Control of nonlinear PDE's

1 Exact controllability for the magnetohydrodynamic equations with control in their fluid part

The magnetohydrodynamic (MHD) equations govern the motion of electrically conducting viscous incompressible fluids in magnetic field. They consist of an elegant and subtle coupling of the Navier–Stokes equations of viscous incompressible fluid flow and the Maxwell equations of electromagnetic field. The mathematical theory of the MHD equations (the concept of weak solution, the functional framework, the methods, the main results of existence and uniqueness of solution, the regularity) is very similar to that of the Navier–Stokes equations and can be found in [8], [9], [10], [11] and [3].

Let us now formulate the main result of exact controllability for the three–dimensional MHD system. Let Ω be a connected bounded open set in $\mathbb{R}^3$ with the boundary $\partial\Omega$ of class C^2 and let $T > 0$. We set $Q = \Omega \times (0, T)$ and $\Sigma = \partial\Omega \times (0, T)$. Let ω be an open subset of Ω (on which the control action will be distributed). The controlled MHD equations we consider is the following:

$$
(1)\quad
\begin{aligned}
&\frac{\partial y}{\partial t} - \nu \Delta y + (y \cdot \nabla) y - (B \cdot \nabla) B + \nabla \left(\frac{1}{2} B^2 \right) \\
&\qquad\qquad\qquad\qquad + \nabla p = f + \chi_\omega u \text{ in } Q, \\
&\frac{\partial B}{\partial t} + \eta \operatorname{curl}(\operatorname{curl} B) + (y \cdot \nabla) B - (B \cdot \nabla) y = P(\chi_\omega u) \text{ in } Q, \\
&\operatorname{div} y = 0, \ \operatorname{div} B = 0 \qquad \text{in } Q, \\
&y = 0, \ B \cdot N = 0, \ (\operatorname{curl} B) \times N = 0 \qquad \text{on } \Sigma, \\
&y(\cdot, 0) = y_0, \ B(\cdot, 0) = B_0 \qquad \text{in } \Omega.
\end{aligned}
$$

Here $y = (y_1, y_2, y_3) : \Omega \times [0,T] \to \mathbb{R}^3$ is the velocity vector field, $p : Q \to \mathbb{R}$ is the (scalar) pressure field, and $B = (B_1, B_2, B_3) : \Omega \times [0,T] \to \mathbb{R}^3$ is the magnetic field. System (1) is controlled through the vector functions $u = (u_1, u_2, u_3) : Q \to \mathbb{R}^3$ and $v = (v_1, v_2, v_3) : Q \to \mathbb{R}^3$. The other symbols in (1) denote known (given) quantities. So, ν and η are the kinematic viscosity and magnetic resistivity coefficients, which are assumed to be positive, $f = (f_1, f_2, f_3) : Q \to \mathbb{R}^3$ is the density of the external forces, $\chi_\omega : \Omega \to \mathbb{R}$ is the characteristic function of ω, and $y_0 : \Omega \to \mathbb{R}^3$ and $B_0 : \Omega \to \mathbb{R}^3$ are the initial configurations of the velocity and magnetic field. Since the left–hand side of the magnetic part of system (1) is proved to be divergence–free and tangential to the boundary, we are forced to put the Leray projector P in the right–hand side in order to "kill" the gradient component of $\chi_\omega v$. The term $(B \cdot \nabla)B - \nabla(\frac{1}{2}B^2) = (\text{curl } B) \times B$ represents the Lorentz force. The physical meaning of the boundary conditions on B (where N denotes the unit outer normal vector to $\partial\Omega$) is that the boundary wall is perfectly conductive. Because of the well–known formula $\text{curl}(\text{curl } B) = \Delta B + \text{grad}(\text{div} B)$, we can replace the term $\text{curl}(\text{curl} B)$ in (1) by $-\Delta B$.

As target solution we consider a weak solution $(\widetilde{y}, \widetilde{B})$ of the uncontrolled version of equations (1); that is, together with some (non–unique) distribution $\widetilde{p}$, $(\widetilde{y}, \widetilde{B})$ satisfies:

