A structural approach to unambiguous state discrimination

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Quantum Information Theory in Düsseldorf

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SECOQC   scala

Outline

- Introduction and motivation:
  Unambiguous State Discrimination (USD)

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- Uniqueness of optimal measurement

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- Four-dimensional solution

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- Impossibility of perfect state discrimination
  $\leftrightarrow$ impossibility of perfect quantum cloning
- Problem for quantum computing:
  perfect read-out of non-orthogonal states impossible
- $\hookrightarrow$ Discrimination of quantum states is fundamental issue in quantum information and quantum computing

Two main strategies for state discrimination

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- Minimum error discrimination (MED):
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- Unambiguous state discrimination (USD):
  No error allowed, but inconclusive answer. Minimize probability to get inconclusive answer.
  (This talk: USD only)

Pure states, equal *a priori* probabilities, "simple" measurements

[Ivanovic 1987, Dieks 1988, Peres 1988]

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Projective measurement, detects $|\psi_2\rangle$ unambiguously:
$P_1 = |\psi_1\rangle\langle\psi_1|, \quad \bar{P}_1 = |\bar{\psi}_1\rangle\langle\bar{\psi}_1|$, with $P_1 + \bar{P}_1 = \mathbb{1}$ and $\langle\psi_1|\bar{\psi}_1\rangle = 0$

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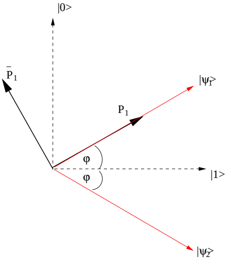

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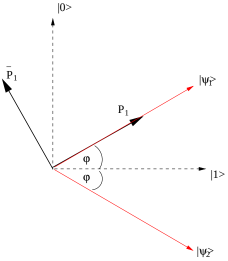

$$p_{\rm succ} = 1 - |\langle\psi_1|\psi_2\rangle|$$

Pure states, unequal *a priori* probabilities, general measurements (POVM)

[Jaeger and Shimony 1995]

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$$E_1 = \mathbb{1} - |\psi_2\rangle\langle\psi_2|, \quad E_2 = \mathbb{1} - |\psi_1\rangle\langle\psi_1|, \quad E_? = \mathbb{1} - E_1 - E_2$$

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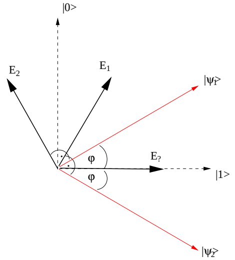

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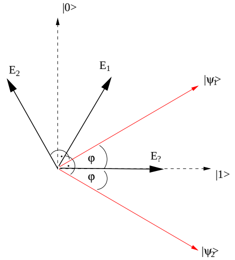
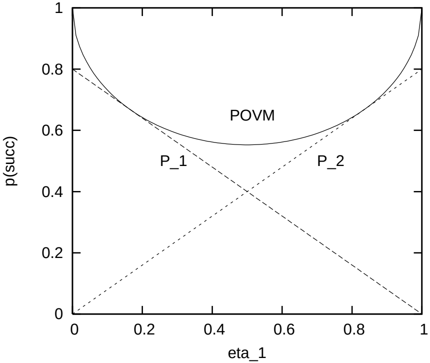


$$p_{\mathrm{succ}} = \begin{cases} 1 - \eta_1 - \eta_2 c^2 & \\ 1 - 2c\sqrt{\eta_1\eta_2} & \text{for} \\ 1 - \eta_2 - \eta_1 c^2 & \end{cases} \quad \begin{array}{l} \eta_1 < \frac{c}{1+c^2} \\ \frac{c}{1+c^2} \leq \eta_1 \leq \frac{1}{1+c^2} \\ \frac{1}{1+c^2} < \eta_1 \end{array}$$

with $c = |\langle\psi_1|\psi_2\rangle|$

Success probability for pure states



here: $c = |\langle\psi_1|\psi_2\rangle| = 0.1$

Unambiguous state discrimination of mixed states

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- *A priori* probabilities: ϱ_i occurs with probability η_i. Abbreviation: $\gamma_i = \eta_i \varrho_i$, with $i = 1, 2$.

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- Maximize the success probability: $p_{\mathrm{succ}} = \mathrm{tr}(E_1\gamma_1) + \mathrm{tr}(E_2\gamma_2)$
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- Choose $\mathcal{H} = \mathrm{supp}\,\gamma_1 + \mathrm{supp}\,\gamma_2$.

