# APPLICATION

LABORATOIRE EUROPÉEN ASSOCIÉ CNRS FRANCO-ROUMAIN
2012

**Title of the Project**

*Inverse scattering transform for the Camassa-Holm equation*


PARTICIPANTS

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## 1. Scientific Project

**Inverse scattering transform for the Camassa-Holm equation.** The Camassa-Holm equation

$$\partial_t u - \partial_t \partial_x^2 u + 2\omega \partial_x u + 3u\partial_x u - 2\partial_x u \partial_x^2 u - u\partial_x^3 u = 0, \tag{1}$$

models the unidirectional propagation of waves over a flat bottom, $u(x,t)$ representing the height of the free surface about a flat bottom, and the constant $\omega$ being related to the critical shallow water speed (see [CH], [CHH]). It describes permanent and breaking waves; its solitary waves are stable solitons when $\omega > 0$, or peakons, when $\omega = 0$. It is also the equation of the geodesic flow for the $H^1$ right-invariant metrics on the Bott-Virasoro group (the case $\omega > 0$) and on the diffeomorphism group of the circle (the case $\omega = 0$) (see [C2], [M]). The equation (1) arises also in the study of motion of a non-Newtonian fluid of second grade in the limit when the viscosity tends to zero, and models the axially symmetric waves in a hyperelastic rod.

If $m = u - \partial_x^2 u$, equation (1) can be expressed as the condition of compatibility between

$$\begin{aligned} \partial_x^2 \psi &= \left(\frac{1}{4} + \lambda(m+\omega)\right)\psi \qquad \text{and} \\ \partial_t \psi &= \left(\frac{1}{2\lambda} - u\right)\partial_x \psi + \frac{1}{2}\partial_x u \psi, \end{aligned} \tag{2}$$

that is, $\partial_t(\partial_x^2\psi) = \partial_x^2(\partial_t\psi)$ is the same as to say that (1) holds.

When one looks for solutions such that $m$ decays at infinity, it turns out that solving the scattering problem for (2) with the time $t$ considered as a parameter leads to solving the Cauchy problem for the Camassa-Holm equation (1), since the eigenvalues and the transmission coefficient for (2) are constants of motion, while the reflexion coefficient and the norming constants have a simple and explicit evolution in $t$. The same procedure can be applied relating the periodic solutions of the Cauchy problem for (1) and the inverse spectral problem for (2). (See [C1], [C2], [C3].)

Consider $\omega > 0$ and then we may assume that $\omega = 1$, by a simple scalling. To briefly present the inverse scattering problem for (2) in this case, consider $t$ fixed and look at the problem

$$\psi'' = \left(\frac{1}{4} + \lambda(m+1)\right)\psi, \tag{3}$$

on the real time, where $m$ is a locally $L^1$ function that is assumed to decay at the infinity in a certain sense. For $\operatorname{Re}\lambda > 0$ denote $k = \sqrt{\lambda - 1/4}$. Then there are unique solutions or (3) such that

$$\psi_1(x,k) \sim \mathrm{e}^{\mathrm{i}kx} \quad \text{when} \quad x \to \infty,$$
$$\psi_2(x,k) \sim \mathrm{e}^{-\mathrm{i}kx} \quad \text{when} \quad x \to -\infty.$$

Then it turns out that there are unique complex constants $R_1(k)$, $R_2(k)$, $T_1(k)$ and $T_2(k)$ determined by $m$ and $\lambda$ and $\omega$ such that

$$\psi_1(x,k) \sim \frac{1}{T_2(k)}\mathrm{e}^{\mathrm{i}kx} + \frac{R_2(k)}{T_2(k)}\mathrm{e}^{-\mathrm{i}kx} \quad \text{when} \quad x \to -\infty,$$
$$\psi_2(x,k) \sim \frac{1}{T_1(k)}\mathrm{e}^{-\mathrm{i}kx} + \frac{R_1(k)}{T_1(k)}\mathrm{e}^{\mathrm{i}kx} \quad \text{when} \quad x \to \infty,$$

