Manipulating Entanglement

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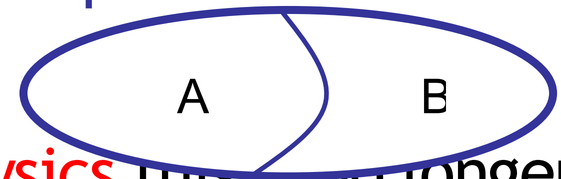
In a classical system: complete information of a system implies a complete description of its individual parts and vice versa.



In quantum physics this is no longer true: If AB is a quantum system, then:

AB in a pure state $|\Psi_{AB}\rangle \not\Rightarrow$ A and B are individually in pure states

A and B can be correlated in a way which has no classical analogue:

A and B are entangled.

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Separable and Entangled States

A pure state $|\Psi_{AB}\rangle$ of a bipartite system AB is separable if it is expressible in the tensor product form:

$$|\Psi_{AB}\rangle = |\phi_A\rangle \otimes |\psi_B\rangle$$

Else it is entangled!

Moreover, $|\Psi_{AB}\rangle$ is a maximally entangled state (MES) if its reduced density matrices are given by completely mixed states

$$\rho_A = \frac{I}{d} = \rho_B$$

: e.g. a Bell state

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\left[|00\rangle + |11\rangle\right]$$

A bipartite mixed state ρ_{AB} is separable if it is of the form

$$\rho_{AB} = \sum_i p_i (\sigma_i^A \otimes \omega_i^B)$$

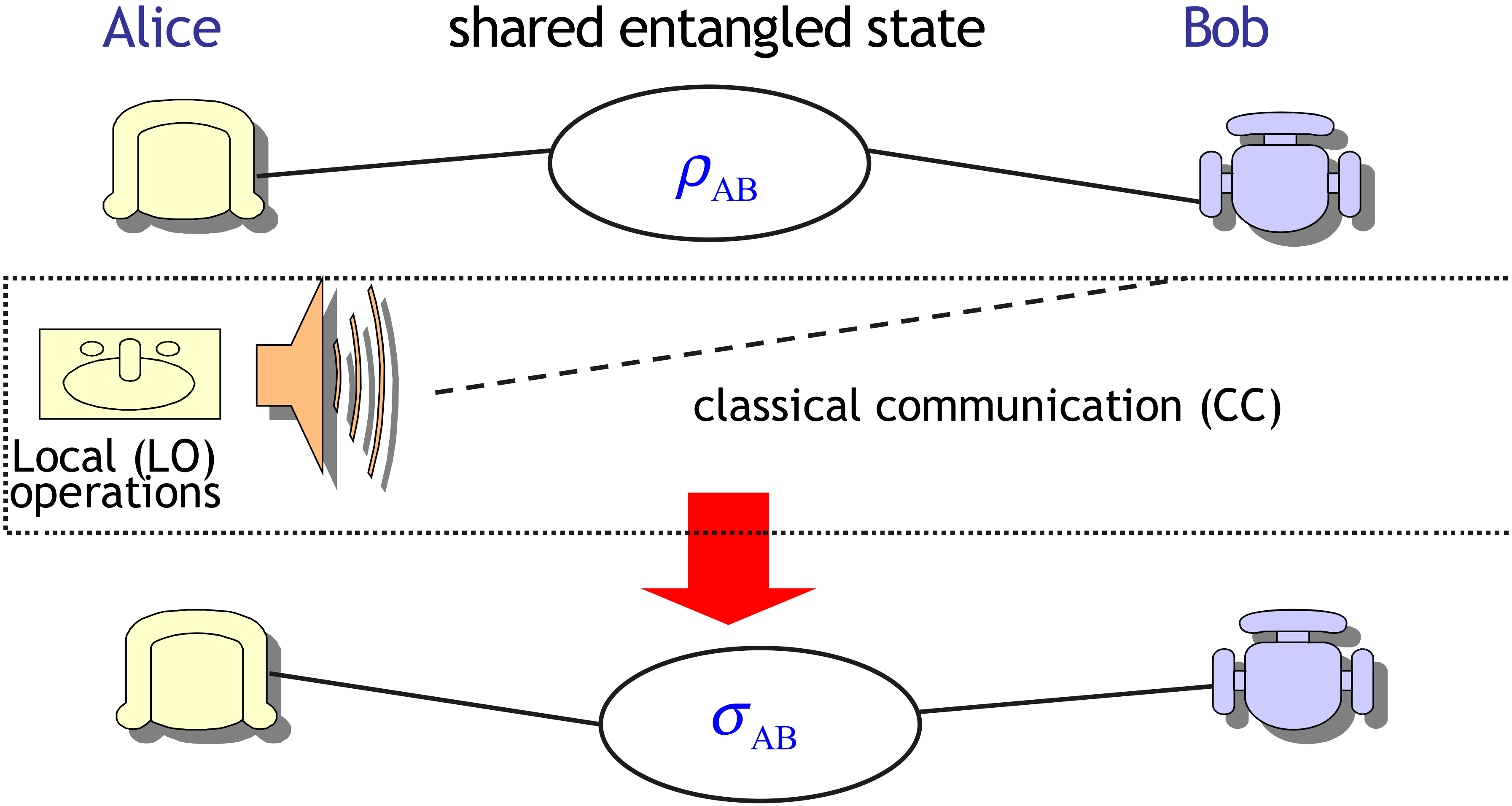
Else it is entangled.

Entanglement plays a crucial role in Quantum Information Theory.

It is a novel resource which can be used to perform tasks which are impossible in the classical realm, e.g., teleportation, superdense coding, quantum cryptography etc.

a fundamental property of entanglement: it cannot be created by local operations and classical communications (LOCC) alone.

However, one can transform one entangled state to another by LOCC alone: this is called as ***entanglement manipulation***



If $E(\rho_{AB})$ denotes entanglement of state ρ_{AB} then :

$$\rho_{AB} \xrightarrow[\text{LOCC}]{} \sigma_{AB} \Rightarrow E(\sigma_{AB}) \leq E(\rho_{AB})$$

An essential property of any quantity that is used to characterise entanglement is that it cannot be increased by LOCC alone

For a bipartite pure state $|\Psi_{AB}\rangle$, one such quantity is its Schmidt number:

$|\Psi_{AB}\rangle$ is entangled if and only if its Schmidt number > 1.

There is no such simple quantity characterising the entanglement of arbitrary bipartite states ρ_{AB}.

However, one can establish asymptotic measures of entanglement for any arbitrary bipartite state ρ_{AB} by considering suitable entanglement manipulations of it.

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Why do we need entanglement manipulations?

to convert the entanglement of a state to a standard form or "currency".

This also allows us to compare the entanglements of two different entangled states.