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  `[Kleinmann, Kampermann and Bruß, in preparation]`
  $\hookrightarrow$ this talk

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Theorem

For $E_?$ optimal,

- $\operatorname{supp} E_? \cap \ker \gamma_1 = \operatorname{supp} E_? \cap \ker \gamma_2 = \{0\}$,
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To remember: For optimal USD measurement $\operatorname{rank} E_?$ is fixed.

The optimality conditions by Eldar *et al.*

Theorem (Eldar, Stojinc & Hassibi, 2004)

$\{E_1, E_2, E_?\}$ is optimal, if and only if there exists a Z, such that

$$Z \geq 0, \quad ZE_? = 0$$

$$\Lambda_i(Z - \gamma_i)\Lambda_i \geq 0, \quad \text{and} \quad \Lambda_i(Z - \gamma_i)E_i = 0 \quad (i = 1, 2).$$

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Z is over-determined and can be eliminated:

Corollary

$E_?$ is optimal, if and only if

$$(\Lambda_1 - \Lambda_2)E_?(\gamma_2 - \gamma_1)E_?(\Lambda_1 + \Lambda_2) \geq 0 \quad \text{and}$$
$$(\Lambda_1 - \Lambda_2)E_?(\gamma_2 - \gamma_1)E_?(\mathbb{1} - E_?) = 0.$$

Hidden equation (from Hermiticity condition of LHS of inequality):

$$\Lambda_1 E_?(\gamma_2 - \gamma_1)E_?\Lambda_2 = 0.$$

Example for strength of Corollary: Single state detection

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Example:

- When $E_1 = 0 \hookrightarrow E_? = \mathbb{1} - \Lambda_2$, i.e $E_?\gamma_1 = \gamma_1$.
- Then $E_?(\mathbb{1} - E_?) = 0$ and $E_?\Lambda_2 = 0$.
- Hence $\Lambda_1(\mathbb{1} - \Lambda_2)(\gamma_2 - \gamma_1)(\mathbb{1} - \Lambda_2)\Lambda_1 \geq 0$ remains.

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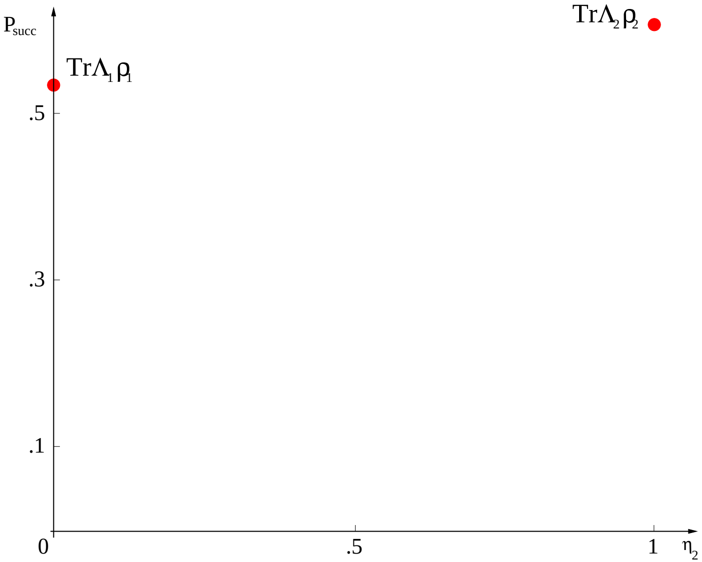
- Hence $\Lambda_1(\mathbb{1} - \Lambda_2)(\gamma_2 - \gamma_1)(\mathbb{1} - \Lambda_2)\Lambda_1 \geq 0$ remains.

Suppose that $\operatorname{supp}\gamma_1 \cap \operatorname{supp}\gamma_2 = \{0\}$.

Single state detection

$E_1 = 0$ is optimal if and only if $\gamma_1(\gamma_2 - \gamma_1)\gamma_1 \geq 0$.

