Construction of Ground States and Resonances in Nonrelativistic QED

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Hydrogen Atom coupled to the Quantized Radiation Field

• **Hilbert space** is tensor product space,

$$\mathcal{H} = \mathcal{H}_{el} \otimes \mathcal{F},$$

where

• the Hilbert space of the electron (in H-atom) is

$$\mathcal{H}_{el} := L^2(\mathbb{R}^3),$$

• and the Fock space

$$\mathcal{F} = \mathcal{F}_b[L^2(\mathbb{R}^3 \times \mathbb{Z}_2)] = \bigoplus_{n=0}^{\infty} \mathcal{F}^{(n)}$$

is the photon Hilbert space,

• where the n-photon sector is

$$\mathcal{F}^{(n)} = \{\psi_n \in \otimes^n L^2 \,|\, \forall \pi : \psi_n(k^{(n)}) = \psi_n(k_\pi^{(n)})\},$$

with $k^{(n)} = (k_1, \ldots, k_n)$ and $k_\pi^{(n)} = (k_{\pi(1)}, \ldots, k_{\pi(n)})$.

• $\mathcal{F}^{(0)} := \mathbb{C}\,\Omega$, where Ω is the vacuum vector.

• On $\mathcal{F}$, we have creation and annihilation operators, obeying CCR: $\forall k, k'$:

$$[a(k), a(k')] = [a^*(k), a^*(k')] = 0,$$
$$[a(k), a^*(k')] = \delta(k - k'),$$
$$a(k)\Omega = 0.$$

• Interacting **Hamiltonian** results from minimal coupling

$$-i\vec{\nabla}_x \mapsto \vec{v} := -i\vec{\nabla}_x - \alpha^{3/2}\vec{A}(\alpha\vec{x}_j),$$

i.e.,

$$H_\alpha \ := \ \big(- i\vec{\nabla}_x - \alpha^{3/2}\vec{A}(\alpha\vec{x}_j)\big)^2 \ + \ V(x) \ + \ H_f \ ,$$

where $V(x) := |x|^{-1}$,

• the vector potential of the quantized radiation field in Coulomb Gauge ($\vec{\varepsilon}(\vec{k},+) \perp \vec{\varepsilon}(\vec{k},-) \perp \vec{k}$ is

$$\vec{A}(\vec{x}) \ :=$$
$$\int \frac{\Lambda(k)\, dk}{2\,\pi^{3/2}\,|\vec{k}|^{1/2}} \left\{ \vec{\varepsilon}(k) e^{-i\vec{k}\cdot\vec{x}} a^*(k) \ + \ \vec{\varepsilon}(k)^* e^{i\vec{k}\cdot\vec{x}} a(k) \right\} ,$$

with UV-Cutoff $\kappa(k) = \mathbf{1}[|\vec{k}| \leq \Lambda]$, and

$$H_f \ = \ \int dk\, \omega(k)\ a^*(k) a(k)$$

is the field energy operator.

• For $\alpha = 0$,

$$
\begin{aligned}
H(0) &= H_{el} \otimes \mathbf{1} \ + \ \mathbf{1} \otimes H_f \ , \\
H_{el} &= -\Delta_x + \frac{1}{|x|} \ .
\end{aligned}
$$

• Spectrum of $H(0)$ is sum of spectra of H_{el} and H_f,
$\sigma[H(0)] = \sigma[H_{el}] + \sigma[H_f] = [E_0 \ , \ \infty)$,

• Ground state energy $\inf \sigma[H(0)] = e_0$ is eigenvalue at the bottom of the continuum,

• Excited eigenval's $e_1, e_2, \ldots$ are embedded in continuum.

Results

1. Existence of Models

- $H(\alpha)$ is semibounded quadratic form, selfadjoint on domain of $H(0)$ [Hiroshima 99];

2. Binding

- Existence of ground state:
  $E_{\mathrm{gs}}(\alpha) := \inf \sigma[H(\alpha)]$ is an eigenvalue,

$$\exists \phi_{\mathrm{gs}}(\alpha) \in \mathcal{H} : \quad H(\alpha)\, \phi_{\mathrm{gs}}(\alpha) \; = \; E_{\mathrm{gs}}(\alpha)\, \phi_{\mathrm{gs}}(\alpha),$$

- Existence of ground state for $0 < |\alpha| \ll 1$ [B + Fröhlich + Sigal 99];
- Existence of ground state $\forall\, \alpha > 0$, provided (HVZ-type) binding condition $(*)$

$$E_{\mathrm{gs}}(\alpha, N, V) < \min_k \{E_{\mathrm{gs}}(\alpha, N-k, V) + E_{\mathrm{gs}}(\alpha, k, 0)\}$$

  holds true [Griesemer + Lieb + Loss 00];
- Condition $(*)$ holds true for
  $N = 1$ [GLL 00],
  $N = 2$ [Chen + Vugalter + Weidl 03],
  $N \geq 1$ [Lieb + Loss 03],