To obtain "standard form" or "currency" for entanglement: define the entanglement of maximally entangled state (MES) of rank M

$$|\Psi^+_{\mathrm{M}}\rangle = \frac{1}{\sqrt{M}}\sum_{k=1}^{M} |e_k^{\mathrm{A}}\rangle |e_k^{\mathrm{B}}\rangle \text{ to be } \qquad E\left(|\Psi^+_M\rangle\right) = \log M \ldots\ldots\ldots (1)$$

This yields a benchmark against which to measure the entanglement of other states.

[Note: take logarithm in (1) is taken to base 2]

Asymptotic measures of the entanglement of any arbitrary bipartite state ρ are then obtained by considering : entanglement manipulations which convert

multiple copies of ρ $\xrightarrow{LOCC}$ multiple Bell pairs (or vice versa)

$\rho^{\otimes n}$ m_n Bell pairs (Entanglement Concentration)

or equivalently, $\rho^{\otimes n}$ a MES of a rank $M_n = 2^{m_n}$;$(n \in N)$

m'_n Bell Pairs $\rho^{\otimes n}$ (Entanglement Dilution)

Denoting a the density matrix of a Bell pair by ω the above transformations can be denoted as follows:

$$\rho^{\otimes n} \xrightarrow[LOCC]{} \omega^{\otimes m_n} \text{......(i)} \qquad \omega^{\otimes m'_n} \xrightarrow[LOCC]{} \rho^{\otimes n} \text{.......(ii)}$$

$$m_n, m'_n, n \in ¥$$

with

Since entanglement cannot be increased by LOCC we have

$$\text{(i)} \Rightarrow E(\omega^{\otimes m_n}) \le E(\rho^{\otimes n}); \text{(ii)} \Rightarrow E(\rho^{\otimes n}) \le E(\omega^{\otimes m'_n})$$

$$\boxed{m_n \le E(\rho^{\otimes n}) \le m'_n \quad Q \quad E(\omega_{m_n}) = m_n}$$

Hence

n

Note: transformations (i) and (ii) cannot be achieved perfectly for finite n. Hence one allows imperfections and

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If $\tau_n : \rho_n \xrightarrow[LOCC]{} \sigma_n$

FIDELITY: $F_n = F(\tau_n(\rho_n), \sigma_n) := \mathrm{Tr}(\tau_n(\rho_n)\sigma_n)$

[final state] [target state]

and we require that $F_n \to 1$ as $n \to \infty$

The asymptotic entanglement measure of the state ρ

$$\varepsilon(\rho) := \lim_{n\to\infty} \frac{1}{n} E(\rho^{\otimes n})$$

We have:

$$\liminf_{n\to\infty} \frac{m_n}{n} \le \varepsilon(\rho) \le \limsup_{n\to\infty} \frac{m_n'}{n} \quad \text{Q} \quad m_n \le E(\rho^{\otimes n}) \le m_n'$$

Thus the entanglement manipulation protocol yields two (different) asymptotic entanglement measures for a bipartite state

(i) the entanglement cost

$$E_C(\rho) = \inf \limsup_{n\to\infty} \frac{m'_n}{n}$$

$$\begin{cases} \omega^{\otimes m'_n} \xrightarrow[LOCC]{} \rho^{\otimes n} \\ F_n \xrightarrow[n\to\infty]{} 1 \end{cases}$$

: the minimum number of Bell pairs needed to create ρ

(ii) the distillable entanglement

$$E_D(\rho) = \sup \liminf_{n\to\infty} \frac{m_n}{n}$$

$$\begin{cases} \rho^{\otimes n} \xrightarrow[LOCC]{} \omega^{\otimes m_n} \\ F_n \xrightarrow[n\to\infty]{} 1 \end{cases}$$

: the maximum number of Bell pairs that can be extracted locally from the state ρ. Hence, $E_D(\rho)$ gives the value of the entangled state ρ as a resource (for an entanglement-based protocol).

For a bipartite pure state $|\Psi_{AB}\rangle$ it is known that

$$E_D\left(|\Psi_{AB}\rangle\right) = S(\rho_A) = S(\rho_B) = E_C\left(|\Psi_{AB}\rangle\right)$$

Here ρ_A and ρ_B: reduced density matrices of the subsystems A and B resply., and $S(\rho_A)$ denotes the von Neumann entropy of ρ_A.

Hence, locally transforming

$$|\Psi_{AB}\rangle^{\otimes n} \leftrightarrow \omega^{\otimes nS(\rho_A)}$$

is an asymptotically reversible process.

Moreover $S(\rho_A)$ is the unique asymptotic entanglement measure for $|\Psi_{AB}\rangle$ since any other entanglement measure E for $|\Psi_{AB}\rangle$ satisfies:

$$E_D \leq E \leq E_C$$

[Donald et al.]

The practical ability of transforming entanglement from one form to another is useful for many applications in Quantum Information Theory.

However, it is not always justified to assume that the entanglement resource available consists of states which are multiple copies (tensor products) of a given entangled state.

In other words, the entanglement resource need not be “memoryless” or “i.i.d.”.

More generally, an entanglement resource is characterized by an arbitrary sequence of bipartite states, which are not necessarily of the tensor product form.

These sequences of bipartite states are considered to exist in Hilbert spaces $H_A^{\otimes n} \otimes H_B^{\otimes n}$ for $n = \{1, 2, 3, \ldots\}$

Our Aim: to establish asymptotic entanglement measures

for arbitrary sequences of bipartite states : $\hat{\rho} = \{\rho_n\}_{n=1}^{\infty}$

The only assumption that we make is that H_A and H_B are finite dimensional

If $\rho_n = \rho^{\otimes n}$ for some state ρ :

then one retrieves the usual memoryless scenario discussed thus far.

In order to establish $E_C(\hat{\rho})$ and $E_D(\hat{\rho})$ for such arbitrary sequences of bipartite states $\hat{\rho} = \{\rho_n\}_{n=1}^{\infty}$, we make use of the so-called Information Spectrum Approach.

This approach was developed in Classical Information Theory by Verdu and Han and was first extended into Quantum Information Theory by Hayashi, Nagaoka & Ogawa.

The Information Spectrum Approach is a powerful method for obtaining the optimal rates of various protocols.

The power of the method lies in the fact that it does not rely on any specific nature of the sources, channels or entanglement resources involved in the protocol.

Spectral Projections

The Quantum Information Spectrum (QIS) approach requires the extensive use of spectral projections.