General form of success probability



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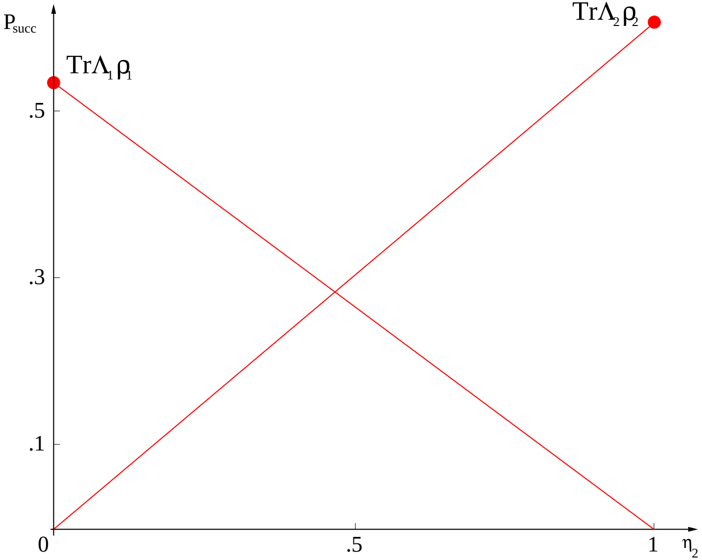

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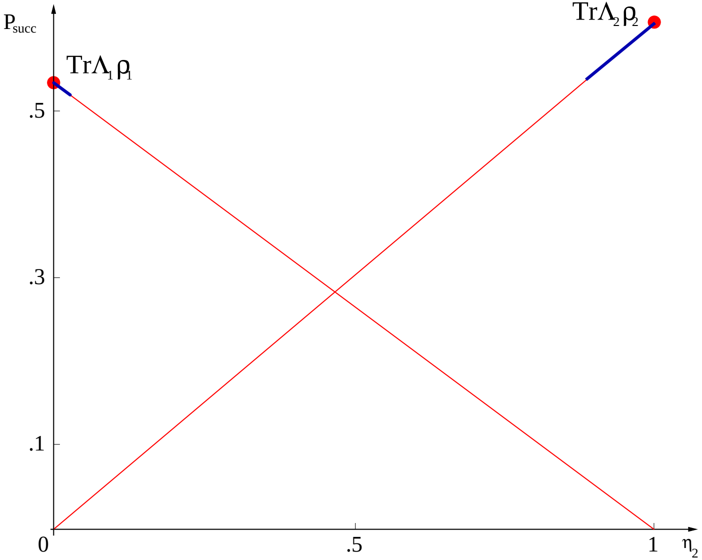

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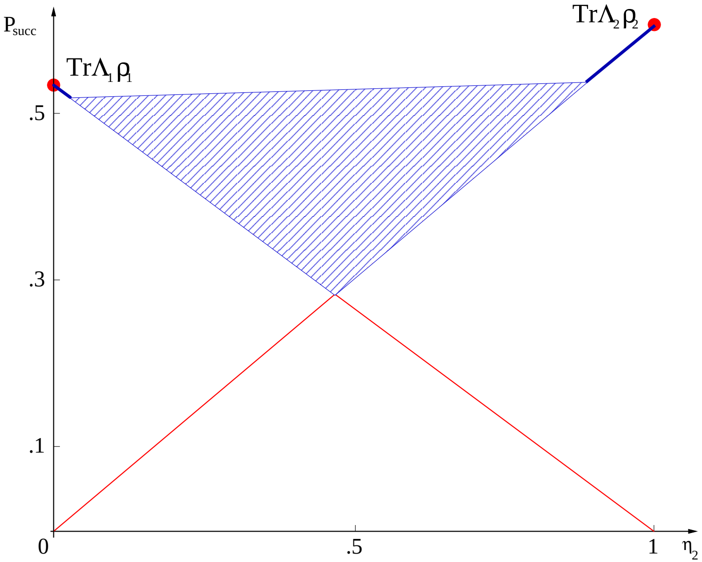

Fidelity form of USD measurements

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E_? = (\gamma_1+\gamma_2)^{-1}\Big\{&\gamma_1\gamma_2+\gamma_2\gamma_1 \\
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\end{aligned}
$$

$\Pi_?$ projector onto $\operatorname{supp} E_?$, Δ projector onto $\ker(\mathbb{1} - E_?)$

Lemma

For $E_?$ optimal, $E_?(\gamma_2-\gamma_1)E_? \stackrel{\text{Eldar}}{=} \Pi_?(\gamma_2-\gamma_1)\Pi_? = \Delta(\gamma_2-\gamma_1)\Delta$.

