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The integrated density of states (IDS) in strong magnetic fields

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1 Self-adjointness

The configuration space is $\mathbb{R}^3$ and denote

$$\mathbb{R}^3 \ni \mathbf{x} = (X_\perp, x_3), \quad X_\perp = (x_1, x_2).$$

The magnetic field $\mathbf{B} = b(0,0,1), b \geq 0$
The vector potential $\mathbf{A} := \frac{1}{2}\mathbf{x}\times\mathbf{B}$ or $\mathbf{A} := \frac{b}{2}(-x_2, x_1, 0)$ (the transversal gauge)
Consider the following magnetic Schrödinger operators

$$H(b) = H_0(b) + V \quad \text{acting on} \quad L^2(\mathbb{R}^3)$$

where

$$H_0(b) = H_{\perp,0}(b) \otimes \mathbb{I} + \mathbb{I} \otimes -\partial_{x_3}^2$$

and

$$H_{\perp,0}(b) = \left(i\frac{\partial}{\partial x_1} - \frac{bx_2}{2}\right)^2 + \left(i\frac{\partial}{\partial x_2} + \frac{bx_1}{2}\right)^2 - b,$$

is the 2-d Landau hamiltonian.

$H_0(b), b \geq 0$ is e.s.a. on $C_0^\infty(\mathbb{R}^3)$, $\sigma(H_0(b))$ is purely absolutely continuous and :

$$\sigma(H_0(b)) = \{2bq\}_{q\in\mathbb{Z}_+}" + "[0,\infty)$$

$2bq$, $q \in \mathbb{Z}_+$ are the Landau levels.

V is a real random electric potential :
Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space, and

$$\mathbb{E}(\cdot) = \int (\cdot) d\mathbb{P}(\omega)$$

denotes the mathematical expectation.

Main assumption : h_S, $S = \mathbb{R}$ or $S = \mathbb{Z}$

$$\Omega \times \mathbb{R}^3 \ni (\omega, x) \mapsto V_\omega(x) \in \mathbb{R}$$

is a S^3- ergodic real random field which is $S-$ ergodic in the direction of the magnetic field (x_3-direction) and :

$$\mathbb{E}\left(\int_{\mathcal{C}} |V_\omega(\mathbf{x})|^4 d\mathbf{x} \right) < \infty; \;\; \mathcal{C} := \left(-\frac{1}{2}, \frac{1}{2} \right)^3$$

If V_ω is $\mathbb{R}^3$-ergodic, then

$$h_{\mathbb{R}} \longrightarrow \mathbb{E}\left(|V_\omega(\mathbf{0})|^4 \right) < \infty.$$

Random field, ergodicity ... $\longrightarrow$ Figotin Pastur (Springer 1992) or Kirsch (Lect. notes. Phys 1989).

Theorem 1.1. *(Hupfer et al, Rev Math. Phys. 2001) Under the condition h_S stated above the operators $H_\omega(b) = H_0(b) + V_\omega, b \geq 0$ is essentially self adjoint on $C_0^\infty(\mathbb{R}^3)$, with probability one.*

2 Exemples

Englisch et al, (Comm.Math.Phys. 1990), Fischer et al, (Jour. Stat Phys 2001), Hupfer et al RMP 2001.......

- Homogeneous $\mathbb{R}^3-$random potential, *Gaussian potentials*: V_ω is a real homogeneous Gaussian random field with $\mathbb{E}(V_\omega(0)) = 0$, a correlation funct. $C(\mathbf{x}) = \mathbb{E}(V_\omega(\mathbf{x})V_\omega(0))$ is continuous at 0 and $C(\mathbf{x}) \to \mathbf{0}$ as $\mathbf{x} \to \infty$. V_ω satisfies $h_\mathbb{R}$

- Homogeneous $\mathbb{Z}^3-$random potential, *Anderson type of potentials*: Let $\{q_j\}_{j\in\mathbb{Z}}$ be a sequence of i.i.d. real random variables and $u : \mathbb{R}^3 \to \mathbb{R}$ be the atomic function s.t.

$$\mathbb{E}(|q|^4) < \infty \quad \text{and} \quad \sum_{j\in\mathbb{Z}} (\int_\mathcal{C} |u(x-j)|^4)^{1/4} < \infty.$$

$$V_\omega(\mathbf{x}) = \sum_{j\in\mathbb{Z}} q_j u(x-j)$$

the Minkowski iniquality (c.f. Rudin Mac Graw Hill 1966), $\to V_\omega = \sum_j q_j u(x-j)$ satisfies $h_\mathbb{Z}$.