For a self-adjoint operator A with spectral decomposition

$$A = \sum_i \lambda_i |i\rangle\langle i|$$

we define the spectral projection on A as

$$\{A \geq 0\} = \sum_{\lambda_i \geq 0} |i\rangle\langle i|$$

:the projector onto the eigenspace of non-negative eigenvalues of A

For 2 operators A and B we can then define

$$\{A \geq B\} = \{A - B \geq 0\}$$

For any given constant γ, one can associate with each sequence of bipartite states $\hat{\rho} = \{\rho_n\}_{n=1}^{\infty}$, a sequence of

orthogonal projectors $\{P_n^\gamma\}_{n=1}^{\infty}$ with $P_n^\gamma = \{\rho_n \geq 2^{-n\gamma} I_n^\gamma\}$

i.e., P_n^γ projects onto the eigenspace of ρ_n corresponding to the eigenvalues which are $\geq 2^{-n\gamma}$

If $\rho_n = \sum_i \lambda_i^n |e_i^n\rangle\langle e_i^n|$: spectral decomposition

$$P_n^\gamma = \sum_{i:\lambda_i^n \geq 2^{-n\gamma}} |e_i^n\rangle\langle e_i^n|$$

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Using these projections, for any sequence $\hat{\rho} = \{ \rho_n \}_{n=1}^{\infty}$ one can define 2 real-valued quantities :

$$\overline{S}(\hat{\rho}) := \inf\left\{ \gamma : \limsup_{n\to\infty} \operatorname{Tr}\left[P_n^{\gamma} \rho_n \right] = 1 \right\}$$: inf-spectral entropy rate

$$\underline{S}(\hat{\rho}) := \sup\left\{ \gamma : \liminf_{n\to\infty} \operatorname{Tr}\left[P_n^{\gamma} \rho_n \right] = 0 \right\}$$: sup-spectral entropy rate

RESULTS: $E_C = \overline{S}(\hat{\rho})$ and $E_D = \underline{S}(\hat{\rho})$

$$\underline{S}(\hat{\rho}) \le \liminf_{n\to\infty} \frac{1}{n} S(\rho_n) \le \limsup_{n\to\infty} \frac{1}{n} S(\rho_n) \le \overline{S}(\hat{\rho})$$

For $\hat{\rho} = \{ \rho^{\otimes n} \}_{n=1}^{\infty}$ we have $\underline{S}(\hat{\rho}) = S(\rho) = \overline{S}(\hat{\rho})$

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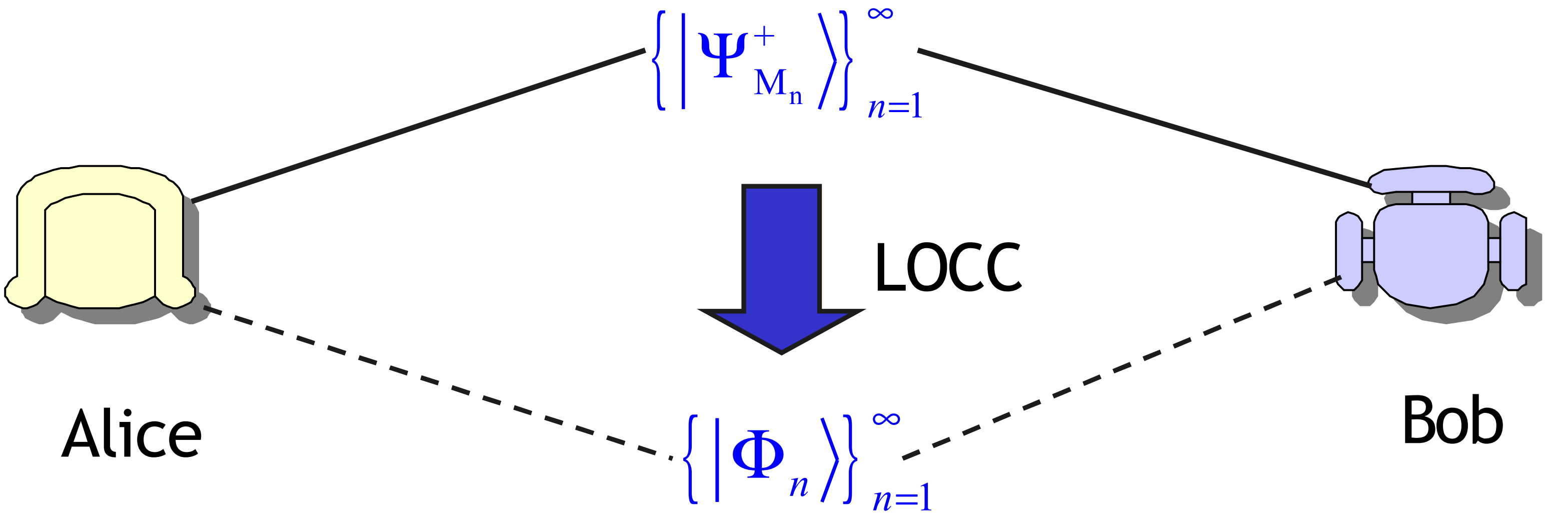
Asymptotic Entanglement Dilution of Pure States



$$\left| \Psi^+_{M_n} \right\rangle = \frac{1}{\sqrt{M_n}} \sum_{k=1}^{M_n} \left| i_A^{(n)} \right\rangle \left| i_B^{(n)} \right\rangle$$: MES of rank M_n

$\left| \Phi_n \right\rangle \in H_A^{\otimes n} \otimes H_B^{\otimes n}$: partially entangled target state

Aim:

$$\left\{ \left| \Psi^+_{M_n} \right\rangle \right\}_{n=1}^{\infty} \xrightarrow{LOCC} \left\{ \left| \Phi_n \right\rangle \right\}_{n=1}^{\infty}$$

Definitions: Achievable rate and Entanglement Cost

Achievable Rate: R is an achievable dilution rate if

$\forall \varepsilon > 0, \exists N$ such that $\forall n \geq N$ an LOCC transformation exists that takes

$$\left|\Psi^+_{M_n}\right\rangle \xrightarrow[LOCC]{} \left|\Phi_n\right\rangle$$

with fidelity $F_n \geq 1-\varepsilon$ and

$$\frac{1}{n}\log M_n \leq R$$

The entanglement cost:

$$E_C = \inf R$$

for the required class of transformations.

Theorem 1: The entanglement cost of a sequence of pure bipartite target states $\{|\Phi_n\rangle\}_{n=1}^{\infty}$ is given by

$$E_C = \overline{S}(\hat{\rho})$$

where $\hat{\rho} = \{\rho_n^A\}_{n=1}^{\infty}$ with $\rho_n^A = \mathrm{Tr}_B |\Phi_n\rangle\langle\Phi_n|$

is the sequence of subsystem states.

