# A VARIATONAL APPROACH TO WATER WAVES IN SHALLOW WATERS

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Abstract. This document contains a brief description of our collaborative research project which is submitted to the CNRS LEA Math-Mode in response to their Call for proposals. The project is to be conducted in 2015.

**Key words and phrases:** variational methods; water-wave problem; shallow-water approximation

# 1. Research context

## 1.1. Participants

- Delia Ionescu-Kruse, Senior Researcher at Simion Stoilow Institute of Mathematics of the Romanian Academy, Romania
- Denys Dutykh, Chargé de Recherche CNRS at LAMA UMR 5127, Université de Savoie Mont-Blanc, France

## 1.2. Field of the study

V.I. Arnold initiated in the 60s – 70s the use of geometric methods in describing the equations of an incompressible ideal fluid in a bounded domain without free boundary [1, 2, 3]. This description of incompressible flows consists in formulating the facts on an infinite dimensional configuration space transferring results from the finite dimensional case of Lie group theory and of classical Riemannian geometry. It turns out that the Euler equations are geodesic equations for the right-invariant metric on the group of volume-preserving diffeomorphisms. The variational formulation gives a meaning to the geodesic equation on the group of diffeomorphisms. For incompressible ideal flows with free surface it becomes more complicated to obtain the solutions of the full water-wave problem as the trajectories of a Hamiltonian system and to show that this system issues naturally from a variational principle.

In the shallow-water regime, that is, for waves whose length is still large compared with the depth of the water in which they propagate, several two-component systems have been derived and studied. One of the characteristic models is the well-known Green–Nagdhi

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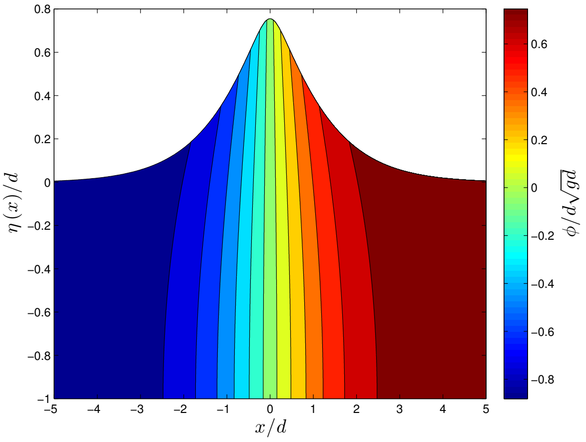


**Figure 1.** *A solitary wave with the velocity potential underneath. Taken from [6].*

(GN) system:

$$\begin{cases} u_t + uu_x + H_x = \frac{1}{3H}\Big[H^3\big(uu_{xx} + u_{xt} - u_x^2\big)\Big]_x \\ H_t + (Hu)_x = 0, \end{cases} \tag{1.1}$$

which models shallow-water waves whose amplitude is not necessarily small. Recently D. Ionescu-Kruse [7] proposed a novel variational derivation of this system. The second equation of the GN system is a transport equation, the free surface is advected, or Lie transported (in the geometry language), by the fluid flow. She showed that the first equation of the GN system yields the critical points of an action functional in the space of paths with fixed endpoints, within the Lagrangian formalism. By the same variational method the known nonlinear integrable two-component Camassa–Holm system has been derived [9]:

$$\begin{cases} u_t + 3uu_x - u_{txx} - 2u_xu_{xx} - uu_{xxx} + HH_x = 0 \\ H_t + (Hu)_x = 0, \end{cases} \tag{1.2}$$

and very recently a new two-component system was obtained [8]:

$$\begin{cases} u_t + 3uu_x + HH_x = \Big[H^2\big(uu_{xx} + u_{xt} - \frac{u_x^2}{2}\big)\Big]_x \\ H_t + (Hu)_x = 0, \end{cases} \tag{1.3}$$

system for which she showed that it has a non-canonical Hamiltonian formulation and she found its exact solitary-wave solutions too. It is important to stress out that in the variational derivation of these three systems, the Lagrangian functions considered are *not* metrics. It is planned to use this approach to continue the quest for new physically sound models.

On the other hand, D. Dutykh in collaboration with D. Clamond developed efficient numerical methods to compute travelling wave solutions to the full Euler equations [5, 6]. These results have been released as an open source project [4]. See Figure 1 for an example of a fully nonlinear solitary wave solution with the velocity potential under it. Moreover, these solutions can be used to assess the accuracy of approximate water wave models.

## 1.3. Project goals

We would like to combine our expertise in order to have a new look on the water wave problem in the shallow water approximation. It will be analyzed from mathematical, variational and numerical points of view. The novelty will consist in the geometric variational approach which exploits deeply the geometrical structure of the full Euler equations. The main goal being to derive approximate models which preserve as much as possible the geometric structure of the parent model.

# 2. Project budget

We would like to ask the LEA Math-Mode international laboratory to cover the fees related to mutual visits of participants to France and to Romania respectively:

**D. Ionescu-Kruse:** One round trip from Bucharest (Romania) to Chambéry (France) and a continuous stay at LAMA during one or two weeks in 2015

**D. Dutykh:** One round trip from Chambéry (France) to Bucarest (Romania) and a continuous stay at IMAR during two or three weeks in 2015

The travel fee is estimated to be around 350€ and the *per diem* is 90€.

# 3. Personal data

- Delia Ionescu-Kruse's personal data:

  **Date and place of birth:** 17$^{\text{th}}$ June 1971, Bucharest, Romania.

  **Academic degrees:** 2002 PhD in Mathematics, University of Bucharest, Thesis-advisors: S. Ianuş (University of Bucharest) and E. Soós (IMAR).

  **Current position:** Senior Researcher at Simion Stoilow Institute of Mathematics of the Romanian Academy.

  **Scientific awards:** *Dimitrie Pompeiu* prize of the Romanian Academy (2008).

- Denys Dutykh's personal data:

  **Date and place of birth:** 17$^{\text{th}}$ August 1982, Pologi, Ukraine.

  **Academic degrees:** 2007 PhD in Applied Mathematics, École Normale Supérieure de Cachan, 2010 Habilitation, Université de Savoie

  **Current position:** Chargé de recherche CNRS

  **Scientific awards:** Prix de la Recherche (2007)

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