- Non-Existence of Gr. St. for
  - Nelson Model [Fröhlich 74, Pizzo 02, Lörinci + Minlos + Spohn 01],
  - Pauli-Fierz Hamiltonian [Arai + Hiroshima + Hirokawa 99, Könenberg 04].
- Key condition for existence/non-existence:

$$\int \frac{\|\vec{G}(k)\|^2}{\omega(k)^2}\, dk \begin{cases} < \infty & \text{gs exists,} \\ = \infty & \text{gs doesn't exist.} \end{cases}$$

- Enhanced binding: no ground state for $\alpha = 0$, but ground state for $\alpha > 0$, not too large. [Hainzl + Vougalter + Vugalter 01, Catto + Hainzl 04]

3. Construction of Ground States

- For Pauli-Fierz Ham. [B + Fröhlich + Sigal 98]
- For Nelson model [Pizzo 02];
- Thm. [B + Fröhlich + Pizzo '06, B + Könenberg '06, B + Pizzo + Shoufan '07]:

$*$ For $0 < |\alpha| \ll 1$ small, $H(\alpha)$ has a unique ground state vector $\Phi_0(\alpha) := \Phi_{\rm gs}(\alpha) \in \mathcal{H}$ at the bottom $E_0(\alpha) := E_{\rm gs}(\alpha) > \infty$ of its spectrum.

$*$ For $0 < |\alpha| \ll 1$ small, the complex dilation $H(\alpha, \theta)$ of $H(\alpha)$ has an resonance eigenvector $\Phi_j(\alpha, \theta)$ with resonance eigenvalue $E_j(\alpha) \in \mathbb{C}^-$ in the vicinity $E_j(\alpha) = e_j + \mathcal{O}(\alpha^3)$ of the excited energy levels e_j, $j = 1, 2, \ldots$.

$*$ Both $\Phi_j(\alpha)$ and $E_j(\alpha)$ are constructed by a convergent iteration procedure (*Pizzo's method*).

$*$ That is, a sequence $(H^{(n)})_{n=0}^{\infty}$ of infrared regularized Hamiltonians with eigenvectors $(\phi^{(n)})_{n=0}^{\infty}$ and corresponding eigenvalues $(E^{(n)})_{n=0}^{\infty}$ is iteratively constructed such that

$$
\begin{aligned}
H^{(0)} &= H(0), \quad H^{(n)} \to H(\alpha), \\
\phi^{(0)} &= \Phi_j(0), \quad \phi^{(n)} \to \Phi_j(\alpha), \\
E^{(0)} &= E_j(0), \quad E^{(n)} \to E_j(\alpha),
\end{aligned}
$$

• <u>Remarks:</u>

∗ Ground state $\Phi_{\mathrm{gs}}(\alpha)$ can be expanded in terms of bare quantities up to $\mathcal{O}(\alpha^L)$, for arbitrary $L \in \mathbb{N}$.

∗ Scattering amplitudes can be expanded in bare quantities, as well.

∗ Imaginary parts $\mathrm{Im}\{E_j(\alpha)\}$ of resonance eigenvalues can be explicitly computed to leading order in α. This yields the inverse <u>life-time</u>

$$\frac{1}{\tau_j} \sim \mathrm{Im}\{E_j(\alpha)\}$$

of the resonances as metastable states.

4. RG based on SFM

• Smooth Feshbach map (SFM):

Let $0 \leq \chi \leq 1$, $\bar{\chi} := \sqrt{1-\chi^2}$, and $[K, \chi] = 0$. Then

$$\dim \mathrm{Ker}(K + W - z) \ = \ \dim \mathrm{Ker}\big(F[K + W - z]\big),$$

where

$$F_{\chi,K}[K + W - z] \ := \\ K - z + \chi W \chi - \chi W \bar{\chi}\big(K - z + \bar{\chi} W \bar{\chi}\big)^{-1} \bar{\chi} W \chi \, .$$

on $\mathrm{Ran}[\chi]$.

• Renormalization group (RG) map $\mathcal{R}$ based on SFM:

∗ Effective Hamiltonian $H^{(n)} =: K^{(n)} + W^{(n)}$ is defined on $\mathbf{1}[H_f < 1]\mathcal{H}$,

∗ where $K^{(n)} = T^{(n)}(H_f)$ and

$$W^{(n)} \ = \\ \sum_{M,N} \int a^*(\xi_1) \cdots a^*(\xi_M) \ w_{M,N}(H_f, \xi, \tilde{\xi}) \ a(\tilde{\xi}_1) \cdots a(\tilde{\xi}_N),$$

∗ and choose $\chi := \mathbf{1}[H_f < \rho]$ with $0 < \rho \ll 1$.

∗ Set

$$H^{(n+1)}(z) \;:=\; \mathcal{R}[H^{(n)}(z)] \,,$$

where $\mathcal{R} = \mathcal{S} \circ F_{\chi, H_f}$ is composition of SFM and rescaling map $\mathcal{S}$ which maps

$$\mathbf{1}[H_f < \rho]\mathcal{H} \;\mapsto\; \mathbf{1}[H_f < 1]\mathcal{H}$$

∗ Show that $W^{(n)} \to 0$, as $n \to \infty$.