Here $\overline{S}(\hat{\rho}) := \inf\left\{\gamma : \limsup_{n\to\infty} \mathrm{Tr}\left[P_n^\gamma \rho_n\right] = 1\right\}$

Hence $\forall \gamma > \overline{S}(\hat{\rho}), \quad \mathrm{Tr}\left[P_n^\gamma \rho_n\right] \xrightarrow[n\to\infty]{} 1$ $\quad P_n^\gamma = \{\rho_n \geq 2^{-n\gamma} I_n^\gamma\}$

i.e., the eigenspace corrs.to eigenvalues of ρ_n^A which are $\geq 2^{-n\overline{S}(\hat{\rho})}$ is a *high probability subspace*

Proof :Let the target state $|\Phi_n\rangle$ have N_n non-zero Schmidt coefficients. Let its Schmidt decomposition be given by

$$|\Phi_n\rangle = \sum_{k=1}^{N_n} \sqrt{\lambda_{n,k}} \, |k_A^{(n)}\rangle |k_B^{(n)}\rangle$$

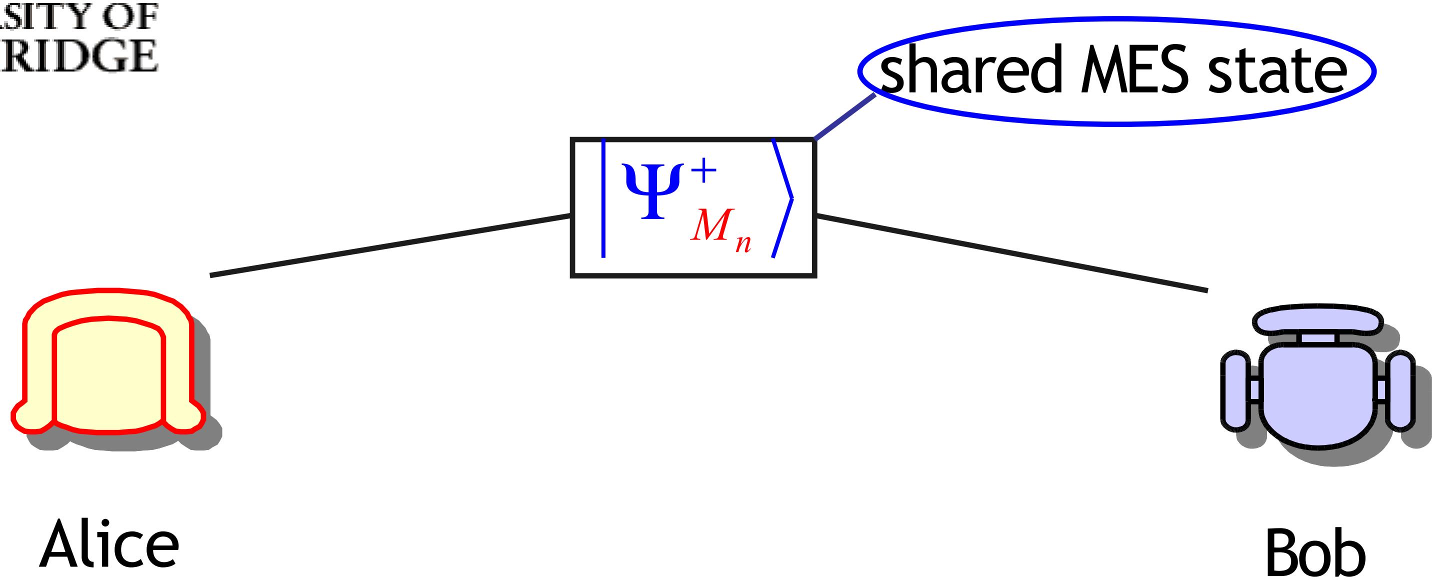
where the Schmidt coefficients $\lambda_{n,k}$ are arranged in decreasing order:

$$\lambda_{n,1} \geq \lambda_{n,2} \geq \ldots . \lambda_{n,N_n}$$

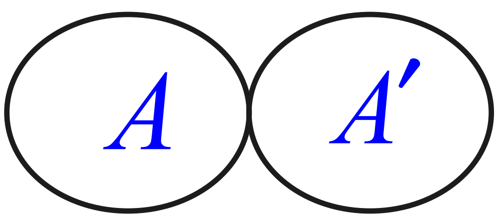
Protocol: Alice has a bipartite system AA' and locally prepares the state

$$|\Phi_n\rangle_{AA'} = \sum_{k=1}^{N_n} \sqrt{\lambda_{n,k}} \, |k_A^{(n)}\rangle |k_{A'}^{(n)}\rangle$$

Then she teleports the state of the subsystem A' to Bob, using her part of the MES $|\Psi_{M_n}^{+}\rangle$

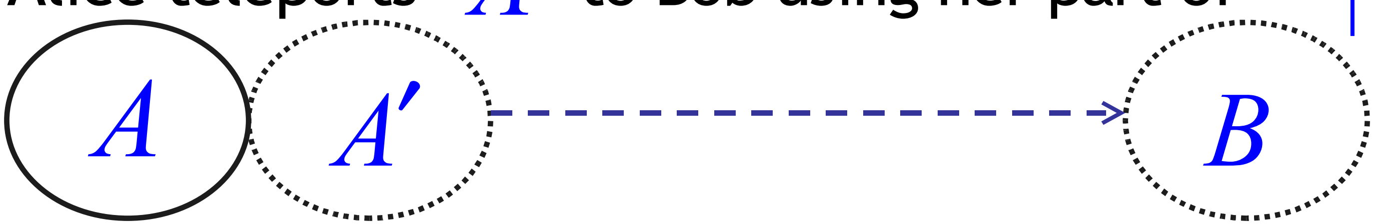


Alice locally prepares AA' in a state



$$|\Phi_n\rangle_{AA'} = \sum_{k=1}^{N_n} \sqrt{\lambda_{n,k}}\, |k_A^{(n)}\rangle |k_{A'}^{(n)}\rangle$$

