Geometrical Objects on Matrix Algebra

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Abstract. In this paper we present some geometric objects as derivations, differential forms, universal differential forms, linear connections and distributions with related objects on matrix algebra using the framework of noncommutative geometry of ρ-algebras.

Keywords: noncommutative geometry, matrix algebras.

1. Introduction

There are studied some differential calculi on the matrix algebra $M_n(\mathbb{C})$ (most of them used the techniques from noncommutative geometry) which have been used in some different areas from mathematics and physics, here we remember some of them: quantum groups ([Coquereaux]), graded matrix algebra ([Grosse]), the noncommutative differential calculi of matrix algebras ([Dubois3], [Dubois1]) and linear connections on them in [Dubois1], [Madore] and [Mourad], finally the ρ-differential calculi and linear connections in [Ciup3].

In this paper we review some geometrical objects on matrix algebra, using techniques of ρ-algebras, from the paper [Ciup3] such: the algebra of forms $\Omega\left(M_n(\mathbb{C})\right)$, the algebra of universal differential forms $\Omega_\alpha\left(M_n(\mathbb{C})\right)$, linear connections on $M_n(\mathbb{C})$. We also introduce and study distributions and related objects on $M_n(\mathbb{C})$ on the algebra of forms $\Omega\left(M_n(\mathbb{C})\right)$ and on the algebra of universal differential forms $\Omega_\alpha\left(M_n(\mathbb{C})\right)$.

The basic idea of noncommutative geometry is to replace an algebra of smooth functions defined on a smooth manifold by an abstract associative algebra A which is not necessarily commutative. In the context of noncommutative geometry the basic role is the generalization of the notion of differential forms (on a manifold). With any associative algebra A over $\mathbb{R}$ or $\mathbb{C}$ one associates a differential algebra which is a $\mathbb{Z}$-graded algebra $\Omega(A) = \oplus_{n\geq 0}\Omega^n(A)$ (where $\Omega^n(A)$ are A-bimodules and $\Omega^0(A) = A$) together with a linear operator $d : \Omega^n(A) \to \Omega^{n+1}(A)$ satisfying $d^2 = 0$ and $d(\omega\omega') = (d\omega)\omega' + (-1)^n\omega d\omega'$ where $\omega \in \Omega^n(A)$. $\Omega(A)$ is also called the (noncommutative) differential calculus on the algebra A.

A generalization of a differential calculus $\Omega(A)$ of an associative algebra A is the ρ-differential calculus associated with a ρ-(commutative) algebra A (where A is a $G-$graded algebra, G is a commutative group and ρ is a twisted cocycle). The differential calculus over a ρ-algebra A was introduced in [Bongaarts] and continued in some recent papers [Ciup1],...,[Ciup7], [Ngakeu1] and [Ngakeu2].

In the second section we review the basic geometrical objects about the ρ-algebras as ρ-derivations, ρ-differential calculi, distributions and linear connections. In the last section we apply the remembered notions on the matrix algebra $M_n(\mathbb{C})$.

2. ρ-ALGEBRAS

In this section we present shortly the class of the noncommutative algebras, namely the ρ-algebras, for more details see [Bongaarts].

Let G be an abelian group, additively written. A ρ-algebra A is a G-graded algebra over that field k (which is $\mathbb{R}$ or $\mathbb{C}$) which is endowed with a cyclic cocycle $\rho : G \times G \to k$ which fulfils the properties

$$\rho(a,b) = \rho(b,a)^{-1} \text{ and } \rho(a+b,c) = \rho(a,c)\rho(b,c), \text{ for any } a,b,c \in G. \tag{1}$$

Notation: From now on, if M is a graded set then $Hg\,(M)$ will stand for the set of homogeneous elements in M. The G-degree of a (nonzero) homogeneous element f of M is denoted as $|f|$.

A G-graded algebra A with a given cocycle ρ will be called ρ-**commutative** (or **almost commutative algebra**) if $fg = \rho(|f|,|g|)gf$ for all $f,g \in Hg\,(A)$.

Example 1. *1) Any usual (commutative) algebra is a ρ-algebra with the trivial group G.*

2) Let be the group $G = \mathbb{Z}$ $(\mathbb{Z}_2)$ and the cocycle $\rho(a,b) = (-1)^{ab}$,. for any $a,b \in G$. In this case any ρ–(commutative) algebra is a super(commutative) algebra.

3) The N-dimensional quantum hyperplane ([Bongaarts], [Ciup1], [Ciup2], [Ciup7]) S_N^q, is the algebra generated by the unit element and N linearly independent elements $x_1, ...x_N$ satisfying the relations:

$$x_i x_j = q x_j x_i, \qquad i < j$$

for some fixed $q \in k$, $q \neq 0$. S_N^q is a $\mathbb{Z}^N-$graded algebra

$$S_N^q = \overset{\infty}{\underset{n_1,...n_N}{\oplus}} (S_N^q)_{n_1...n_N},$$

with $(S_N^q)_{n_1...n_N}$ the one-dimensional subspace spanned by products $x^{n_1}\cdots x^{n_N}$. The $\mathbb{Z}^N-$degree of these elements is denoted by

$$|x^{n_1}\cdots x^{n_N}| = n = (n_1,...n_N).$$

Define the function $\rho : \mathbb{Z}^N \times \mathbb{Z}^N \to k$ as

$$\rho(n,n') = q^{\sum\limits_{j,k=1}^{N} n_j n'_k \alpha_{jk}},$$

with $\alpha_{jk} = 1$ for $j < k$, 0 for $j = k$ and -1 for $j > k$. It is obvious that S_N^q is a ρ-commutative algebra.

4) Quantum torus T_q^N is generated by 1,$x_1, ..., x_N, x_1^{-1}, ..., x_N^{-1}$ and relations

$$x_i x_j = q x_j x_i, \quad i < j; \quad x_i x_i^{-1} = 1.$$

The $\mathbb{Z}^N$ graduation of T_q^N is the extension of that of the quantum hyperplane S_N^q and $\left|x_i^{-1}\right| = -\left|x_i\right|$.