$$
(2) \quad
\begin{aligned}
&\frac{\partial \widetilde{y}}{\partial t} - \nu \Delta \widetilde{y} + (\widetilde{y} \cdot \nabla)\widetilde{y} - (\widetilde{B} \cdot \nabla)\widetilde{B} \\
&\qquad\qquad + \nabla\left(\frac{1}{2}\widetilde{B}^2\right) + \nabla \widetilde{p} = f \text{ in } Q, \\
&\frac{\partial \widetilde{B}}{\partial t} + \eta \text{ curl}(\text{curl } \widetilde{B}) + (\widetilde{y} \cdot \nabla)\widetilde{B} - (\widetilde{B} \cdot \nabla)\widetilde{y} = 0 \text{ in } Q, \\
&\text{div } \widetilde{y} = 0, \ \text{div } \widetilde{B} = 0 \qquad \text{in } Q, \\
&\widetilde{y} = 0, \ \widetilde{B} \cdot N = 0, \ (\text{curl } \widetilde{B}) \times N = 0 \qquad \text{on } \Sigma,
\end{aligned}
$$

in the distribution sense. The best result of exact controllability known for the MHD system (1) is contained in the following statement.

Theorem 1. *Let $f \in (L^2(Q))^3$. If $(\widetilde{y}, \widetilde{B})$ is a weak solution of* (2) *which satisfies*

$$
(3) \qquad (\widetilde{y}, \widetilde{B}) \in (L^\infty(Q))^6 \ \text{and} \ \left(\frac{\partial \widetilde{y}}{\partial t}, \frac{\partial \widetilde{B}}{\partial t}\right) \in L^2(0,T; (L^\infty(\Omega))^6),
$$

then there exists $\delta > 0$ *such that, for any* $(y_0, B_0) \in (H \cap (L^4(\Omega))^3)^2$ *satisfying*

$$|y_0 - \widetilde{y}(\cdot, 0)|_{(L^4(\Omega))^3} + |B_0 - \widetilde{B}(\cdot, 0)|_{(L^4(\Omega))^3} \leq \delta,$$

we can find $(u, v) \in (L^2(Q))^6$ *and a corresponding weak solution* (y, B) *of system* (1) *which also satisfies*

$$y(\cdot, T) = \widetilde{y}(\cdot, T) \text{ and } B(\cdot, T) = \widetilde{B}(\cdot, T) \text{ a. e. in } \Omega.$$

In the above statement H is the standard space of all the weakly divergence–free vector fields in $(L^2(\Omega))^3$ which are tangential to the boundary. Theorem 1 (together with its proof) is found in [7]. It is almost the analogue for the MHD system of the result of exact controllability for the Navier–Stokes equations established by E. Fernandez-Cara, S. Guerrero, O. Imanuvilov, and J.-P. Puel in [4]. We point out that in [4] the condition on $\widetilde{y}$ which corresponds to the second regularity condition in (3) is somewhat weaker:

$$\frac{\partial \widetilde{y}}{\partial t} \in L^2(0, T; (L^\tau(\Omega))^3) \text{ for } \tau > \frac{6}{5}.$$

However, for the important case of stationary target solutions, this condition (like the second one in (3)) is automatically satisfied, and Theorem 1 becomes the perfect analogue of the result in [4] for the MHD equations. Previous results of exact controllability for the MHD system (1) were obtained in [1], [2], and [5]. A similar result of exact controllability for the two-dimensional MHD equations was established in [6].

The strategy of proving Theorem 1 is quite standard. With the help of an infinite–dimensional variant of the local inversion theorem, the local exact controllability of the controlled MHD equations (1) (asserted by Theorem 1) is reduced to a global exact controllability property for the following linearization of equations (1) around the target solution:

$$
(4)\quad
\begin{aligned}
&\frac{\partial y}{\partial t} - \nu \Delta y + (\widetilde{y}\cdot\nabla)y + (y\cdot\nabla)\widetilde{y} - (\widetilde{B}\cdot\nabla)B - (B\cdot\nabla)\widetilde{B} \\
&\qquad\qquad\qquad + \nabla(\widetilde{B}\cdot B) + \nabla p = f + \chi_\omega u \qquad \text{in } Q,\\
&\frac{\partial B}{\partial t} + \eta \text{ curl(curl } B) + (\widetilde{y}\cdot\nabla)B + (y\cdot\nabla)\widetilde{B} \\
&\qquad\qquad\qquad - (\widetilde{B}\cdot\nabla)y - (B\cdot\nabla)\widetilde{y} = P(\chi_\omega v) \text{ in } Q,\\
&\text{div } y = 0,\ \text{div } B = 0 \qquad \text{in } Q,\\
&y = 0,\ B\cdot N = 0,\ (\text{curl } B)\times N = 0 \qquad \text{on } \Sigma,\\
&y(\cdot,0) = y_0,\ B(\cdot,0) = B_0 \qquad \text{in } \Omega.
\end{aligned}
$$