Alice teleports A' to Bob using her part of $|\Psi^+_{M_n}\rangle$



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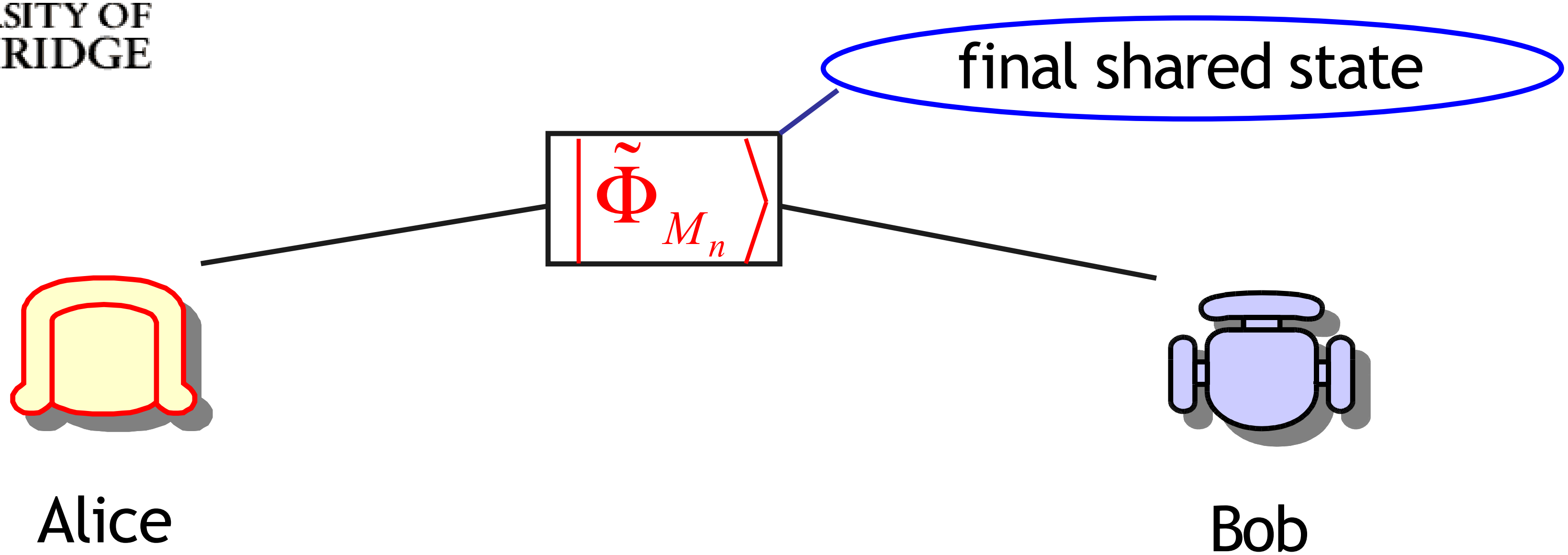

If $M_n \geq N_n$ the teleportation can be done perfectly and the final shared state is the desired target state:

$$\left|\tilde{\Phi}_{M_n}\right\rangle = \left|\Phi_n\right\rangle_{AB} = \sum_{k=1}^{N_n} \sqrt{\lambda_{n,k}} \left|k_A^{(n)}\right\rangle \left|k_B^{(n)}\right\rangle$$

The subsystem A' is now referred to as B since it is now in Bob's possession.

In this case the fidelity : $F_n = 1$

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However, if $M_n < N_n$, then Alice can perfectly teleport only the (unnormalized) truncated state

$$\left|\tilde{\Phi}_{M_n}\right\rangle = \sum_{k=1}^{M_n} \sqrt{\lambda_{n,k}} \left|k_A^{(n)}\right\rangle \left|k_B^{(n)}\right\rangle$$

Note : only the M_n largest Schmidt coefficients of the target state $|\Phi_n\rangle$ are retained in the teleported state

This is the "quantum scissors effect": if the quantum state to be teleported lives in a space of a dimension higher than the rank of the MES shared between the 2 parties, then the higher dimensional terms in the expansion of the state are "cut-off".

Hence, for $M_n < N_n$ the final shared state between Alice and Bob after the teleportation can be expressed as

$$|\tilde{\Phi}_{M_n}\rangle\langle\tilde{\Phi}_{M_n}| + \sigma_n^{AB}$$

where σ_n^{AB} is an unnormalized error state.

Fidelity for $M_n < N_n$

Using Uhlmann's Theorem we prove that

$$F_n = F\left(|\tilde{\Phi}_{M_n}\rangle\langle\tilde{\Phi}_{M_n}| + \sigma_n^{AB}, |\Phi_n\rangle\langle\Phi_n|\right)$$

(final state) (target state)

$$\geq \left|\langle\Phi_n|\tilde{\Phi}_{M_n}\rangle\right| = \mathrm{Tr}\left(Q_{M_n}^A \rho_n^A\right)$$

$Q_{M_n}^A$:= orthogonal projection onto the M_n largest eigenvalues of $\rho_n^A = \mathrm{Tr}_B |\Phi_n\rangle\langle\Phi_n|$

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Fidelity (for $M_n < N_n$)

$$\mathrm{F_n} \geq \mathrm{Tr}\left(Q^A_{M_n}\rho^A_n\right)$$

$Q^A_{M_n}$:= orthogonal projection onto the M_n largest eigenvalues of the reduced state ρ^A_n

CLAIM: By choosing M_n appropriately we can ensure:

$$F_n \xrightarrow[n\to\infty]{} 1$$

i.e., in spite of truncation of the state under teleportation, unit fidelity achieved asymptotically!!

PROOF: Define a projection operator $P_n^\gamma := \{ \rho_n^A \geq 2^{-n\gamma} I_n^A \}$

Rank of P_n^γ satisfies: $\mathrm{Tr} P_n^\gamma \leq 2^{n\gamma}$

Why?

$$\mathrm{Tr}\left[P_n^\gamma \left(\rho_n^A - 2^{-n\gamma} I_n^A \right) \right] \geq 0$$

$$\Rightarrow \mathrm{Tr} P_n^A \leq 2^{n\gamma}$$

Note that P_n^γ is the projection used in defining the

$$\overline{S}(\hat{\rho}) := \inf \left\{ \gamma : \limsup_{n\to\infty} \mathrm{Tr}\left[P_n^\gamma \rho_\mathrm{n} \right] = 1 \right\}$$ where $\rho_n = \rho_n^A$

Hence $\forall \gamma > \overline{S}(\hat{\rho})$ we have $\mathrm{Tr}\left[P_n^\gamma \rho_\mathrm{n} \right] \xrightarrow[n\to\infty]{} 1$

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We saw that $F_n \geq \mathrm{Tr}\left(Q^A_{M_n} \rho^A_n\right)$

(Q1) How can we prove that $F_n \xrightarrow[n\to\infty]{} 1$?

(A1) By proving that:

$\mathrm{Tr}\left(Q^A_{M_n} \rho^A_n\right) \geq \mathrm{Tr}(P^\gamma_n \rho^A_n)$ with $\gamma > \bar{S}(\hat{\rho})$ ....(a)

(Q2) Why?