The matrix

$$S(k) = \begin{pmatrix} T_1(k) & R_2(k) \\ R_1(k) & T_2(k) \end{pmatrix}$$

is the scattering matrix determined by $m$ and $\omega$, $R_1(k)$ and $R_2(k)$ are the reflection coefficients, whereas $T_1(k)$ and $T_2(k)$ are the transmission coefficients. $S(k)$ is an unitary matrix. Under good conditions on the decay of $m$, there may be an infinite number of eigenvalues $\{\lambda_j\}_j$ for (3), and all lie in $(-\infty, 1/4)$. The inverse scattering problem requires the determination of $m$ (uniqueness, construction, characterization and dependence on the scattering data) for the scattering data: the scattering matrix, the eigenvalues and the norming constants.

The inverse scattering problem for (2) is especially difficult when $m$ is not assumed to be regular. In this case, the problem cannot be reduced to a Schrödinger type problem, as in [DT]. Also, further complexity is added by the fact that $m + \omega$ has negative values on a set of non-zero measure in the interesting case when wave breaking holds.

The uniqueness part of the inverse scattering problem for (2) was very recently solved in [BBW]. The present project aims to get a method of construction of $m$ from the reflexion coefficient, eigenvalues and norming constant, and to investigate the dependence on the data, by developing the method used in [SWG] to treat a simpler, but similar problem. These results would be then used for the study of the Cauchy problem for the Camassa-Holm equation.

The team of the project has been formed precisely by considering the experience of the participants in non-linear partial differential equations and fluid mechanics, on one hand, and scattering and inverse scattering, on the other hand.

**References.**


[BBW ] C. Bennewitz, M. Brown, R. Weikard. Scattering and inverse scattering for a left-definite Sturm-Liouville problem. Preprint 2011.

[CH ] R. Camassa, D. Holm, An integrable shallow water equation with peakon solitons. *Phys. Rev. Lett.* **71** (1993), 1661–1664.

[CHH ] R. Camassa, D. Holm, J. Hyman, A new integrable shallow water equation. *Adv. Appl. Mech.* **31** (1994), 1–33.

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[SWG ] J. Sylvester, T. Winebrenner, F. Gyles-Colwell, Layer stripping for the Helmholtz equation *SIAM J. Appl. Math.* **50** (1996), 736–754.

## 2. Activities to be supported by the project

One research visit to IMAR of Renata Bunoiu, 16-20 September 2012.
Financial support is needed to cover travel, accommodation and local expenses.
**Required financial support** 900 €,
as follows

- 400 € for travel expenses (flight tickets and train and bus tickets)
- 500 € for accommodation and local expenses.

## 3. Curricula Vitae

### Renata Bunoiu

Surname : Bunoiu (married Schiltz)
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Age : 41
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Professional address: LMAM, Université de Lorraine - Metz. Ile du Saulcy,
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Position : Maître de Conférences, Université de Lorraine - Metz, since September 1998

Education. 1997 PhD in Mathematics, Université de Metz (trés honorable)
Thesis *Sur quelques problémes mathématiques en mécanique des fluides.*
Supervisor Prof. Jeannine Saint Jean Paulin. 1994 D.E.A. de Mathématiques (mention bien), supported by a Tempus stipendium, at Université de Metz
Thesis *Sur les équations de bioconvection.* Supervisor Prof. Jeannine Saint Jean Paulin, Université de Metz.

Publications.