5) The algebra of matrix $M_n(\mathbb{C})$ ([Ciup3]) is a $\rho-$commutative as follows:

$$\text{Let } p = \begin{pmatrix} 1 & 0 & ... & 0 \\ 0 & \varepsilon & ... & 0 \\ ... & & & \\ 0 & 0 & ... & \varepsilon^{n-1} \end{pmatrix} \text{ and } q = \begin{pmatrix} 0 & 0 & ... & 0 & 1 \\ \varepsilon & 0 & ... & 0 & 0 \\ 0 & \varepsilon^2 & ... & 0 & 0 \\ ... & & & & \\ 0 & 0 & ... & \varepsilon^{n-1} & 0 \end{pmatrix}$$

$p, q \in M_n(\mathbb{C})$, where $\varepsilon^n = 1$, $\varepsilon \neq 1$. Then $pq = \varepsilon qp$ and $M_n(\mathbb{C})$ is generated by the set $B = \{p^a q^b | a, b = 0, 1, ..., n-1\}$.
It is easy to see that $p^a q^b = \varepsilon^{ab} q^b p^a$ and $q^b p^a = \varepsilon^{-ab} p^a q^b$ for any $a, b = 0, 1, ..., n-1$. Let $G := \mathbb{Z}_n \oplus \mathbb{Z}_n$, $\alpha = (\alpha_1, \alpha_2) \in G$ and $x_\alpha := p^{\alpha_1} q^{\alpha_2} \in M_n(\mathbb{C})$. Denoting by $\rho(\alpha, \beta) = \varepsilon^{\alpha_2 \beta_1 - \alpha_1 \beta_2}$ results that $x_\alpha x_\beta = \rho(\alpha, \beta) x_\beta x_\alpha$, for any $\alpha, \beta \in G$, $x_\alpha, x_\beta \in B$.

It is obvious that the map $\rho : G \times G \to \mathbb{C}$, $\rho(\alpha, \beta) = \varepsilon^{\alpha_2 \beta_i - \alpha_1 \beta_2}$ is a cocycle and that $M_n(\mathbb{C})$ is a ρ-commutative algebra.

2.1. ρ-derivations.

Definition 1. *([Ciup3]) Let $\alpha, \beta \in G$. A ρ-**derivation of the order** (α, β) is a linear map $X : A \to A$, which fulfils the properties:*

1) $X : A_* \to A_{*+\beta}$,

2) $X(fg) = (Xf)g + \rho(\alpha, |f|) f(Xg)$,for any $f \in A_{|f|}$ and $g \in A$.

The left product between the element $f \in A$ and a derivation X of the order (α, β) is defined in a natural way: $fX : A \to A$ by $(fX)(g) = fX(g)$, for any $g \in A$. Remark that fX is a derivation of the order $(|f| + \alpha, |f| + \beta)$ if and only if the algebra A is ρ-commutative.

Next we study the case when A is a ρ-commutative algebra.

Let X be a ρ-derivation of the order (α, β) and X' a ρ-derivation of the order (α', β'). The ρ-bracket of X and X' is $[X, X'] = X \circ X' - \rho(\alpha, \beta') X' \circ X$ and satisfies the following property: $[X, X']$ is a ρ-derivation of the order $(\alpha + \alpha', \beta + \beta')$ if and only if $\rho(\alpha, \beta)\rho(\alpha', \beta') = 1$.

Definition 2. *([Bongaarts]) We say that $X : A \to A$ is a ρ-**derivation** if it has the order $(|X|, |X|)$, i.e. $X : A_* \to A_{*+|X|}$ and $X(fg) = (Xf)g + \rho(|X|, |f|) f(Xg)$ for any $f \in A_{|f|}$ and $g \in A$.*

It is known ([Bongaarts]) that the ρ-commutator $[X,Y]_\rho = XY - \rho(|X|,|Y|)YX$ of two ρ-derivations is again a ρ-derivation and the linear space of all ρ-derivations is a ρ-Lie algebra, denoted by ρ-Der$A$.

One verifies immediately that for such an algebra A, ρ-Der$A$ is not only a ρ-Lie algebra but also a left A-module with the action of A on ρ-Der$A$ defined by $(fX)g = f(Xg)$, for $f,g \in A$ and $X \in \rho$-Der$A$.

Let M be a G-graded left module over a ρ-commutative algebra A, with the usual properties, in particular $|f\psi| = |f| + |\psi|$ for $f \in A$, $\psi \in M$. Then M is also a right A-module with the right action on M defined by $\psi f = \rho(|\psi|,|f|)f\psi$, for any $\psi \in Hg(M)$ and $f \in Hg(A)$. In fact M is a bimodule over A, i.e. $f(\psi g) = (f\psi)g$ for any $f,g \in A$, $\psi \in M$.

2.2. Differential calculi on a ρ-algebra. Next we generalize the classical notions of differential graded algebra and the differential graded superalgebras by defining so called **differential graded $\rho-$algebras**.

Denote by G'=$\mathbb{Z} \times G$ and define the cocycle $\rho' : G' \times G' \to k$ as follow

$$\rho'((n,\alpha),(m,\beta)) = (-1)^{nm}\rho(\alpha,\beta).$$

It is obvious the function ρ' is a satisfies the properties (1) and (**??**).

Definition 3. *We say that* $\Omega = \bigoplus_{(n,\beta)\in G'} \Omega^n_\beta$ *is a* ρ-***differential graded algebra*** *(**DG*** $\rho-$***algebra**) if there is an element* $\alpha \in G$ *and a map* $d : \Omega^n_\beta \to \Omega^{n+1}_\beta$ *of degree* $(1,\alpha) \in G'$ *and the* G'*-degree* $|d|'$ *=(1,0) such that:* $d^2 = 0$ *and*

$$d(\omega\theta) = (d\omega)\theta + (-1)^n \rho(\alpha,|\omega|)\omega d\theta \tag{2}$$

for any $\omega \in \Omega^n_{|\omega|}$ *and* $\theta \in \Omega$.

If we denote $|\omega|' = (n,|\omega|)$ the G'-degree of $\omega \in \Omega^n_{|\omega|}$, then the last equality is:

$$d(\omega\theta) = (d\omega)\theta + \rho'(|d|',|\omega|')\omega d\theta.$$

Results that Ω is a ρ'-algebra.