(A2) Because $\mathrm{Tr}\left(P^\gamma_n \rho^A_n\right) \xrightarrow[n\to\infty]{} 1$

(Q3) How can we choose M_n such that (a) holds ?_

Eigenvalues of ρ_n^A *in decreasing order*

labels 1 2 . . . . M_n

$Q_{M_n}^A$ *projects onto first* M_n *eigenvalues*

values $2^{-n\gamma}$

P_n^γ *projects onto all eigenvalues* $\geq 2^{-n\gamma}$

(there are $\leq 2^{n\gamma}$ such values $\mathrm{Tr} P_n^\gamma \leq 2^{n\gamma}$)

If we choose $M_n \geq 2^{n\gamma}$ then $\mathrm{Tr}\left(Q_{M_n}^A \rho_n^A\right) \geq \mathrm{Tr}(P_n^\gamma \rho_n^A)$

&

$$F_n \geq \mathrm{Tr}\left(Q_{M_n}^A \rho_n^A\right) \geq \mathrm{Tr}\left(P_n^\gamma \rho_n^A\right) \xrightarrow[n\to\infty]{} 1 \quad \text{for} \quad \gamma > \overline{S}(\hat{\rho})$$

If the rank M_n of the initial shared MES $|\Psi^{+}_{M_n}\rangle$ is:

$$M_n = \lceil 2^{n\gamma} \rceil \text{ with } \gamma > \overline{S}(\hat{\rho}) \text{ , then } F_n \xrightarrow[n\to\infty]{} 1$$

Hence, a rate

$$R = \frac{1}{n} \log M_n > \overline{S}(\hat{\rho})$$

is achievable!

Weak converse: A rate $R < \overline{S}(\hat{\rho})$ is not achievable

Hence, entanglement cost:

$$E_C = \inf R = \overline{S}(\hat{\rho})$$

∎

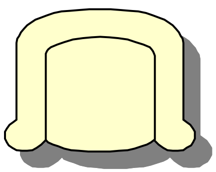
Schematic summary of protocol for entanglement dilution

Aim:

$$\left\{ \left| \Psi^{+}_{M_n} \right\rangle \right\}_{n=1}^{\infty} \xrightarrow{LOCC} \left\{ \left| \Phi_n \right\rangle \right\}_{n=1}^{\infty}$$

where

$$\left| \Phi_n \right\rangle = \sum_{k=1}^{N_n} \sqrt{\lambda_{n,k}} \left| k_A^{(n)} \right\rangle \left| k_B^{(n)} \right\rangle$$



(1) Locally prepares AA' in state $\left| \Phi_n \right\rangle$

(2) She teleports A' to Bob

(k) If $M_n \geq N_n$ then $F_n = 1$

(l) If $M_n < N_n$ then $F_n \xrightarrow[n \to \infty]{} 1$ if we choose M_n

such that $\frac{1}{n} \log M_n > \overline{S}(\hat{\rho})$

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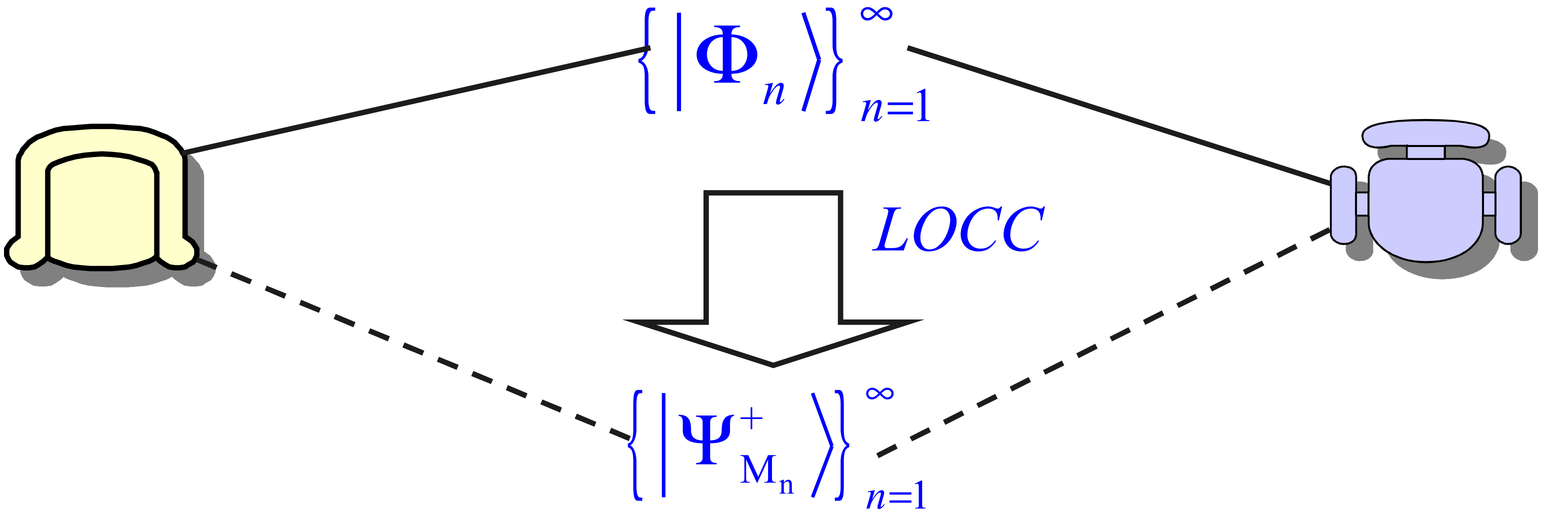
Asymptotic Entanglement Concentration of Pure States



$\left\{ |\Phi_n\rangle \right\}_{n=1}^{\infty}$: partially entangled pure states

AIM: $\left\{ |\Phi_n\rangle \right\}_{n=1}^{\infty} \xrightarrow{LOCC} \left\{ |\Psi^{+}_{M_n}\rangle \right\}_{n=1}^{\infty}$

If the fidelity of this LOCC transformation: $F_n \xrightarrow[n\to\infty]{} 1$

then, any $R \leq \frac{1}{n} \log M_n$ is an achievable rate:

Distillable entanglement: $E_D = \sup R$

THEOREM (Hayashi): For the entanglement concentration protocol $\left\{ \left| \Phi_n \right\rangle \right\}_{n=1}^{\infty} \xrightarrow{LOCC} \left\{ \left| \Psi^+_{M_n} \right\rangle \right\}_{n=1}^{\infty}$

$$E_D = \underline{S}(\hat{\rho})$$

where $\hat{\rho} = \left\{ \rho^A_{\Phi_n} \right\}_{n=1}^{\infty}$ with $\rho^A_{\Phi_n} = \mathrm{Tr}_B \left| \Phi_n \right\rangle \left\langle \Phi_n \right|$

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Proof: Let initial shared state: $$|\Phi_n\rangle = \sum_k \sqrt{\lambda_{n,k}}\,|k_A^{(n)}\rangle |k_B^{(n)}\rangle$$

Define projection operators $$P_n^\gamma := \{\rho_n^A \geq 2^{-n\gamma} I_n^A\}$$

and $$\bar{P}_n^\gamma = I_n^A - P_n^\gamma := \{\rho_n^A < 2^{-n\gamma} I_n^A\}$$

Note : P_n^γ is the operator used in defining $\underline{S}(\hat{\rho})$:

$$\underline{S}(\hat{\rho}) := \sup\left\{\gamma : \liminf_{n\to\infty} \mathrm{Tr}\left[P_n^\gamma \rho_n^A\right] = 0\right\}$$ for $\hat{\rho} = \{\rho_n^A\}_{n=1}^\infty$