• Preservation of soft photon sum rules (SR) under RG map:

$$[S, H] = 0 \;\;\Rightarrow\;\; [S, \mathcal{R}(H)] = 0 \,.$$

5. Pizzo's Scale Decomposition

- Define a decreasing energy scales by

$$\sigma_n \ := \ \Lambda \, \beta^n,$$

- Divide momentum space into corresponding slices, for $1 \leq m < n \leq \infty$,

$$\mathcal{K}_n \ := \ \big\{ (\vec{k}, \tau) \in \mathbb{R}^3 \times \mathbb{Z}_2 \, \big| \, \sigma_n \leq \omega(\vec{k}) \big\} \, ,$$
$$\mathcal{K}_n^m \ := \ \big\{ (\vec{k}, \tau) \in \mathbb{R}^3 \times \mathbb{Z}_2 \, \big| \, \sigma_n \leq \omega(\vec{k}) < \sigma_m \big\} \, ,$$

- with corresponding L^2-spaces

$$h_n \ := \ L^2\big[\mathcal{K}_n\big] \quad \text{and} \quad h_n^m \ := \ L^2\big[\mathcal{K}_n^m\big]$$

- and corresponding Fock spaces

$$\mathcal{F}_n \ := \ \mathcal{F}\big[h_n\big] \quad \text{and} \quad \mathcal{F}_n^m \ := \ \mathcal{F}\big[h_n^m\big] \, .$$

- This gives rise to an momentum scale decomposition of the photon Fock space

$$\mathcal{F} \ = \ \mathcal{F}_\infty \ \cong \ \mathcal{F}_n \otimes \mathcal{F}_{n+1}^n \otimes \mathcal{F}_\infty^{n+1} \, .$$

• Similarly, we decompose the Hamiltonian as

$$
\begin{aligned}
\vec{v}_n \;&:=\; -i\vec{\nabla}_x \,+\, a^*(\vec{G}_n) \,+\, a(\vec{G}_n)\,, \\
\check{H}_n \;&:=\; \int \mathbf{1}(\sigma_n \leq |k|)\, \omega(k)\; a^*(k)a(k)\; dk\,, \\
\check{H}_n^m \;&:=\; \int \mathbf{1}(\sigma_n \leq |k| < \sigma_m)\, \omega(k)\; a^*(k)a(k)\; dk\,, \\
H_n \;&:=\; \vec{v}_n^{\,2} \,-\, V(x) \,+\, \check{H}_n\,,
\end{aligned}
$$

• which implies that

$$
\begin{aligned}
W_{n+1}^n \;&:=\; H_{n+1} - H_n \;=\; (\vec{v}_{n+1})^2 \,-\, (\vec{v}_n)^2 \\
&=\; 2\, a^*(\vec{G}_{n+1}^n)\cdot \vec{v}_n \,+\, 2\,\vec{v}_n \cdot a(\vec{G}_{n+1}^n) \\
&\qquad + \big(a^*(\vec{G}_{n+1}^n) + a(\vec{G}_{n+1}^n)\big)^2\,,
\end{aligned}
$$

where, e.g.,

$$
a^*(\vec{G}_{n+1}^n) \;:=\; \int \frac{\mathbf{1}[\sigma_{n+1} \leq \omega(k) < \sigma_n]\, dk}{2\,\pi^{3/2}\,|\vec{k}|^{1/2}}\, \vec{\varepsilon}(k) e^{-i\vec{k}\cdot\vec{x}} a^*(k)\,.
$$

• Projection P_n onto approx. ground state or resonance:

$$
P_n \;=\; \frac{-1}{2\pi i} \int_{|z|=\sigma_n} \frac{dz}{H_n - \widehat{E}_n - z}
$$

• Inductive step $n \to n+1$ by Neumann series expansions:

$$P_{n+1} \;=\; \frac{-1}{2\pi i} \int_{|z|=\sigma_{n+1}} \frac{dz}{H_{n+1} - \widehat{E}_n - z}$$

and

$$\frac{1}{H_{n+1} - \widehat{E}_n - z} \;=\; \sum_{k=0}^{\infty} \frac{1}{H_n + \check{H}_{n+1}^n - \widehat{E}_n - z} \Big\{ - W_{n+1}^n \Big(\frac{1}{H_n + \check{H}_{n+1}^n - \widehat{E}_n - z} \Big) \Big\}^k$$

• Existence of each P_n is easy - convergence $P_n \to P_j(\alpha)$ and $\widehat{E}_n \to E_j(\alpha)$ is difficult

• For this use that W_{n+1}^n is “small” and

$$i\,[H_n\,,\,\vec{x}] \;=\; 2\,\vec{v}_n,$$

which only holds for the minimally coupled model.