3 The Integrated density of states

Figotin Pastur, Kirsch (lect. notes in Math 1985)

Denote by $\chi_{\mathcal{A}}$ the characteristic function of the set A,

Theorem 3.1. *(Hupfer et al, Rev Math. Phys. 2001, Figotin-Pastur) Under the condition h_S stated above, the IDS associated to the operators $H_\omega(b), b \geq 0$,*

$$\rho(E) := \mathbb{E}\{\mathrm{Tr}(\chi_{\mathcal{C}}\ \chi_{(-\infty,E)}(H_\omega(b))\chi_{\mathcal{C}})\}$$

exists for all energies $E \in \mathbb{R}$.

Remark: With probability one we have:

$$\rho(E) = \lim_{\Lambda_i \uparrow \mathbb{R}^3} \frac{\mathrm{N}_{\Lambda_i}(\omega, E)}{|\Lambda_i|}$$

for all $E \in \mathbb{R}$ where ρ is continuous. This is true for sequences $\{\Lambda_i\}_{i\in\mathbb{N}}$ of domains of $\mathbb{R}^3 \to \mathbb{R}^3$ (in the Fischer sense).

$\mathrm{N}_{\Lambda_i}(\omega, E)\sharp :=$ of eigenvalues $< E$ of the Dirichlet restrict. of $H_\omega(b)$ on Λ_i

Also with probability one

$$\rho(E) = \lim_{\Lambda_i \uparrow \mathbb{R}^3} \frac{\mathrm{Tr}(\chi_{\Lambda_i}\ \chi_{(-\infty,E)}(H_\omega(b)\chi_{\Lambda_i}))}{|\Lambda_i|}$$

for all $E \in \mathbb{R}$ where ρ is continuous.

4 $\mathbb{R}^3-$ergodic random electric potentials

Recall that here we are in the regime $b \to \infty$.

Avron, Herbst and Simon (Comm. Math. Phys. 1981) :

for large b the quantum particle will move in tight landau level in the $x_1 - x_2$ plane and will be bound in the x_3 direction by the potential (Coulomb) V : the motion becomes $1 - d$ in the direction of the magnetic field $\mathbf{B}$*.*

$$H_\omega(b) \longrightarrow H_{eff} = -\partial_{x_3}^2 + V_{eff} \quad \text{on} \quad L^2(\mathbb{R}_{x_3})$$

Two questions: -definition of V_{eff}?
-the sense of the convergence?

$\longrightarrow$

-the " stability of the Matter" see also Bamgartner, Solovej, Lieb, Ruskaï, ... and more recently by Brummelhuis and Duclos.

- the eigenvalue asymptotics, see Raikov (1998), Ivrii (1989)...

-The IDS, Kirsch, Raikov (Ann Henri Poincare 2000), Raikov (Op. Theory 2001), Warzel (PHD 2001).
Suppose first $V_\omega = 0$, then the IDS is given by :

$$\rho(E) := \frac{b}{2\pi^2} \sum_{q=0}^{\infty} (E - 2bq)_+^{1/2}.$$

let $\lambda_1 \leq \lambda_2$ fixed. Different asymptotic as $b \to \infty$ of

$$\rho(\mathcal{E} b + \lambda_2) - \rho(\mathcal{E} b + \lambda_1)$$

1- "near a given landau level": $\mathcal{E} \in 2\mathbb{Z}_+$

2-"far from the landau levels" : $\mathcal{E} \notin 2\mathbb{Z}_+$

In the case 1 the following holds :

$$\lim_{b\to\infty} b^{-1} \left(\rho(\mathcal{E} b + \lambda_2) - \rho(\mathcal{E} b + \lambda_1)\right) = \frac{1}{2\pi^2} \left((\lambda_2)_+^{1/2} - (\lambda_1)_+^{1/2} \right),$$

while in the case 2:

$$\lim_{b\to\infty} b^{-1/2} \left(\rho(\mathcal{E} b + \lambda_2) - \rho(\mathcal{E} b + \lambda_1)\right) = \frac{\lambda_2 - \lambda_1}{4\pi^2} \sum_{q=0}^{[\mathcal{E}/2]} (\mathcal{E} - 2q)^{-1/2}.$$

Remark

- The second relation is a typical "free behavior", in this case the leading term of the IDS is order $b^{1/2}$, while near a given Landau level it is of order b .

- Let $\rho_{\|}$ be the IDS associated to $H_{\|} = -\partial^2_{x_3}$ i.e. $\rho_{\|}(\lambda) = (1/\pi)\lambda_+^{1/2}$. the first case can be read as

$$\lim_{b\to\infty} b^{-1} \left(....\right)) = \frac{1}{2\pi} \left(\rho_{\|}(\lambda_2) - \rho_{\|}(\lambda_1)\right),$$

So in the case $V = 0$, $V_{eff} = 0$ and this is a convergence relation evoqued above.