Hence for $\gamma < \underline{S}(\hat{\rho})$:

$$\mathrm{Tr}\,(P_n^\gamma \rho_n^A) \xrightarrow[n\to\infty]{} 0$$

$$\mathrm{Tr}\,(\bar{P}_n^\gamma \rho_n^A) \xrightarrow[n\to\infty]{} 1$$

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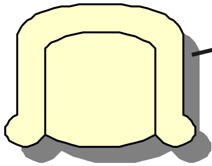
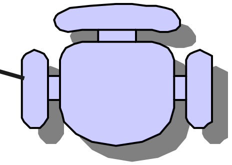
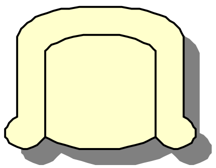


$|\Phi_n\rangle$



initial shared state

PROTOCOL:



(1) Does a von Neumann measurement corrs. to $P_n^\gamma, \overline{P}_n^\gamma$ on her part of shared state $|\Phi_n\rangle$

If outcome corrs. to P_n^γ

Failure!

Protocol aborted!

Probability= $\mathrm{Tr}\,(P_n^\gamma \rho_n^A)$

If outcome corrs. to $\overline{P}_n^\gamma$

Success!

Probability= $\mathrm{Tr}\,(\overline{P}_n^\gamma \rho_n^A)$

If the outcome corrs. to $\bar{P}_n^\gamma$: post-measurement state:

$$|\Phi_n\rangle_{AB} \propto (\bar{P}_n^\gamma \otimes I_n^B)|\Phi_n\rangle_{AB} \propto \sum_{k:\lambda_{n,k}<2^{-n\gamma}} \sqrt{\lambda_{n,k}}\, |k_A^{(n)}\rangle |k_B^{(n)}\rangle$$

since $\bar{P}_n^\gamma := \{\rho_n^A < 2^{-n\gamma} I_n^A\}$

Normalized post-meas. state:

$$|\Psi_n\rangle = \sum_{k:\lambda_{n,k}<2^{-n\gamma}} \frac{\sqrt{\lambda_{n,k}}}{\sqrt{\mathrm{Tr}(\bar{P}_n^\gamma \rho_n^A)}} |k_A^{(n)}\rangle |k_B^{(n)}\rangle$$

square root of eigenvalues of $\rho_{\Psi_n}^A \left(= \mathrm{Tr}_B |\Psi_n\rangle\langle\Psi_n|\right)$

each eigenvalue of $\rho_{\Psi_n}^A \leq \dfrac{2^{-n\gamma}}{\mathrm{Tr}(\bar{P}_n^\gamma \rho_n^A)}$

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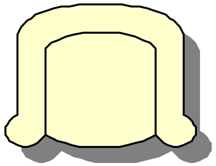


Alice's measurement (if successful) takes

$$|\Phi_n\rangle \longrightarrow |\Psi_n\rangle$$

We wanted: $|\Phi_n\rangle \xrightarrow{LOCC} |\Psi^+_{\mathrm{M_n}}\rangle$

(Q) Is there an LOCC operation that will take:

$$|\Psi_n\rangle \xrightarrow[LOCC]{} |\Psi^+_{M_n}\rangle \ ?$$

(A) Yes! Use Nielsen's majorization theorem.

By *Nielsen's Majorization Theorem* $|\Psi_n\rangle \xrightarrow[LOCC]{} |\Psi^+_{M_n}\rangle$

iff $\lambda_{\Psi_n} \prec \lambda_{\Psi^+_{M_n}}$ ....(1)

$\lambda_{\Psi_n}, \lambda_{\Psi^+_{M_n}}$: vectors of Schmidt coefficients of $|\Psi_n\rangle, |\Psi^+_{M_n}\rangle$

i.e., vectors of eigenvalues of $\rho^A_{\Psi_n}, \rho^A_{\Psi^+_{M_n}}$

each eigenvalue of $\rho^A_{\Psi_n}$ $\leq \dfrac{2^{-n\gamma}}{\sqrt{\mathrm{Tr}(P^\gamma_n \rho^A_n)}}$; $\lambda_{\Psi^+_{M_n}} = \left(\frac{1}{M_n}, \frac{1}{M_n}, \ldots\ldots, \frac{1}{M_n}, \right)$

$\therefore$ (1) holds if we choose M_n such that :

$$\frac{2^{-n\gamma}}{\mathrm{Tr}(P^\gamma_n \rho^A_n)} \leq \frac{1}{M_n} \Rightarrow \boxed{M_n \leq 2^{n\gamma}\,\mathrm{Tr}(\mathrm{P}^\gamma_{\mathrm{n}} \rho^A_n)}$$

We need: $M_n \leq 2^{n\gamma}\,\mathrm{Tr}(\mathrm{P}_n^\gamma \rho_n^A)$; Let

$$M_n = \left\lfloor 2^{n\gamma}\,\mathrm{Tr}(\bar{P}_n^\gamma \rho_n^A) \right\rfloor$$

If $\gamma < \underline{S}(\hat{\rho})$ where $\hat{\rho} = \{\rho_n^A\}_{n=1}^\infty$ then

Probability of failure: $\mathrm{Tr}\,(P_n^\gamma \rho_n^A) \xrightarrow[n\to\infty]{} 0$

Probability of success: $\mathrm{Tr}\,(\bar{P}_n^\gamma \rho_n^A) \xrightarrow[n\to\infty]{} 1$

Achievable rate: $R \leq \frac{1}{n}\log M_n < \underline{S}(\hat{\rho})$

Weak Converse: A rate $R > \underline{S}(\hat{\rho})$ is not achievable

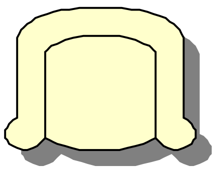
Distillable Entanglement: $E_D = \underline{S}(\hat{\rho})$

■

Schematic summary: protocol for entanglement concentration

Aim : $\left\{ \left| \Phi_n \right\rangle_{AB} \right\}_{n=1}^{\infty} \xrightarrow{LOCC} \left\{ \left| \Psi^+_{M_n} \right\rangle_{AB} \right\}_{n=1}^{\infty}$

PROTOCOL:



(1) Does a von Neumann measurement corrs. to $P_n^\gamma, \bar{P}_n^\gamma$ on her part of shared state $\left| \Phi_n \right\rangle$

If outcome corrs. to P_n^γ

Failure!

If outcome corrs. to $\bar{P}_n^\gamma$

Success!