The global exact controllability for the linear system (4) is equivalent to an observability inequality for the following adjoint of (4):

$$
(5)\quad
\begin{aligned}
&\frac{\partial z}{\partial t} + \nu\Delta z + (\nabla z + {}^t\nabla z)\widetilde{y} \\
&\qquad\qquad - (\nabla C - {}^t\nabla C)\widetilde{B} + \nabla q = g \text{ in } Q,\\
&\frac{\partial C}{\partial t} - \eta \text{ curl(curl } C) + (\nabla C - {}^t\nabla C)\widetilde{y} \\
&\qquad\qquad - (\nabla z + {}^t\nabla z)\widetilde{B} + \nabla r = G \text{ in } Q,\\
&\text{div } z = 0,\ \text{div } C = 0 \qquad \text{in } Q,\\
&z = 0,\ C\cdot N = 0,\ (\text{curl } C)\times N = 0 \ \text{ on } \Sigma.
\end{aligned}
$$

Here ∇z (for instance) is the matrix $(\partial z_i/\partial x_j)_{i,j=1}^n$ and ${}^t\nabla z$ denotes its transposition: ${}^t\nabla z = (\partial z_j/\partial x_i)_{i,j=1}^n$. So $(\nabla z + {}^t\nabla z)\widetilde{y}$ signifies the product of the square matrix $\nabla z + {}^t\nabla z$ and the column matrix $\widetilde{y}$:

$$
((\nabla z + {}^t\nabla z)\widetilde{y})_i = \sum_{j=1}^n \left(\frac{\partial z_i}{\partial x_j} + \frac{\partial z_j}{\partial x_i}\right)\widetilde{y}_j.
$$

The needed observability inequality for equations (4) easily follows if a certain global Carleman–type estimate for the same equations is available. So a suitable Carleman estimate for (4) is the key tool in this approach. The establishment of such an estimate is the most difficult step in the proof of Theorem 1.

The physical significance of the controllability result expressed by Theorem 1 is however questionable because of the following reasons. First, in the physical literature, it is always found only the MHD equations with its magnetic part *homogeneous*. So we may ask ourselves what physical meaning could have a control action in the right–hand side of the second equation (magnetic part) of system (1). More than this, for the reason we have explained earlier, in the same equation we were forced to introduce the Leray projector P. Of course, it can be viewed as one of the known quantities which define the system to control. (We could imagine some physical device which simulates the mathematical action of P.) But could P have physical significance in the context we have specified at all? Finally, the Leray projector P instantly spreads the action of the localized control function $\chi_\omega v$ into the whole region Ω. Physically, this is not too realistic. For all these reasons, it is desirable to be able to drive the solution of the MHD equations towards the target by acting only in the fluid part of the system.

So, instead of the controlled MHD system (1), we shall consider the following one:

$$
\begin{aligned}
&\frac{\partial y}{\partial t} - \nu \Delta y + (y \cdot \nabla) y - (B \cdot \nabla) B \\
&\qquad\qquad + \nabla \left(\frac{1}{2} B^2 \right) + \nabla p = f + \chi_\omega u && \text{in } Q, \\
&\frac{\partial B}{\partial t} + \eta \, \text{curl}(\text{curl } B) + (y \cdot \nabla) B - (B \cdot \nabla) y = 0 && \text{in } Q, \\
&\text{div } y = 0, \ \text{div } B = 0 && \text{in } Q, \\
&y = 0, \ B \cdot N = 0, \ (\text{curl } B) \times N = 0 && \text{on } \Sigma, \\
&y(\cdot, 0) = y_0, \ B(\cdot, 0) = B_0 && \text{in } \Omega.
\end{aligned}
\tag{6}
$$

This time, as target, we take a stationary (steady state) solution $(\widetilde{y}, \widetilde{B})$ of the uncontrolled version of (6). So, together with some scalar function (distribution) $\widetilde{p}$, $(\widetilde{y}, \widetilde{B})$ satisfies:

$$
(7)\qquad
\begin{aligned}
&-\nu\Delta\widetilde{y} + (\widetilde{y}\cdot\nabla)\widetilde{y} - (\widetilde{B}\cdot\nabla)\widetilde{B} \\
&\qquad\qquad +\nabla\left(\frac{1}{2}\widetilde{B}^2\right) + \nabla\widetilde{p} = f \text{ in } Q,\\
&\eta \text{ curl}(\text{curl } \widetilde{B}) + (\widetilde{y}\cdot\nabla)\widetilde{B} - (\widetilde{B}\cdot\nabla)\widetilde{y} = 0 \text{ in } Q,\\
&\text{div } \widetilde{y} = 0, \ \text{div } \widetilde{B} = 0 \qquad\qquad\qquad \text{in } Q,\\
&\widetilde{y} = 0, \ \widetilde{B}\cdot N = 0, \ (\text{curl}\widetilde{B})\times N = 0 \qquad \text{on } \Sigma.
\end{aligned}
$$

