L^2-Sobolev estimates for the backscattering transform

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The backscattering data

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- *Wave operators*: $W_\pm = \lim_{t\to\pm\infty} e^{itH_v} e^{-itH_0}$ exist and are complete
- *Scattering operator:* $S = W_+^* W_-$.

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$$(\mathcal{F}Af)(\lambda\cdot) = \hat{A}(\lambda^2)(\mathcal{F}f)(\lambda\cdot),$$
where $\mathcal{F}$ is the Fourier transformation.
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- *Scattering amplitude*: $\hat{A}(\lambda^2; \omega, \omega')$
- *"Classical" backscattering data*: $\hat{A}(\lambda^2; \omega, -\omega)$, $\omega \in \mathbb{S}^{n-1}$ and $\lambda > 0$.

The real backscattering data

$$A\varphi = -(2\pi)^{-1} \lim_{\varepsilon \searrow 0} \int_{-\infty}^{\infty} e^{-\varepsilon|t|} e^{itH_0} (W_+^* v) e^{-itH_0} \varphi dt.$$

The "classical" backscattering data can be identified with

$$(B_{\text{class}} v)(x) = \mathcal{F}^{-1}(\xi \to \overline{\mathcal{F}(W_+^* v)\mathcal{F}^*(-\xi/2, \xi/2)})$$
$$= 2^n \int v(x-y) W_+(x-y, x+y)\, dy.$$

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Problem 1. Find v (singularities of v) from $B_{\text{class,re}} v$.

A formula for $W_+ + W_-$

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- $K_0(t) = \frac{\sin t|D|}{|D|}$, $t \geq 0$.
- $K_N(t) = \int_0^t K_{N-1}(t-s) v K_0(s)\, ds$.

  Then

$$K_v(t) = \sum_N (-1)^N K_N(t),$$

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- $|x-y| \leq t$ in the support of $K_v(t,x,y)$ (with $=$ when $v = 0$)

- Define

$$G = -\int_0^\infty K_v(t) v \dot{K}_0(t)\, dt, \quad v \in L^q_{\mathrm{cpt}}.$$

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- $\{\lambda_j\} \subset (-\infty, 0]$ eigenvalues of H_v counted with multiplicities, with $\{f_j\}$ the corresponding orthonormal set of eigenfunctions. There exist $g_j \in C^\infty(\mathbb{R}^n)$ such that

$$(W_+ + W_-)/2 = I + G + \sum f_j \otimes g_j.$$

The backscattering transformation and the problem

Definition (A. Melin (99))

The *backscattering transform* of $v \in L^q_{\mathrm{cpt}}$:

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- $C_0^\infty(\mathbb{R}^n) \ni v \mapsto Bv \in C^\infty(\mathbb{R}^n)$ is entire analytic in v:

$$Bv = \sum_1^\infty B_N v, \quad B_1 v = v.$$

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- $B_N v$ is the value at $(v, \dots, v)$ of an N-linear singular integral operator.

Continuity problems and smoothning

Aim: Find $(C_0^\infty(\mathbb{R}^n) \subseteq) X \subseteq Y$ Banach spaces of functions such that there is $C > 0$

- $\|B_N v\|_X \leq C^N \|v\|_Y^N$, for every N and $v \in Y$, or
- $\|B_N v\|_{X_{\mathrm{loc}}} \leq C^N \|v\|_Y^N$, for every N and $v \in Y$ with compact support.

First result

- The regularity of $B_N v$ increases with N in the sense of the next theorem.

Theorem (A. Melin (03))

Let $q > n$ and k be a nonnegative integer. Then there is a positive integer $N_0 = N_0(n, q, k)$ such that if $R_1, R_2 > 0$ there is a $C = C(n, k, R_1, R_2, q)$ such that

$$\|\Delta^k B_N v\|_{L^2(B(0,R_1))} \leq C^N \|v\|_{L^q}^N / N!, \qquad N \geq N_0,$$

whenever $v \in L^q$ has support in the ball $B(0, R_2)$.

Formula for $B_N v$

We have

$$(B_N v)(x) = \int \cdots \int v(x + x_1) \dots v(x + x_N)$$
$$E_N(x_N - x_1, , \dots, x_{N-2} - x_{N-1}, x_{N-1} + x_N)\, dx_1 \dots dx_N$$

when $v \in C_0^\infty(\mathbb{R}^n)$, $N \geq 2$.
Here $E_N \in \mathcal{D}'((\mathbb{R}^n)^N)$ is constructed iteratively.