(1) R. Bunoiu, J. Saint Jean Paulin, Fluide á viscosité non linéaire dans un domaine de faible épaisseur dans le cas de lubrification. *C. R. Acad. Sci. Paris*, t. **323**, Série I (1996), 1097–1102.
(2) R. Bunoiu, Sur un probléme mathématique en mécanique des fluides. *Rev. de l'Assoc. Femmes et Mathématiques* (1997).
(3) R. Bunoiu, J. Saint Jean Paulin, Linear flow in porous media with double periodicity. *Portugaliae Mathematica*, **56** (1999), 221–238.
(4) R. Bunoiu, Nonlinear viscous flow through a thin slab in the lubrication case. *Rev. Roumaine Math. Pures Appl.* **45** (2000), no. 4, 577–591 (2001).
(5) R. Bunoiu, K. Taous, On a thermal problem in domains with cuts. Proceedings of the conference *Homogenization and applications to material sciences, September 15-19, 2001, Timişoara*, Editura Universităţii de Vest, Timisoara, 2001.

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**(10)** D. Borisov, R. Bunoiu, G. Cardone**, On a waveguide with frequently alternating boundary conditions: homogenized Neumann condition. *Ann. Henri Poincaré* 11 (2010), no. 8, 1591–1627.**
**(11)** D. Borisov, R. Bunoiu, G. Cardone**, On a waveguide with an infinite number of small windows. *C. R. Math. Acad. Sci. Paris* 349 (2011), no. 1-2, 53–56.**
**(12)** R. Bunoiu, G. Cardone, T. Suslina**, Spectral approach to homogenization of an elliptic operator periodic in some directions. *Math. Methods Appl. Sci.* 34 (2011), no. 9, 1075–1096,**
**(13)** D. Borisov, R. Bunoiu, G. Cardone**, Homogenization and asymptotics for a waveguide with an infinite number of closely located small windows. Problems in mathematical analysis. No. 58. *J. Math. Sci.* (N. Y.) 176 (2011), no. 6, 774–785**

Preprints.

- R. Bunoiu, G. Cardone, C. Perugia, *Unfolding Method for the Homogenization of Bingham Flow.* (Submitted.)
- D. Borisov. R. Bunoiu, G. Cardone, *Waveguide with non-periodically alternating Dirichlet and Robin conditions : homogenization and asymptotics.* (Submitted.)

Lectures at international meetings.

- 28.05-31.05 1996 28-ème Congrés National d'Analyse Numérique, La Londe les Maures, France.
- 20.07-03.08 1996 Optimisation of nonlinear systems and of free boundaries - International Workshop, Constanta, Roumanie.
- 30.01-01.02 1997 Forum des Jeunes Mathématiciennes, Paris.
- 15.09-20.09 1997 Analysis and Control of Differential Systems - International Workshop, Constanta, Roumanie.
- 31.08-04.09 1998 4-ème Colloque Franco-Roumain, Metz.
- 11.02-31.02 1999 Summer school in Partial Differential Equations, Temuco, Chile.
- 22.03-24.03 1999 International Conference on Optimisation,Trèves, Germany.
- 12.04-16.04 1999 Congrès GAMM, Université de Metz.
- 17.05-21.05 1999 31-Congrès National d'Analyse Numérique, Bonascre, France.
- 15.09-19.09 2001 Homogenization and Applications to Material Sciences, Timisoara.
- 02.09-06.09 2002 6-ème Colloque Franco-Roumain, Perpignan, France.
- 10.10-11.10 2002 Interregional Congress in Mathematics, Namur, Belgium.
- 13.10-14.10 2005 Interregional Congress in Mathematics, Liège, Belgium.
- 24.07-28.07 2006 Asymptotic Behaviour in Fluid Mechanics, EPFL, Lausanne, Suisse.
- 28.08-02.09.2006 8-ème Colloque Franco-Roumain, Chambéry, France.
- 12.09-14.09.2011 Workshop on Fluid Dynamics in Porous Media, Coimbra, Portugal.

- 21.05-25.05.2012 7-the European Conference on Elliptic and Parabolic problems, Gaeta, Italie.

## Ingrid Alma Beltiţă

Name : BELTIŢĂ INGRID ALMA
Date of birth and place: November 16, 1970, Baia Mare, Romania

EDUCATION.