Following the same approach as in the case of controllability of system (1), we expect to finally reduce the exact controllability of the MHD equations (6) to a special observability inequality for the adjoint system (5), in which some *global* weighted L^2 norms of z and C are estimated in terms of a certain *local* weighted L^2 norm of z. In removing the local weighted L^2 norm of C from the right–hand side of an intermediate observability inequality for (5), the coupling term $({}^t\nabla C)\widetilde{B}$ in the first equation of system (5) turns out to be crucial. This term can be rewritten as $-({}^t\nabla\widetilde{B})C$, because $({}^t\nabla C)\widetilde{B} = -({}^t\nabla\widetilde{B})C$ as linear functional on H. So, multiplying it by C locally (that is, multiplying the first equation in (6) by C locally) and then integrating, one can obtain a local weighted L^2 norm of C, which is expected to absorb all the other similar norms in the right–hand side. Assuming that the magnetic component $\widetilde{B}$ of the stationary target solution $(\widetilde{y}, \widetilde{B})$ (which satisfies (7)) is of class C^1 (it may be much less regular but so we simplify the discussion), this idea seems to work if, at some point $x_0 \in \Omega$, $\widetilde{B}$ satisfies

$$\det({}^t\nabla\widetilde{B})(x_0) \neq 0.$$

Then the small subregion of Ω on which we can act through the control function u (in order to reach the target) is a (or any) sufficiently small open neighborhood ω of x_0 such that

$$(8)\qquad \det({}^t\nabla\widetilde{B}) \neq 0 \text{ on } \overline{\omega}.$$

The local exact controllability result for the controlled MHD equations (6) which could be derived in this way is the main objective of this project.

The question is then how physically significant is condition (8). We may ask whether other considerations involving the key coupling term $({}^t\nabla C)\widetilde{B}$

could produce the observability inequality leading to a similar controllability result for system (6) under a suitable condition imposed (this time) on $\widetilde{B}$ (instead of $\nabla\widetilde{B}$). The simplest idea would be to locally multiply the first equation in (6) by a convenient first–order differential expression in C (obtained by applying an appropriate first–order differential operator to C), instead of a local multiplication by C, and then (after integration) to employ a Poincaré–type inequality and, possibly, something like Korn's inequality.

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2 Control and stabilization of coupled systems in fluid dynamics

Viscous incompressible flows are modelled by the well known Navier-Stokes system. When supplementary physical effects take place, then the new model is a coupling between Navier-Stokes system and other equations. For example one has a coupling with Maxwell equations if one considers fluids with magnetic properties (one then obtains magnetohydrodynamic equations). Another example is the Boussinesq system where thermal effects are considered:

$$\begin{cases} y_t - \Delta y + (y \cdot \nabla)y + \nabla p = f + \theta \overline{e_3} \\ \theta_t + y \cdot \nabla\theta - \Delta\theta = g \end{cases} \tag{1}$$

Here y is the velocity field of the fluid, θ is the temperature, f represents an external field of forces and g is the external heat flow applied to the system.

Our main goal is the study of control problems associated to such systems
There is a vast literature concerning the analysis and the control of Navier-Stokes, MHD, Boussinesq. We added in the bibliography a selection of papers representative for the control problems where the controllers are localized in subdomains. What is common in the cited bibliography is the fact that the controllers act in all the equations of the system.

We are thus interested in the study of control problems for coupled systems in fluid dynamics, with controllers acting only in a part of the equations. For example, in the case of Boussinesq system one may want to control the system only by heating the fluid. In this general approach we intend to consider different problems:

1. Approximate controllability with controllers distributed in a subdomain and acting only in the velocity field or only in the heat equation.
2. Feedback stabilization
3. Exact controllability

Another related field of problems is the domain of inverse problems, where a quantity entering the system has to be determined, in an indirect way, by knowing some extra information about the solution.

Control and inverse problems are intimately related through Carleman inequalities which have to be deduced in an appropriate form.

In this context we are also interested in considering fluids with variable density or compressible fluids.

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