The distribution E_N

Define

$$Q_1(x;t) = k_0(x;t),$$

$N \geq 2:$

$$Q_N(x_1,\dots,x_N;t) = \int Q_{N-1}(x_1,\dots x_{N-1};t-s)Q_1(x_N;s)\,ds,$$

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Then set

$$E_N(x_1, \ldots, x_N) = -\int Q_{N-1}(x_1, \ldots, x_{N-1}; t) \dot{k}_0(x_N, t)\, dt.$$

Further properties of E_N

- $E_2 = 4^{-1}(i\pi)^{1-n}\delta^{(n-2)}(x^2 - y^2)$ and it is the unique solution to $(\Delta_x - \Delta_y)E_2(x, y) = \delta(x, y)$ such that
  - $E_2(x, y) = -E_2(y, x)$
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  - $E_2(x, y) = -E_2(y, x)$
  - E_2 is rotation invariant separately in both variables
- $N > 2$:

$$
\begin{aligned}
&E_N(x_1, \dots, x_N) = \\
&= (E_2(x_1, x_2) \otimes \delta(x_3, \dots, x_N)) * (\delta(x_1) \otimes E_{N-1}(x_2, \dots, x_N)) \\
&- (E_2(x_1, x_2) \otimes \delta(x_3, \dots, x_N)) * (\delta(x_2) \otimes E_{N-1}(x_1, x_3, \dots, x_N))
\end{aligned}
$$

  - E_N is rotation invariant separately in all variables
  - E_N is symmetric in $x_1, \dots, x_{N-1}$
  - $|x_N| = |x_1| + \cdots + |x_{N-1}|$ on the support of E_N.

Another result

Theorem

Assume $n = 3$ and $s > 1$. Then there exists $C = C(s)$ such that

$$\|B_N v\|_{L^2} \leq C^N \|\langle x \rangle^s v\|_{L^2}^N$$

when $v \in C_0^\infty(\mathbb{R}^n)$.

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Proof relies on:

$$E_N(x_1, x_2, \ldots, x_N) = c^N \frac{\dot{k}_0(|x_1| + \cdots + |x_{N-1}|; x_N)}{|x_1| \cdots |x_{N-1}|}$$

Smoothning properties of B_N

Theorem

Let $R,\ R_1 > 0$ be fixed. Assume

$$\sigma \geq (n-3)/2,\ 0 \leq \alpha \leq \sigma - (n-3)/2,$$
$$N_0 \geq 2 \text{ integer },\ N_0 > \alpha + 1,\ N_0 \geq 2.$$

Then there exists C that depends only on R, R_1, σ, α and N_0 such that

$$\|B_N v\|_{H_{(\sigma+\alpha)}(B(0,R_1))} \leq C^N \|v\|^N_{H_{(\sigma)}(\mathbb{R}^n)}$$

for every $N \geq N_0$ and $v \in C_0^\infty(B(0,R))$.

Consequences

Corollary

Assume $R, R_1 > 0$ *are fixed and*

$$q > n, \sigma \geq (n-3)/2,\ 0 \leq \alpha < 1,\ \alpha \leq \sigma - (n-3)/2.$$

If $v \in H_{(\sigma)} \cap L^q$ *is compactly supported then*

$$v - Bv \in H_{(\sigma+\alpha),\mathrm{loc}}.$$

Main ingredients of the proof

- $B_N v = \beta_N v$ on $B(0, R_1)$ where

$$\mathcal{F}(\beta_N v)(\xi) = C^N \int \cdots \int_{(\mathbb{R}^n)^{N-1}} \widehat{U}_N(\theta_1, \ldots, \theta_{N-1}, \xi/2)$$
$$\hat{v}(\xi/2 + \theta_1)\hat{v}(\theta_2 - \theta_1) \ldots \hat{v}(\xi/2 - \theta_{N-1})\, d\vec{\theta},$$

  where

$$U_N = \Phi(x_N) E_N$$

  for some $\Phi \in C_0^\infty(\mathbb{R}^n)$.

- $|\hat{U}_2(\xi, \eta)| \le C\langle|\xi| + |\eta|\rangle^{-1}\langle|\xi| - |\eta|\rangle^{-1}$.

The quadratic term

Assume $N = 2$:

Theorem

Assume s, σ, a satisfy

$$0 \le a < (n-1)/2, \quad a > 1/4,$$
$$0 \le \alpha < 1, \quad \alpha \le (n-3)/2 - \sigma,$$
$$a + \alpha \le (n-1)/2.$$

Then

$$\|\langle x\rangle^{2a-1/2}\langle D\rangle^{\sigma+\alpha}B_2(v)\| \le C\|\langle x\rangle^{a}\langle D\rangle^{\sigma}v\|^2.$$

Remark: When $1/2 < a$, $2a - 1/2 > a$.

References

- *A. Melin*, (1999, 2003, 2004).
- *G. Eskin, J. Ralston*, (1991, 1989, 1992).
- *A. Greenleaf, G. Uhlmann*, (1993).
- *M.S. Joshi*, (1998).
- *L. Päivärinta, E. Somersalo* (1991).
- *A. Ruiz, A. Vargas*, (2005).
- *P. Stefanov*, (1992).
- *G. Uhlmann*, (2001).