz_1 = -\gamma z_1 dt + \gamma x_1 dt - \sqrt{2\gamma T_1} dw_1 \\ dz_2 = -\gamma z_1 dt + \gamma x_2 dt - \sqrt{2\gamma T_2} dw_2 \\ dy_2 = -\partial_{x_2} V dt + z_2 dt \\ dx_2 = y_2 dt. \end{cases}$$

where $V(x_1, x_2) = V_p(x_1) + V_p(x_2) + V_c(x_1 - x_2)$.

- Parameters : $D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \gamma T \end{bmatrix}$ and $b(x) = \begin{bmatrix} y \\ -\partial_x V + z \\ \gamma(x - z) \end{bmatrix}$.
- Scaling $T_1 = \alpha_1 h$, $T_2 = \alpha_2 h$, and $\times h$:

$$\begin{aligned} h\partial_t f + \gamma\alpha_1(-h\partial_{z_1}).(h\partial_{z_1} + (z_1 - x_1)/\alpha_1)f \\ + \gamma\alpha_2(-h\partial_{z_2}).(h\partial_{z_2} + (z_2 - x_2)/\alpha_2)f \\ + (y\partial_x f - (\partial_x V - z)\partial_y)f = 0. \end{aligned}$$

Difficult to exhibit a Maxwellian (although existence OK under additional conditions [EPR99]) Restriction to the case of same temperature for the bathes.

- Maxwellian : $\mathcal{M}_\alpha = C^{-1} e^{-(V(x)+y^2/2+z^2/2-zx)/\alpha h}$.
- Conjugation :

$$
\begin{aligned}
& h\partial_t u \\
& + \gamma\alpha_1 \left(-h\partial_{z_1} + \frac{1}{2\alpha}(z_1 - x_1) \right) . \left(h\partial_{z_1} + \left(\frac{1}{\alpha_1} - \frac{1}{2\alpha} \right) (z_1 - x_1) \right) \\
& + \gamma\alpha_2 \left(-h\partial_{z_2} + \frac{1}{2\alpha}(z_2 - x_2) \right) . \left(h\partial_{z_2} + \left(\frac{1}{\alpha_2} - \frac{1}{2\alpha} \right) (z_2 - x_2) \right) \\
& + (yh\partial_x f - (\partial_x V - z)h\partial_y)u = 0.
\end{aligned}
$$

Difficult to exhibit a Maxwellian (although existence OK under additional conditions [EPR99]) Restriction to the case of same temperature for the bathes.

- Maxwellian : $\mathcal{M}_\alpha = C^{-1} e^{-(V(x)+y^2/2+z^2/2-zx)/\alpha h}$.
- Conjugation :

$$
\begin{aligned}
& h\partial_t u \\
& + \gamma\alpha_1 \left(-h\partial_{z_1} + \frac{1}{2\alpha}(z_1 - x_1) \right) . \left(h\partial_{z_1} + \left(\frac{1}{\alpha_1} - \frac{1}{2\alpha} \right) (z_1 - x_1) \right) \\
& + \gamma\alpha_2 \left(-h\partial_{z_2} + \frac{1}{2\alpha}(z_2 - x_2) \right) . \left(h\partial_{z_2} + \left(\frac{1}{\alpha_2} - \frac{1}{2\alpha} \right) (z_2 - x_2) \right) \\
& + (yh\partial_x f - (\partial_x V - z)h\partial_y)u = 0.
\end{aligned}
$$

Difficult to exhibit a Maxwellian (although existence OK under additional conditions [EPR99]) Restriction to the case of same temperature for the bathes.

- Maxwellian : $\mathcal{M}_\alpha = C^{-1} e^{-(V(x)+y^2/2+z^2/2-zx)/\alpha h}$.
- Conjugation :

$$
\begin{aligned}
& h\partial_t u \\
& + \gamma\alpha_1 \left(-h\partial_{z_1} + \frac{1}{2\alpha}(z_1 - x_1)\right) \cdot \left(h\partial_{z_1} + \left(\frac{1}{\alpha_1} - \frac{1}{2\alpha}\right)(z_1 - x_1)\right) \\
& + \gamma\alpha_2 \left(-h\partial_{z_2} + \frac{1}{2\alpha}(z_2 - x_2)\right) \cdot \left(h\partial_{z_2} + \left(\frac{1}{\alpha_2} - \frac{1}{2\alpha}\right)(z_2 - x_2)\right) \\
& + (yh\partial_x f - (\partial_x V - z)h\partial_y)u = 0.
\end{aligned}
$$

Difficult to exhibit a Maxwellian (although existence OK under additional conditions [EPR99]) Restriction to the case of same temperature for the bathes.

- Maxwellian : $\mathcal{M}_\alpha = C^{-1} e^{-(V(x)+y^2/2+z^2/2-zx)/\alpha h}$.
- Conjugation :

$$
\begin{aligned}
& h\partial_t u \\
& + \gamma\alpha_1 \left(-h\partial_{z_1} + \frac{1}{2\alpha}(z_1 - x_1)\right) . \left(h\partial_{z_1} + \left(\frac{1}{\alpha_1} - \frac{1}{2\alpha}\right)(z_1 - x_1)\right) \\
& + \gamma\alpha_2 \left(-h\partial_{z_2} + \frac{1}{2\alpha}(z_2 - x_2)\right) . \left(h\partial_{z_2} + \left(\frac{1}{\alpha_2} - \frac{1}{2\alpha}\right)(z_2 - x_2)\right) \\
& + (yh\partial_x f - (\partial_x V - z)h\partial_y)u = 0.
\end{aligned}
$$

- Supersymetry : $A = \alpha \begin{bmatrix} 0 & Id & 0 \\ -Id & 0 & 0 \\ 0 & 0 & \gamma Id \end{bmatrix}$ and $\phi_\alpha = \left(V(x) + y^2/2 + z^2/2 - zx \right) /2/\alpha$.
- Potential : V Morse, $\partial^\alpha V(x) = \mathcal{O}(1)$ and for example, V_p of double well type and V_c of simple well type.

- Supersymetry : $A = \alpha \begin{bmatrix} 0 & Id & 0 \\ -Id & 0 & 0 \\ 0 & 0 & \gamma Id \end{bmatrix}$ and $\phi_\alpha = \left(V(x) + y^2/2 + z^2/2 - zx\right)/2/\alpha$.
- Potential : V Morse, $\partial^\alpha V(x) = \mathcal{O}(1)$ and for example, V_p of double well type and V_c of simple well type.