Example 2. *1) In the case when the group* G *is trivial then* Ω *is the classical differential graded algebra.*

2) In the case when the group G *is* $\mathbb{Z}_2$ *and the map* ρ *is given by* $\rho(a,b) = (-1)^{ab}$ *then* Ω *is a differential graded superalgebra (see [Kastler1], [Kastler2]).*

Definition 4. *Let A be a ρ-algebra. $\left(\Omega(A) = \underset{(n,\alpha)\in G'}{\oplus} \Omega_\alpha^n(A), d\right)$ is a ρ-differential calculus over A if $\Omega(A)$ is a ρ-differential graded algebra, $\Omega(A)$ is an A-bimodule and $\Omega^0(A) = A$.*

The first example of a ρ-differential calculus over the ρ-commutative algebra A is the algebra of forms $(\Omega(A), d)$ of A from [Bongaarts].

The second example of a ρ-differential calculus over a ρ-algebra is the universal differential calculus of A from the next paragraph.

The algebra of forms of a ρ-algebra. In this paragraph we construct the algebra of forms $\Omega(A)$ of an almost commutative algebra A (see [Bongaarts]).

The algebra of forms of an the ρ-algebra A is given in the classical manner: $\Omega^0(A) := A$, and $\Omega^p(A)$ for $p = 1, 2, ...,$ as the G-graded space of p-linear maps $\alpha_p : \times^p \rho\text{-Der}A \to A$, p-linear in sense of left A-modules

$$\alpha_p(fX_1, ..., X_p) = f\alpha_p(X_1, ..., X_p), \tag{3}$$

$$\alpha_p(X_1, ..., X_j f, X_{j+1}, ..., X_p) = \alpha_p(X_1, ..., X_j, fX_{j+1}, ...X_p) \tag{4}$$

and ρ-alternating

$$\alpha_p(X_1, ..., , X_j, X_{j+1}, ..., X_p) = -\rho(|X_j|, |X_{j+1}|)\alpha_p(X_1, ..., , X_{j+1}, X_j, ..., X_p) \tag{5}$$

for $j = 1, ..., p-1$; $X_k \in \rho\text{-Der}(A)$, $k = 1, ..., p$; $f \in A$ and Xf is the right A-action on ρ-Der$A$.

$\Omega^p(A)$ is in natural way a G-graded right A-module with

$$|\alpha_p| = |\alpha_p(X_1, ..., X_p)| - (|X_1| + ... + |X_p|) \tag{6}$$

and with the right action of A defined as

$$(\alpha_p f)(X_1, ..., X_p) = \alpha_p(X_1, ..., X_p)f. \tag{7}$$

From the previous considerations, it follows that $\Omega(A) = \oplus_{p=0}^\infty \Omega^p(A)$ is again a G-graded A-bimodule.

One defines exterior differentiation as a linear map $d : \Omega^p(A) \to \Omega^{p+1}(A)$, for all $p \geq 0$, as

$$df(X) = X(f),$$

and for $p = 1, 2, ...,$

$$
\begin{aligned}
d\alpha_p(X_1,...,X_{p+1}) \ : &= \sum_{j=1}^{p+1}(-1)^{j-1}\rho(\sum_{i=1}^{j-1}\left|X_i\right|,\left|X_j\right|)X_j\alpha_p(X_1,...,\widehat{X}_j,...,X_{p+1}) \\
&+ \sum_{1\leq j<k\leq p+1}(-1)^{j+k}\rho(\sum_{i=1}^{j-1}\left|X_i\right|,\left|X_j\right|)\rho(\sum_{i=1}^{j-1}\left|X_i\right|,\left|X_k\right|)\times \\
&\times\rho(\sum_{i=j+1}^{k-1}\left|X_i\right|,\left|X_k\right|)\alpha_p([X_j,X_k]_\rho,...,X_1,...,\widehat{X}_j,...,\widehat{X}_k,...X_{p+1}).
\end{aligned}
$$

One can show that d has degree 0, and that $d^2=0$.

There is an exterior product $\Omega^p(A)\times\Omega^q(A)\to\Omega^{p+q}(A)$, $\left(\alpha_p,\beta_q\right)\mapsto\alpha_p\wedge\beta_q$, defined by the ρ-antisymmetrization formula:

$$
\alpha_p\wedge\beta_q\left(X_1,...,X_{p+q}\right)=
$$
$$
\sum_\sigma sign(\sigma)\left(\rho\text{-factor}\right)\alpha_p\left(X_{\sigma(1)},...,X_{\sigma(p)}\right)\beta_q\left(X_{\sigma(p+1)},...,X_{\sigma(p+q)}\right)
$$

The sum is over all permutations σ of the cyclic group S_{p+q} so that $\sigma(1)<...<\sigma(p)$ and $\sigma(p+1)<...<\sigma(p+q)$. The ρ-factor is the product of all $\rho(\left|X_{\sigma(j)}\right|,\left|\alpha_p\right|)$ for $p+1\leq j\leq p+q$ and all $\rho(\left|X_{\sigma(j)}\right|,\left|X_{\sigma(k)}\right|)^{-1}$ for $j<k$ and $\sigma(j)>\sigma(k)$.

$\Omega(A)$ is a $G'-$graded algebra with the group $G'=\mathbb{Z}\times G$. Denote the G' degree of α_p as $|\alpha_p|'=(p,|\alpha_p|)$. It is easy to see that the map $\rho':G'\times G'\to k$ defined by $\rho'((p,a),(q,b))=(-1)^{pq}\rho(a,b)$ is a cocycle and that $\Omega(A)$ is a ρ'-commutative algebra. Moreover, the map d is a ρ'-derivation of $\Omega(A)$ with G'-degree $|d|'=(+1,0)$.

The algebra of universal differential forms of a ρ-algebra. Next we present our construction of algebra of universal differential forms $\Omega_\alpha A$ of the ρ-algebra A (not necessarily ρ-commutative) for a given element $\alpha\in G$.

Let α be an arbitrary element of G. By definition the algebra of **universal differential forms** (also called the **algebra of noncommutative differential forms**) of the ρ-algebra A is the algebra $\Omega_\alpha A$ generated by the algebra A and the symbols da, $a\in A$ which satisfies the following relations:

1. da is linear in a.
2. the $\rho-$Leibniz rule: $d(ab)=d(a)b+\rho(\alpha,|a|)adb$.
3. $d(1)=0$.