Fidelity form of USD measurements

- $\operatorname{supp}\gamma_1 \cap \operatorname{supp}\gamma_2 = \{0\}$ (otherwise: reduction, see Raynal *et al* 2003)
- Any USD measurement satisfies

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\begin{aligned}
E_? = (\gamma_1+\gamma_2)^{-1}\Big\{&\gamma_1\gamma_2+\gamma_2\gamma_1 \\
&+\sqrt{\gamma_1}\sqrt{\sqrt{\gamma_1}[\gamma_2 - E_?(\gamma_2-\gamma_1)E_?]\sqrt{\gamma_1}}\sqrt{\gamma_1} \\
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The optimal USD measurement is unique.

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- $\operatorname{supp}\gamma_1 \cap \operatorname{supp}\gamma_2 = \{0\}$ and $\operatorname{supp}\gamma_1 \cap \ker\gamma_2 = \{0\}$ and $\ker\gamma_1 \cap \operatorname{supp}\gamma_2 = \{0\}$. `[Raynal, Lütkenhaus and van Enk 2003]`

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$\operatorname{rank}\Delta = 0$: The optimal solution is already given by $(*)$.
$\operatorname{rank}\Delta = 1$: We have $\Lambda_1\,\Delta(\gamma_2-\gamma_1)\Delta\,\Lambda_2 = 0$.

$$\Longrightarrow \left\{ \begin{array}{l} \left.\begin{array}{l}\operatorname{supp}\Delta \subseteq \operatorname{supp}\gamma_1 \\ \operatorname{supp}\Delta \subseteq \operatorname{supp}\gamma_2\end{array}\right\rangle \quad \hookrightarrow \text{unknown vector in 2 dim} \\ \Delta(\gamma_2-\gamma_1)\Delta = 0 \qquad\quad \hookrightarrow (*). \end{array}\right.$$

Solution in four dimensions (ii)

$\mathrm{rank}\,\Delta = 2$: Then $\mathrm{rank}\,E_1 + \mathrm{rank}\,E_2 = 2$ and $E_i^2 = E_i$

$$\Longrightarrow \begin{cases} \left.\begin{array}{l} \mathrm{rank}\,E_1 = 0 \\ \mathrm{rank}\,E_2 = 0 \end{array}\right\rangle & \hookrightarrow \text{single state detection} \\ \mathrm{rank}\,E_1 = 1 = \mathrm{rank}\,E_2 & \hookrightarrow \mathrm{supp}\,E_2 = \mathrm{supp}(\Lambda_2 - \Lambda_2 E_1 \Lambda_2) \\ & \quad \hookrightarrow \text{unknown vector in 2 dim} \end{cases}$$

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Remaining step: Show that the equation only has a **finite** number of solutions.

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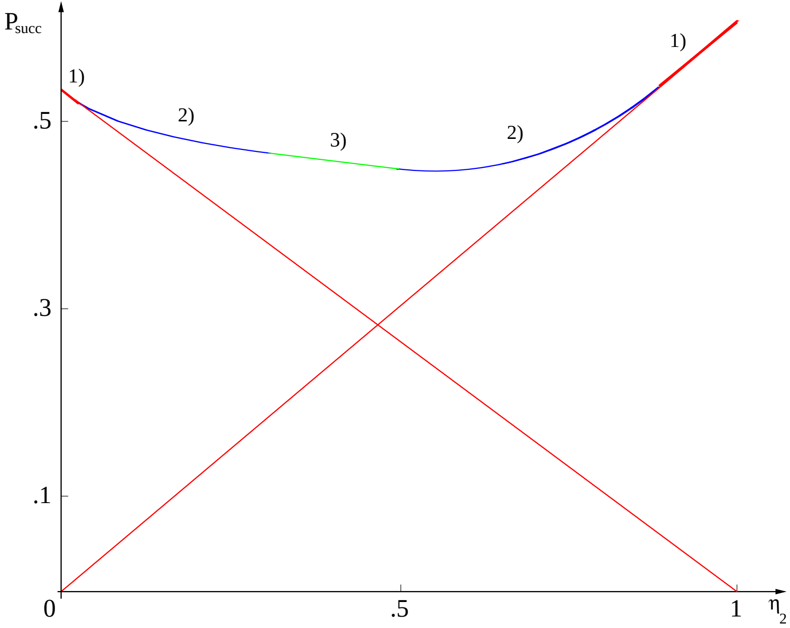
4) The "fidelity form" $(\Delta(\gamma_2 - \gamma_1)\Delta = 0$, $\mathrm{rank}\, \Delta = 0)$.
   [Herzog & Bergou 2005, Raynal and Lütkenhaus 2005]

Typical example in four dimensions



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