- Graduated- Faculty of Mathematics, University of Bucharest, 1994.
- Master Degree in Harmonic Analysis, Faculty of Mathematics, University of Bucharest, 1995.
- Ph. D. in Partial Differential Equations, University of Bucharest (2005). Thesis: *Inverse scattering problems for stratified media.* Supervisor Prof. V. Iftimie

PROFESSIONAL CAREER. Since 1995: Researcher, Institute of Mathematics of the Romanian Academy

EXPERIENCE.

Research stages. CMI Marseilles (1999), Osaka University, Kiryu University (2000), CPT Marseilles (2001, 2002), Institute for Electrosciences, Lund (2006; supported by Short Visit Grant 1006, within the SPECT Programme, ESF), Instituto de Investigaciones en Matemáticas Aplicadas y en Sistemas, UNAM, Mexico, Departament of Mathematics, Faculdad de Sciencias, Universidad de Chile (Chile) 2010.

Temporary position. Ålborg University (May 2001)

Lectures in International Meetings.

- *Spectral theory for Schödinger operators with boundary conditions on a half-space*: Workshop New results in quantum mechanics and related topics (September 1998, Bucharest)
- *Inverse scattering problems for layered media*: Summer School New Analytic and Geometric Methods in Inverse Problems (July 2000, Heriot-Watt University, Edinburgh).
- *On an abstract radiation condition*: Workshop Spectral and Scattering Theory and Related Fields (RIMS Kyoto, 2000).
- *Mourre theory and inverse scattering problems for layered media*: Mini-Workshop in Mathematical Physics, Århus University (May 29, 2001).
- *Inverse scattering problems for Schrödinger operators with magnetic fields.* Workshop Quantum Hamiltonians with Magnetic Fields (Bucharest, 8 - 14 September 2002).
- *Multilinear estimates in backscattering theory*: Conference Operator Theory, Analysis and Mathematical Physics OTAMP 2006 (15–22 June 2006, Lund).
- $L^2$ *-Sobolev estimates for the backscattering transfromation*: QMath10 Conference, Moieciu (September 2007).
- *Local smoothing for the backscattering transform.* Second International Conference on Pseudo-Differential Operators and Related Topics, Växjö, Sweden (June 2008).
- *Magnetic Weyl calculus on coadjoint orbits of some semidirect products of Lie groups.* Generalized Functions GF2009, Wien (31 August- 04 September 2009).
- *Weyl-Pedersen calculus on coadjoint orbits of nilpotent Lie groups.* 10-ème Colloque Franco-Roumaine de Mathématiques Appliqu'ees, 26–31 August 2010, Poitiers.
- Mini-course "Inverse problem of conductivity", Ålborg, 2001.

- Mini-course "Local smoothing for the backscattering transform", Madrid 2011, Special Trimester on Inverse Problems: Theoretical and Numerical Aspects of Inverse Problems and Scattering Theory.

VARIOUS.

- *Translations:*
  Analiză Convexă (Convex Analysis), by R. Tyrrell Rockafellar, Texte Matematice Esentiale, Theta, Bucuresti 2002 (with D. Beltiţă).
  Analiză Matriceală (Matrix Analysis), by Roger A. Horn, Charles R. Johnson, Texte Matematice Esentiale, Theta, Bucuresti 2001 (with D. Beltiţă, R.-N. Gologan)
- *Local organizer* (together with G. Nenciu and R. Purice) of the QMath10 Conference in Moieciu
- *Editor* (together with G. Nenciu and R. Purice) of the volume *Mathematical results in quantum mechanics*. Proceedings of the QMath10 conference, Moieciu, Romania, 10–15 September 2007. Hackensack, NJ: World Scientific. (2008).

PUBLICATIONS.

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- I. Beltiţă, D. Beltiţă, Faithful representations of infinite-dimensional nilpotent Lie algebras. Preprint arXiv:1108.5563.
- I. Beltiţă, D. Beltiţă, Boundedness for Weyl-Pedersen calculus on flat coadjoint orbits. Preprint arXiv: 1203.0974.