Let $\Omega_\alpha^n A$ the space of $n-$forms $a_0da_1...da_n$, $a_i\in A$ for any $0\leq i\leq n$. $\Omega_\alpha^n A$ is an A-bimodule with the left multiplication

$$
a(a_0da_1...da_n)=aa_0da_1...da_n, \tag{8}
$$

and the right multiplication is given by:

$$
(a_0da_1...da_n)a_{n+1}= \tag{9}
$$

$$=\sum_{i=1}^{n}(-1)^{n-i}\rho(\alpha,\sum_{j=i+1}^{n}|a_j|)(a_0da_1...d(a_ia_{i+1})...da_{n+1})+$$
$$+(-1)^n\rho(\alpha,\sum_{i=1}^{n}|a_j|)a_0a_1da_2...da_{n+1}$$

$\Omega_\alpha A=\bigoplus_{n\in\mathbb{Z}}\Omega_\alpha^n A$ is a $\mathbb{Z}$-graded algebra with the multiplication $\Omega_\alpha^n A\cdot\Omega_\alpha^m A\subset\Omega_\alpha^{n+m}A$ given by:

$$(a_0da_1...da_n)(a_{n+1}da_{n+2}...da_{m+n})= \tag{10}$$
$$=((a_0da_1...da_n)a_{n+1})da_{n+2}...da_{m+n}).$$

for any $a_i\in A$, $0\leq i\leq n+m$, $n,m\in\mathbb{N}$.

We define the $G-$degree of the n-form $a_0da_1...da_n$ in the following way

$$|a_0da_1...da_n|=\sum_{i=0}^{n}|a_i|\,.$$

It is obvious that $|\omega_n\cdot\omega_m|=|\omega_n|+|\omega_m|$ for any homogeneous forms $\omega_n\in\Omega_\alpha^n A$ and $\omega_m\in\Omega_\alpha^m A$.

$\Omega_\alpha A$ is a $G'=\mathbb{Z}\times G$-graded algebra with the G' degree of the n-form $a_0da_1...da_n$ thus $|a_0da_1...da_n|'=(n,\sum_{i=0}^{n}|a_i|)$.

We may define the cocycle $\rho':G'\times G'\to k$ on the algebra $\Omega_\alpha A$ thus:

$$\rho'(|\omega_n|',|\omega_m|')=(-1)^{nm}\rho(|\omega_n|,|\omega_m|) \tag{11}$$

for any $\omega_n\in\Omega_\alpha^n A$, $\omega_m\in\Omega_\alpha^m A$. It is obvious that $\Omega_\alpha A$ is a ρ'-algebra. Remark that G'-degree of the map d is $(1,0)$ i.e. $d:\Omega_{|\omega|}^n\to\Omega_{|\omega|}^{n+1}$, and the $G'-$degree of an element $x\in A$ is $|x|'=(0,|x|)$.

Theorem 1. *([Ciup3]) 1) $d:\Omega_\alpha^* A\to\Omega_\alpha^{*+1}A$ satisfies:*

$$d(\omega\theta)=(d\omega)\theta+(-1)^n\rho(\alpha,|\omega|)\omega d\theta$$

for any $\omega\in\Omega_\alpha^n A$, $\theta\in\Omega_\alpha^m A$.

2) $(\Omega_\alpha A,d)$ is a ρ-differential calculus over A.

Example 3. *In the case when the group G is trivial then A is the usual associative algebra and $\Omega_\alpha A$ is the algebra of universal differential forms of A.*

Example 4. *If the group G is $\mathbb{Z}_2$ and the cocycle is from example 2 then A is a superalgebra. In the case when $\alpha=1$, $\Omega_\alpha A$ is the superalgebra of universal differential forms of A from [Kastler2].*

2.3. Connections on a ρ-bimodule over a ρ-algebra. Let A be a ρ-algebra and M a ρ-bimodule on A.

Definition 5. *([Ciup2]) A **linear connection** on M is a linear map of $\rho-\mathrm{Der}A$ into the linear endomorphisms of M, $\nabla : \rho-\mathrm{Der}A \to End(M)$, so that one has:*

$$\nabla_X : M_p \to M_{p+|X|}, \tag{12}$$

with

$$\nabla_{aX}(m) = a\nabla_X(m) \text{ and } \nabla_X(ma) = \rho(|X|, |m|)mX(a) + \nabla_X(m)a \tag{13}$$

if we use the right structure of M or $\nabla_X(am) = X(a)m + \rho(|X|, |a|)a\nabla_X(m), if M$ is considered a left bimodule, for all $p \in G$, $a \in A$, $X \in Hg(\rho\text{-Der}A)$ and $m \in Hg(M)$.

We say that the distribution $\mathcal{D}$ in the ρ-algebra A over the ρ-differential calculus $(\Omega(A) = \oplus_{n\geq 0}\Omega^n(A), d)$ is **parallel** with respect to the connection $\nabla : \rho\text{-Der}A \to End(\Omega^1(A))$ if

$$\nabla_X(m) = 0, \text{ for any } X \in \rho\text{-Der}A \text{ and for any } m \in \mathcal{D}.$$

The **curvature** R of the connection ∇ on M is defined in a natural way

$$R : (\rho\text{-Der}A) \times (\rho\text{-Der}A) \to End(M);\ (X, Y) \longmapsto R_{X,Y}$$

given by:

$$R_{X,Y}(m) = \nabla_X\nabla_Y - \rho(|X|, |Y|)\nabla_Y\nabla_X(m) - \nabla_{[X,Y]_\rho}(m) \tag{14}$$

for any $X, Y \in \rho\text{-Der}A$, and $m \in M$, where $[X, Y]_\rho = X \circ Y - \rho(|X|, |Y|)Y \circ X$.

Theorem 2. *If the algebra A is ρ-commutative, then the curvature of any connection ∇ has the following properties:*

1) *A-linearity: $R_{aX,Y}(m) = aR_{X,Y}$,*
2) *$R_{X,Y}$ is right A-linear: $R_{X,Y}(ma) = R_{X,Y}(m)a$,*
3) *$R_{X,Y}$ is left A-linear: $R_{X,Y}(am) = \rho(|X| + |Y|, |a|)R_{X,Y}(m)$,*
4) *R is a ρ-symmetric map: $R_{X,Y} = -\rho(|X|, |Y|)R_{Y,X}$*

for any $a \in A_{|a|}$, $m \in M$, $X, Y \in \rho\text{-}DerA$.