$\left| \Phi_n \right\rangle \longrightarrow \left| \Psi_n \right\rangle$

LOCC (ii)

If_ $\gamma < \underline{S}(\hat{\rho})$:

(i) Prob.of success_ $\xrightarrow[n\to\infty]{} 1$ & $\left| \Psi^+_{M_n} \right\rangle$

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Summary

Entanglement Dilution

$$\left\{ \left| \Psi^{+}_{M_n} \right\rangle_{AB} \right\}_{n=1}^{\infty} \xrightarrow{LOCC} \left\{ \left| \Phi_n \right\rangle_{AB} \right\}_{n=1}^{\infty}$$

Entanglement cost

$$E_C = \overline{S}(\hat{\rho})$$

where $\hat{\rho} = \left\{ \rho^{A}_{\Phi_n} \right\}_{n=1}^{\infty}$ with $\rho^{A}_{\Phi_n} = \mathrm{Tr}_B \left| \Phi_n \right\rangle \left\langle \Phi_n \right|$

Entanglement Concentration $\left\{ \left| \Phi_n \right\rangle_{AB} \right\}_{n=1}^{\infty} \xrightarrow{LOCC} \left\{ \left| \Psi^{+}_{M_n} \right\rangle_{AB} \right\}_{n=1}^{\infty}$

Distillable entanglement

$$E_D = \underline{S}(\hat{\rho})$$

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Any sequence of bipartite pure states $\left\{ |\Phi_n\rangle_{AB} \right\}_{n=1}^{\infty}$ for which

$$\underline{S}(\hat{\rho}) = \lim_{n\to\infty} \frac{1}{n} S(\rho_n) = \overline{S}(\hat{\rho})$$

: *information stable on its subsystems*

asymptotic entanglement measure:

$$E_C = E_D = \lim_{n\to\infty} \frac{1}{n} S(\rho_n)$$

here $\rho_n = \mathrm{Tr}_B |\Phi_n\rangle\langle\Phi_n|$

e.g. if $\left\{ |\Phi_n\rangle_{AB} \right\}_{n=1}^{\infty} = \left\{ |\varphi\rangle_{AB}^{\otimes n} \right\}_{n=1}^{\infty}$ sequences of memoryless/i.i.d. states

then $\hat{\rho} = \left\{ \rho^{\otimes n} \right\}_{n=1}^{\infty}$ with $\rho = \mathrm{Tr}_B |\varphi\rangle\langle\varphi|$

and

$$\underline{S}(\hat{\rho}) = \overline{S}(\hat{\rho}) = S(\rho)$$

Hence, $\boxed{E_C = E_D = S(\rho)}$:unique entanglement measure

However, $\exists$ sequences $\left\{ |\Phi_n\rangle_{AB} \right\}_{n=1}^{\infty}$ of bipartite pure states for which the corrs. sequence of subsystem states are

not information stable: $\underline{S}(\hat{\rho}) \neq \overline{S}(\hat{\rho})$

e.g. sequences for which the reduced states are:

$$|\Psi_n\rangle = \sqrt{t}\,|\varphi_1\rangle^{\otimes n} + \sqrt{(1-t)}\,|\varphi_2\rangle^{\otimes n}$$

with $t \in (0,1)$ and $S(\sigma) < S(\omega)$

For such sequences $E_D = S(\sigma) < S(\omega) = E_C$

Hence, the asymptotic entanglement measure is unique only for information stable sequences !

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SUMMARY

For an arbitrary sequence of pure bipartite states $\left\{ |\Phi_n\rangle_{AB} \right\}_{n=1}^{\infty}$

$$\hat{\rho} = \left\{ \mathrm{Tr}_B |\Phi_n\rangle\langle\Phi_n| \right\}_{n=1}^{\infty}$$

entanglement cost $E_C = \overline{S}(\hat{\rho})$;

distillable entanglement $E_D = \underline{S}(\hat{\rho})$

$E_C = E_D$ only for sequences of states which are

information stable, i.e., for which $\underline{S}(\hat{\rho}) = \overline{S}(\hat{\rho})$

only such sequences have a unique asymptotic entanglement measure.

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NOTE: The quantities $\underline{S}(\hat{\rho}), \quad \overline{S}(\hat{\rho})$

are obtainable from 2 fundamental quantities: the *spectral divergence rates:*

$$\overline{D}(\hat{\rho}\,\|\,\hat{\omega}) := \inf\left\{\gamma : \limsup_{n\to\infty} \mathrm{Tr}\left[\{\Pi_n(\gamma)\geq 0\}\,\Pi_n(\gamma)\right] = 0\right\}$$

$$\underline{D}(\hat{\rho}\,\|\,\hat{\omega}) := \sup\left\{\gamma : \liminf_{n\to\infty} \mathrm{Tr}\left[\{\Pi_n(\gamma)\geq 0\}\,\Pi_n(\gamma)\right] = 1\right\}$$

Here $\hat{\rho} = \{\rho_n\}_{n=1}^{\infty}$, $\hat{\omega} = \{\omega_n\}_{n=1}^{\infty}$ and

$$\Pi_n(\gamma) = \rho_n - 2^{n\gamma}\omega_n$$

By substituting $\hat{\omega} = \hat{I} = \{I_n\}_{n=1}^{\infty}$ we get

$$\underline{S}(\hat{\rho}) = -\overline{D}(\hat{\rho}\,\|\,\hat{I})$$

and

$$\overline{S}(\hat{\rho}) = -\underline{D}(\hat{\rho}\,\|\,\hat{I})$$

From $\overline{D}(\hat{\rho} \| \hat{\omega})$ and $\underline{D}(\hat{\rho} \| \hat{\omega})$ we obtain

$$\underline{S}(\hat{\rho}) = -\overline{D}(\hat{\rho} \| \hat{I})$$

and

$$\overline{S}(\hat{\rho}) = -\underline{D}(\hat{\rho} \| \hat{I})$$

by substituting $\hat{\omega} = \hat{I} = \{I_n\}_{n=1}^{\infty}$

The spectral divergences rates can be viewed as generalizations of the quantum relative entropy:

$$S(\rho \| \omega) = \mathrm{Tr}\, \rho \log \rho - \mathrm{Tr}\, \rho \log \omega$$

since

$$S(\rho) = -S(\rho \| I)$$

$$S(A \| B) = -S(\rho^{AB} \| I^{A} \otimes \rho^{B})$$

The Quantum Information Spectrum Method provides a unifying mathematical framework for evaluating the optimal rates of various information theoretic tasks e.g. entanglement manipulation, data compression, data transmission, dense coding etc.

OPEN PROBLEMS

Use the Quantum Information Spectrum Method to:

Find the quantum capacity of an arbitrary quantum channel.

Find the optimal rates for various other informations theoretic protocols, such as, distributed quantum compression, quantum capacity in the presence of feedback,
etc., using arbitrary sources, channels and entanglement resources.