Consider the "effective" operator: $H_{\|,\omega} := -\frac{d^2}{dx_3^2} + v(x_3)$ with

$$v(x_3) = v_\omega(x_3) := V_\omega(0, x_3)$$

Due to the our assumption about V_ω, $H_{\|,\omega}$ is almost-surely e.s.a. on $C_0^\infty(\mathbb{R})$ (see Figotin-Pastur); Denote by $\rho_\|$ the IDS of $H_{\|,\omega}$. We have:

Theorem 4.1. *Let V_ω a random electric potential satisfying $h_\mathbb{R}$. Then we have*

i) if $\mathcal{E} \in 2\mathbb{Z}_+$, and $\lambda_1, \lambda_2 \in \mathbb{R}$, $\lambda_1 < \lambda_2$,

$$\lim_{b\to\infty} b^{-1} \left(\rho(\mathcal{E}b + \lambda_2) - \rho(\mathcal{E}b + \lambda_1)\right) = \frac{1}{2\pi}\left(\rho_\|(\lambda_2) - \rho_\|(\lambda_1)\right).$$

ii) and if $\mathcal{E} \notin 2\mathbb{Z}_+$,

$$\lim_{b\to\infty} b^{-1/2} \left(\rho(\mathcal{E}b + \lambda_2) - \rho(\mathcal{E}b + \lambda_1)\right) = \frac{\lambda_2 - \lambda_1}{4\pi^2} \sum_{q=0}^{[\mathcal{E}/2]} (\mathcal{E} - 2q)^{-1/2}.$$

Remark

- $V_{eff} = V(0, x_3)$

-Under additional assumptions the result ”near the landau level” is proved by Kirsch-Raikov for the first landau level (AHP-2000)and for higher landau levels by Raikov (Op. Theory 2001).

5 $\mathbb{Z}^3-$ergodic random electric potentials: the suspension method

See e.g. Kirsch (lect notes in Math 1989) and Figotin Pastur (springer 1992).
Suppose now V_ω a real random electric field satisfying $h_{\mathbb{Z}}$. Let θ be a real random variable uniformly distributed on $\Omega_0 = (-1/2, 1/2)^3$ and denote by

$$(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}) = (\Omega, \mathcal{F}, \mathbb{P}) \otimes (\Omega_0, \mathcal{F}_0, \mathbb{P}_0)$$

$\mathcal{F}_0$, being the Borel sigma algebra on Ω_0 and $\mathbb{P}_0$ the Lebesgue measure. Define

$$\tilde{V}_{\omega,\theta}(\mathbf{x}) = V_\omega(\mathbf{x}+\theta), \quad \omega \in \Omega, \quad \theta \in \left(-\frac{1}{2}, \frac{1}{2}\right)^3, \quad \mathbf{x} \in \mathbb{R}^3.$$

Then the potential $\tilde{V}_{\omega,\theta}$ is $\mathbb{R}^3$-ergodic and $\mathbb{R}$-ergodic in the direction of the magnetic field on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$. Moreover, due to the unitary equivalence of $\tilde{H}_\omega := H_0(b) + \tilde{V}_{\omega,\theta}$ and $H_\omega = H_0(b) + V_\omega$, we have

$$\varrho(\lambda) = \tilde{\varrho}(\lambda), \quad \lambda \in \mathbb{R}.$$

where $\tilde{\rho}$ is the IDS associated to $\tilde{H}_\omega$. Then the theorem follows for $\mathbb{Z}^3$-ergodic potentials from the theorem applied to the potential $\tilde{V}_\omega$.

Notice that now near the landau levels our relation read as

$$\lim_{b\to\infty} b^{-1}\left(\rho(\mathcal{E}b+\lambda_2)-\rho(\mathcal{E}b+\lambda_1)\right) = \frac{1}{2\pi}\left(\tilde{\rho}_{\|}(\lambda_2)-\tilde{\rho}_{\|}((\lambda_1)\right).$$

with

$$\tilde{\rho}_{\|}(\lambda) = \int_{\left(-\frac{1}{2},\frac{1}{2}\right)^2} \rho_{\|}(\lambda;\theta_\perp)d\theta_\perp \quad \lambda \in \mathbb{R}. \tag{5.1}$$

6 Periodic Potentials

The suspension procedure holds for real periodic potential (see e.g. Figotin Pastur): $\tilde{V}$ is again a periodic potential but with a random value at the origin. So our theorem holds in this case.

However we expect here the weaker condition :

$$\int_{\mathcal{C}} |V_\omega(\mathbf{x})|^2 d\mathbf{x} < \infty$$

i.e. V is uniformly locally L^2.