In the case when the bimodule M is ρ-Der$A$ then the **torsion** of the connection ∇ as the map

$$T_\nabla : (\rho\text{-Der}A) \times (\rho\text{-Der}A) \to \rho\text{-Der}A$$

defined by

$$T_\nabla(X, Y) = [\nabla_X Y, \nabla_Y X]_\rho - [X, Y]_\rho$$

for any homogeneous $X, Y \in \rho\text{-Der}A$.

Remark 1. *If the group G is $\mathbb{Z}_2$ and the cocycle is from example 2 then A is a superalgebra. In this case we obtain the same definition of linear connections as in [**?**].*

Remark 2. *The noncommutative geometry of ρ-algebras may be view as a natural generalization of fermionic differential calculus.*

2.4. Distributions. Let A be a ρ-algebra and $(\Omega(A), d)$ a ρ-differential calculus over A.

Definition 6. *A distribution $\mathcal{D}$ in the ρ-algebra A over the ρ-differential calculus $(\Omega(A), d)$ is an A-sub-bimodule $\mathcal{D}$ of $\Omega(A)$.*

The distribution $\mathcal{D}$ is globally integrable if there is a ρ-subalgebra B of A such that $\mathcal{D}$ is the space generated by AdB and $(dB)A$.

Remark 3. *Let us assume that A is generated as algebra by n homogeneous coordinates $x_1, x_2, ..., x_n$ and the ρ-differential calculus $(\Omega(A), d)$ by the differentials $dx_1, dx_2, ..., dx_n$ with some relations between them. In this case any globally integrable distribution $\mathcal{D}$ is generated by a subset of p elements, denoted by I of $\{1, ..., N\}$, such that $\mathcal{D}$ is generated by $x_j y_i$ and $y_i x_j$ for any $j \in \{1, ..., N\}$ and $i \in I$. In this situation we say that the distribution $\mathcal{D}$ has the dimension p. For other examples than $M_n(\mathbb{C})$ of these kind of spaces, see [Ciup2], [Ciup5], [Ciup6], [Ciup7].*

Definition 7. *We say that the distribution $\mathcal{D}$ over the ρ-differential calculus $(\Omega(A), d)$ is* ***parallel*** *with respect to the connection $\nabla : \rho\text{-Der}A \to End(\Omega A)$ if*

$$\nabla_X(m) = 0, \text{ for any } X \in \rho\text{-Der}A \text{ and for any } m \in \mathcal{D}.$$

3. APPLICATIONS TO THE MATRIX ALGEBRA

In this section we apply the geometrical objects which are defined in the previous section to the particular case of the matrix algebra $M_n(\mathbb{C})$.

3.1. Derivations. We denote by ρ-Der$M_n(\mathbb{C})$ the set of ρ-derivations of the algebra $M_n(\mathbb{C})$ and it is generated by the elements $\frac{\partial}{\partial p^{\alpha_1}}, \frac{\partial}{\partial q^{\alpha_2}}$, with $\alpha = (\alpha_1, \alpha_2) \in G$, which acts on the basis $\{p^{\alpha_1} q^{\alpha_2} | (\alpha_1, \alpha_2) \in G\}$ like partial derivatives:

$$\frac{\partial}{\partial p^k}(p^{\alpha_1} q^{\alpha_2}) = \frac{\alpha_1}{k} p^{\alpha_1 - k} q^{\alpha_2} \text{ and } \frac{\partial}{\partial p^k}(q^{\alpha_2}) = 0 \text{ of } G-\text{degree } (-k, 0) \tag{15}$$

and

$$\frac{\partial}{\partial q^k}(q^{\alpha_2} p^{\alpha_1}) = \frac{\alpha_2}{k} q^{\alpha_2 - k} p^{\alpha_1} \text{ and } \frac{\partial}{\partial q^k}(p^{\alpha_1}) = 0, \text{ of } G-\text{degree } (0, -k), \tag{16}$$

for any $(\alpha_1, \alpha_2) \in G$. Remark that the first relation from (16) is equivalent with

$$\frac{\partial}{\partial q^k}(p^{\alpha_1} q^{\alpha_2}) = \frac{\alpha_2}{k} \varepsilon^{\alpha_1 k} p^{\alpha_1} q^{\alpha_2 - k} \tag{17}$$

From an easy calculus we obtain that the applications from the equations (15) and (16) are ρ-derivations. Results that ρ-Der$M_n(\mathbb{C})$ is a $M_n(\mathbb{C})$-bimodule generated by $2n-1$ elements and the ρ-bracket of the ρ-derivations is zero i.e. $[\frac{\partial}{\partial p^{k_1}}, \frac{\partial}{\partial q^{k_2}}] = 0$.

Then any $X \in \rho$-Der$M_n(\mathbb{C})$ is given by the following relation:

$$X = \sum_{\alpha=(\alpha_1,\alpha_2)\in G} (\frac{\partial}{\partial p^{\alpha_1}} X^{\alpha_1} + \frac{\partial}{\partial q^{\alpha_2}} X^{\alpha_2}), \tag{18}$$

where X^{α_1}, $X^{\alpha_2} \in M_n(\mathbb{C})$. We denote the derivation from 18 using the following compact form:

$$X = \sum_{\alpha \in G} \partial_\alpha X^\alpha. \tag{19}$$

3.2. The algebra of forms of $M_n(\mathbb{C})$. In this section we use the construction of the algebra of forms of a ρ-commutative algebra from [Bongaarts] for defining our construction of the algebra of forms of the algebra $M_n(\mathbb{C})$. Thus we obtain a new differential calculus on the matrix algebra.

We denote by $\Omega^p(M_n(\mathbb{C}))$ the space of $p-$forms and

$$\Omega(M_n(\mathbb{C})) = \bigoplus_{p\in\mathbb{Z}} \Omega^p(M_n(\mathbb{C}))$$

the algebra of forms of $M_n(\mathbb{C})$.

The bimodule $\Omega^1(M_n(\mathbb{C}))$ is also free of rank $2n$ with the basis dual to the basis $\{\partial_\alpha |\ \alpha \in G\} := \left\{\frac{\partial}{\partial p^i}, \frac{\partial}{\partial q^j} |\ i,j = \overline{1,n}\right\}$ of the bimodule ρ-Der$(M_n(\mathbb{C}))$.The basis of $\Omega^1(M_n(\mathbb{C}))$ is $\{d_\alpha |\ \alpha \in G\} := \left\{d_{p^i},\ d_{q^j} |\ i,j = \overline{1,n}\right\}$ with the relations:

$$d_{p^i}(\frac{\partial}{\partial p^j}) = 0 \text{ for } i \neq j,\ d_{p^i}(\frac{\partial}{\partial p^i}) = 1 \text{ and } d_{p^i}(\frac{\partial}{\partial q^j}) = 0, \tag{20}$$

$$d_{q^i}(\frac{\partial}{\partial q^j}) = 0 \text{ for } i \neq j,\ d_{p^i}(\frac{\partial}{\partial p^i}) = 1 \text{ and } d_{q^i}(\frac{\partial}{\partial p^j}) = 0. \tag{21}$$

For an easier writing the relations (20) and (21) can be written in the following compact form:

$$d_\alpha(\partial_\beta) = 0 \text{ for } \alpha \neq \beta, \text{ and } d_\alpha(\partial_\alpha) = 1. \tag{22}$$

Remark that the $G-$degree of the 1-form d_{p^k} is $\left|d_{p^k}\right| = (k,0)$ and of d_{q^k} is $\left|d_{q^k}\right| = (0,k)$.

An arbitrary 1-form α_1 can be written in the following way:

$$\alpha_1 = \sum_{\alpha \in G} d_\alpha A_\alpha := \sum_{i=1}^{n} d_{p^i} A_{p^i} + \sum_{j=1}^{n} d_{q^j} A_{q^j}$$

where $A_{p^i} = \alpha_1(\frac{\partial}{\partial p^i}) \in M_n(\mathbb{C})$ and $A_{q^j} = \alpha_1(\frac{\partial}{\partial q^j}) \in M_n(\mathbb{C})$, for $i, j = \overline{1, n}$, or using the compact form we have: $A_\alpha = \alpha_1(\partial_\alpha) \in M_n(\mathbb{C})$, for $\alpha \in G$.

Because $\Omega^1 (M_n(\mathbb{C}))$ is of finite rank $2n$, $\Omega^p (M_n(\mathbb{C}))$ is the pth exterior power of $\Omega^1 (M_n(\mathbb{C}))$, in the sense of $M_n(\mathbb{C})-$modules, and is again free of the rank $(p, 2n)$. An arbitrary $p-$form α_p can be written as

$$\alpha_p = \frac{1}{p!}(-1)^{\frac{p(p-1)}{2}} \sum_{i_1,\ldots,i_p=1}^{p} d_{\alpha_{i_1}} \wedge \ldots \wedge d_{\alpha_{i_{kp}}} A_{i_1 \ldots i_p},$$

with

$$A_{i_1 \ldots i_p} = \alpha_p(\partial_{\alpha_{i_1}}, \ldots, \partial_{\alpha_{i_p}}) \in M_n(\mathbb{C}).$$

From these considerations the algebra $\Omega (M_n(\mathbb{C}))$ is generated by the elements p^i, q^j for $i, j = \overline{1, n}$ and their differentials d_{p^i}, d_{q^j}, for $i, j = \overline{1, n}$ with the relations:

$$p^i q^j = \varepsilon^{ij} q^j p^i, \quad p^i p^j = p^j p^i \tag{23}$$

$$d_{p^i} d_{q^j} = -\varepsilon^{ij} d_{q^j} d_{p^i}, \quad d_{p^i} d_{p^j} = d_{p^j} d_{p^i} \tag{24}$$

and

$$p^i d_{q^j} = \varepsilon^{ij} d_{q^j} p^i, \quad q^i d_{p^j} = \varepsilon^{-ij} d_{p^j} q^i, \quad d_{p^i} p^j = p^j d_{p^i}, \quad q^i d_{q^j} = d_{q^j} q^i \tag{25}$$

3.3. The algebra of universal differential forms of $M_n(\mathbb{C})$. In this paragraph we present our construction of the algebra of universal differential forms of $M_n(\mathbb{C})$, using the construction from the paragraph 2.2.

Let $\alpha = (\alpha_1, \alpha_2) \in G = \mathbb{Z}_n \times \mathbb{Z}_n$ an arbitrary element. $\Omega^1_\alpha M_n(\mathbb{C})$ is the $M_n(\mathbb{C})-$bimodule generated by the elements adb, with $a, b \in M_n(\mathbb{C})$ which satisfies the properties:

1) $d(a + b) = da + db$,
2) $d(ab) = (da)b + \rho(\alpha, |a|)adb$,
3) $d1 = 0$, for any $a, b \in M_n(\mathbb{C})$, where 1 is the unit from $M_n(\mathbb{C})$.

From an easy computation we obtain the following relations:

Proposition 3. *1)* $p^k dp = (dp)p^k$ *and* $q^l dq = (dq)q^l$

1) $dp^k = \begin{cases} 0 \text{ if } k=0 \\ (1+\varepsilon^{-\alpha_2}+\varepsilon^{-2\alpha_2}+...+\varepsilon^{(k-1)\alpha_2})p^{k-1}dp \text{ if } k\in\{1,...,n-1\} \end{cases}$

2) $dp^s = \begin{cases} 0 \text{ if } k=0 \\ (1+\varepsilon^{-\alpha_1}+\varepsilon^{-2\alpha_1}+...+\varepsilon^{(s-1)\alpha_1})p^{s-1}dp \text{ if } s\in\{1,...,n-1\} \end{cases}$

For an easier writing we make the following notations:

$$\varepsilon^{\alpha}_{(k,0)} = \begin{cases} 0 \text{ if } k=0 \\ 1+\varepsilon^{-\alpha_2}+\varepsilon^{-2\alpha_2}+...+\varepsilon^{-(k-1)\alpha_2} \text{ if } k\in\{1,...,n-1\} \end{cases}$$

and

$$\varepsilon^{\alpha}_{(0,s)} = \begin{cases} 0 \text{ if } s=0 \\ 1+\varepsilon^{-\alpha_1}+\varepsilon^{-2\alpha_1}+...+\varepsilon^{-(s-1)\alpha_1} \text{ if } s\in\{1,...,n-1\} \end{cases}$$

From the properties of the derivation d and from the proposition 6 we obtain:

Proposition 4. $d(p^kq^s) = \varepsilon^{\alpha}_{(k,0)}p^{k-1}(dp)q^s + \varepsilon^{-\alpha_2}\varepsilon^{\alpha}_{(0,s)}p^kq^{s-1}dq,$
for any $k, s \in \{0, ..., n-1\}$.

Putting together the propositions 6 and 7 we obtain the structure of the $M_n(\mathbb{C})$-bimodule $\Omega^1_\alpha M_n(\mathbb{C})$:

Theorem 5. $\Omega^1_\alpha M_n(\mathbb{C})$ *is generated by the elements* $p^i, q^j, dp^k, dq^s, i, j, k, s \in \{0, ..., n-1\}$ *with the relations:*

1) $p^kq^s = \varepsilon^{ks}q^sp^k,$

2) $p^kdp^l = (dp^l)p^k$ *and* $q^sdq^l = (dq^l)q^s,$

3) $dp^k = \varepsilon^{\alpha}_{(k,0)}p^{k-1}dp$ *and* $dq^s = \varepsilon^{\alpha}_{(0,s)}q^{s-1}dq$

4) $d(p^kq^s) = \varepsilon^{\alpha}_{(k,0)}\varepsilon^s p^{k-1}q^s(dp) + \varepsilon^{-\alpha_2}\varepsilon^{\alpha}_{(0,s)}p^kq^{s-1}dq,$

for any $k, l, s \in G$.

The $M_n(\mathbb{C})-$bimodule $\Omega^k_\alpha M_n(\mathbb{C})$ is again free and an arbitrary element $\omega_k \in \Omega^k_\alpha M_n(\mathbb{C})$ can be written

$$\omega^k = \sum_{l+s=k} A_{l+s}(dp)^l(dq)^s, \tag{26}$$

where $A_{l,s} \in M_n(\mathbb{C})$.

From these considerations we obtain the following theorem which gives the structure of the algebra $\Omega_\alpha M_n(\mathbb{C})$.

Theorem 6. *The algebra* $\Omega_\alpha M_n(\mathbb{C})$ *is generated by the elements* $p^i, q^j, (dp)^k := P^k, (dq)^s := Q^s$, $i, j \in \{0, ..., n-1\}$,$k, s \in \mathbb{Z}$ *with the relations:*

1) $p^iq^j = \varepsilon^{ij}q^jp^i$, $P^kQ^s = (-1)^{ks}\varepsilon^{ks}Q^sP^k,$

2) $p^kP^s = P^sp^k$, $q^kQ^s = Q^sq^k$

3) $p^kQ^s = \varepsilon^{ks}Q^sp^k$, $q^kP^s = \varepsilon^{-ks}P^sq^k$.

3.4. Linear connections on $M_n(\mathbb{C})$. Next we introduce linear connections on the algebra $M_n(\mathbb{C})$. A linear connection on $M_n(\mathbb{C})$ is a linear map

$$\nabla : \rho - \mathrm{Der} M_n(\mathbb{C}) \to \mathrm{End}(\rho - \mathrm{Der} M_n(\mathbb{C}))$$
$$\nabla_X : (\rho - \mathrm{Der} M_n(\mathbb{C}))_* \to (\rho - \mathrm{Der} M_n(\mathbb{C}))_{*+|X|} \tag{27}$$

satisfying the equations (13). Any linear connection is well defined if are given the **connections coefficients** on the basis $\{\partial_\alpha | \alpha \in G\}$:

$$\partial_\sigma \Gamma^\sigma_{\alpha,\beta} = \nabla_{\partial_\alpha} \partial_\beta \tag{28}$$

for any $\alpha, \beta \in G$.

The **curvature** R of the connection ∇ is given by the **curvature coefficients:** $R^\sigma_{\alpha,\beta,\tau}$

$$\partial_\sigma R^\sigma_{\alpha,\beta,\tau} = [\nabla_{\partial_\alpha}, \nabla_{\partial_\beta}](\partial_\tau) - \nabla_{[\partial_\alpha, \partial_\beta]}(\partial_\tau). \tag{29}$$

From an easy computation it follows that

$$\begin{aligned} R^\sigma_{\alpha,\beta,\tau} &= \partial_\alpha(\Gamma^\sigma_{\beta,\tau}) - \rho(\alpha, \left|\Gamma^\mu_{\beta,\tau}\right|)\Gamma^\sigma_{\alpha,\mu} - \\ &\quad -\rho(\alpha,\beta)(\partial_\beta \Gamma^\sigma_{\alpha,\tau} - \rho(\beta, \left|\Gamma^\mu_{\beta,\sigma}\right|)\Gamma^\sigma_{\beta,\mu}) \end{aligned} \tag{30}$$

for any $\alpha, \beta, \tau, \sigma \in G$.

The torsion of the connection ∇ is well defined by the torsion coefficients:

$$T(\partial_\alpha, \partial_\beta) = \partial_\sigma T^\sigma_{\alpha,\beta} \tag{31}$$

and the relations between connections coefficients and the **torsion coefficients** are:

$$T^\sigma_{\alpha,\beta} = \Gamma^\sigma_{\alpha,\beta} - \rho(\alpha,\beta)\Gamma^\sigma_{\beta,\alpha}. \tag{32}$$

Linear connections on $\Omega^1_\alpha\left(M_n(\mathbb{C})\right)$. Any linear connection ∇ on the $M_n(\mathbb{C})$-bimodule $\Omega^1_\alpha(M_n(\mathbb{C}))$ is given by the connection coefficients thus (using the compact formula):

$$\nabla_{\partial_\beta} d_\alpha p = \Gamma^{p,p}_\beta d_\alpha p + \Gamma^{p,q}_\beta d_\alpha q,$$

Γ^p_β, $\Gamma^q_\beta \in M_n(\mathbb{C})$.

For example we have

$$\nabla_{\frac{\partial}{\partial p^i}} (d_\alpha p) = \Gamma^{p,p}_i d_\alpha p + \Gamma^{p,q}_i d_\alpha q$$

and

$$\begin{aligned} \nabla_{\frac{\partial}{\partial p^i}} \left(p^k d_\alpha p\right) &= \frac{\partial}{\partial p^i}\left(p^k\right) d_\alpha p + \rho\left(\left(-i,0\right),\left(k,0\right)\right) p^k \nabla_{\frac{\partial}{\partial p^i}} \left(d_\alpha p\right) = \\ &= \frac{k}{i} p^{k-i} d_\alpha p + p^k \Gamma^{p,p}_i d_\alpha p + \Gamma^{p,q}_i d_\alpha q. \end{aligned}$$

3.5. Distributions. In this subsection we introduce distributions on the matrix algebra $M_n(\mathbb{C})$ over the differential calculi $\Omega\left(M_n(\mathbb{C})\right)$ and $\Omega_\alpha\left(M_n(\mathbb{C})\right)$. In each of these situation we give characterizations of globally integrable distributions and globally integrable distributions parallel with respect to a connection ∇ on $\Omega^1\left(M_n(\mathbb{C})\right)$ and $\Omega^1_\alpha\left(M_n(\mathbb{C})\right)$.

Distributions on $\Omega\left(M_n(\mathbb{C})\right)$. From the definition 6 a distribution $\mathcal{D}$ on $\Omega\left(M_n(\mathbb{C})\right)$ is a $M_n(\mathbb{C})$ ρ-sub-bimodule of $\Omega\left(M_n(\mathbb{C})\right)$. The distribution $\mathcal{D}$ is globally integrable if there is a subspace B of $M_n(\mathbb{C})$ so that $\mathcal{D}$ is generated by $M_n(\mathbb{C})d\left(B\right)$, so the determination of this kind of distributions is reduced to the determination of subalgebras from $M_n(\mathbb{C})$.

Let $D_n = \{k \in \mathbb{N} \text{ such that } k|n\}$ be the set of all natural divisors of n. Then for any subalgebra B of $M_n(\mathbb{C})$ there are $k, s \in D_n$ such that B is generated by the set $\left\{p^{k\cdot i}q^{s\cdot j},\ i,j \in \mathbb{Z}\right\}$. Consequently we have the following result.

Proposition 7. *For any globally integrable distribution $\mathcal{D}$ of $\Omega\left(M_n(\mathbb{C})\right)$ there are $k, s \in D_n$ such that $\mathcal{D}$ is generated by the elements p^i, q^j for $i, j = \overline{1,n}$ and the differentials $d_{p^{i\cdot k}}$, $d_{q^{j\cdot s}}$, for $i, j = \overline{1,n}$.*

Remark 4. *Without any confusion the previous proposition may be written using the compact form (19 and 22): for any globally integrable distribution $\mathcal{D}$ of $\Omega\left(M_n(\mathbb{C})\right)$ there is a subgroup H of G such that $\mathcal{D}$ is generated by the elements $a_\alpha db_\beta$, with a_α, $b_\beta \in M_n(\mathbb{C})$ with $\alpha \in G$ and $\beta \in H$.*

Remark 5. *If $\mathcal{D}$ is a globally integrable distribution of $\Omega\left(M_n(\mathbb{C})\right)$ of the dimension p then p is a divisor of n^2.*

It is obvious that a linear connection ∇ on the $M_n(\mathbb{C})$-bimodule $\Omega^1(M_n(\mathbb{C}))$ is given by their connection coefficients, again denoted by, $\Gamma^\sigma_{\alpha,\beta} \in M_n(\mathbb{C})$ and they are given by the following equation:

$$\nabla_{\partial_\alpha} d_\beta = \Gamma^\sigma_{\alpha,\beta} d_\sigma, \tag{33}$$

for any $\alpha, \beta \in G$.

Proposition 8. *Any globally integrable and parallel distribution $\mathcal{D}$ with respect to a connection ∇ : ρ-Der$M_n(\mathbb{C})$ $\rightarrow$End$(\Omega^1\left(M_n(\mathbb{C})\right))$ of dimension p is given by the following equations:*

$$\Gamma^\sigma_{\alpha,\beta} = 0 \tag{34}$$

for a subgroup H of G and for any $\sigma, \alpha \in G$, $\beta \in H$.

Distributions on $\Omega_\alpha(M_n(\mathbb{C}))$. Let $\alpha \in G$. Any distribution $\mathcal{D}$ on $\Omega_\alpha(M_n(\mathbb{C}))$ is a $M_n(\mathbb{C})$ ρ-sub-bimodule of $\Omega_\alpha(M_n(\mathbb{C}))$.

Using the structure of $\Omega_\alpha(M_n(\mathbb{C}))$ (theorem 6) we have that any globally integrable distribution $\mathcal{D}$ on $\Omega_\alpha(M_n(\mathbb{C}))$ is one of the following subalgebras: $M_n(\mathbb{C})$, $M_n(\mathbb{C})d_\alpha p$, $M_n(\mathbb{C})d_\alpha q$ and $\Omega_\alpha(M_n(\mathbb{C}))$, consequently any globally integrable distribution on $\Omega_\alpha(M_n(\mathbb{C}))$ has the dimension 0, 1 or 2.

Any globally integrable and parallel distribution $\mathcal{D}$ with respect to a connection $\nabla : \rho\text{-Der}M_n(\mathbb{C}) \to \mathrm{End}(\Omega^1_\alpha(M_n(\mathbb{C})))$ of dimension 1 is given by the following equations:

$$\Gamma_\beta^{p,p} = \Gamma_\beta^{p,q} = 0 \tag{35}$$

if $\mathcal{D}$ is $M_n(\mathbb{C})d_\alpha p$ and

$$\Gamma_\beta^{q,p} = \Gamma_\beta^{q,q} = 0$$

if is $M_n(\mathbb{C})d_\alpha q$, for any $\beta \in G$.

Conclusions and remarks. In this paper we present the principal notions from the (noncommutative) geometry as differential calculus and linear connections and distributions no the matrix algebra $M_n(\mathbb{C})$ using methods of ρ-algebras.

There are more geometrical objects to introduce on matrix algebra such as: tensors, metrics, simpletic forms etc.

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