# Bisets, fusion systems, and modular representation theory

April 22, 2012

## 1 Characteristic idempotents and the double Burnside ring

For finite groups $G$ and $H$, a $(G, H)$-biset is a set with commuting left $G$-action and right $H$-action. One constructs the double Burnside group $B(G, H)$, respectively $B_{\mathrm{fr}}(G, H)$, as the Grothendieck group of isomorphism classes of $(G, H)$-bisets, respectively bi-free $(G, H)$-bisets. Moreover, $B(G, G)$ has a multiplicative structure given by the direct product over $G$, and $B_{\mathrm{fr}}(G, G)$ is a subring of $B(G, G)$. Tensoring $B(G, H)$ or $B(G, G)$, over $\mathbb{Z}$, with a commutative ring $k$, one obtains $kB(G, H)$, respectively $kB(G, G)$, the double Burnside module, respectively the double Burnside algebra over $k$.

To a saturated fusion system $\mathcal{F}$ over a finite $p$-group $S$ one associates a characteristic biset in $B_{\mathrm{fr}}(S, S)$, defined by Linckelmann and Webb, and constructed by Broto, Levi and Oliver [BLO2]. Ragnarsson [R] proved that iterated powers of a characteristic biset converge to a characteristic idempotent in $\mathbb{Z}_{(p)}B_{fr}(S, S)$, and that this idempotent is unique. Ragnarsson and Stancu [RS] showed that the saturated fusion systems over $S$ and the idempotents in $\mathbb{Z}_{(p)}B_{fr}(S, S)$ satisfying Frobenius reciprocity are in bijective correspondence. This correspondence sends a saturated fusion system to its characteristic idempotent. It allows translating properties between the two approaches and provides insight into the nature of saturated fusion systems and their role in representation theory and stable homotopy. Applications to stable splittings and to retractive transfer triples can be given in this context.

Understanding (characteristic) idempotents of $kB(G, G)$ amounts to understanding the representation theory of this algebra. Webb [W], and, recently, using different methods, Boltje and Danz [BD] prove that the algebra $kB_{\mathrm{fr}}(G, G)$ is semisimple if $k$ is a field of characteristic 0. Ragnarsson and Stancu [RS] also extend the bijective correspondence to one between the fusion systems (not necessarily saturated) over $S$ and a set of idempotents on $\mathbb{Q}B_{\mathrm{fr}}(G, G)$ satisfying the Frobenius reciprocity. Now, the simple $kB(G, G)$-modules arise as evaluation at $G$ of simple biset functors. A biset functor is a functor from the category of finite groups, with morphisms between $G$ and $H$ given by $kB(H, G)$, into the category of $k$-modules. Bouc [B1] gives a parametrization of the simple biset functors in terms of pairs $(H, V)$, where $H$ is a finite group and $V$ is a simple module over the algebra of outer automorphisms of $H$. In a work in progress, Bouc, Stancu and Thevenaz [BST] found a stratification of the functor $B(-, H)$ and, for the quotients of this stratification, their largest semisimple quotient. In particular, one obtains a formula for the dimensions of simple $kB(G, G)$ modules in terms of the rank of some bilinear form. Interesting problems here, besides understanding the structure of the radical of $kB(G, G)$, is knowing when the evaluation of a simple biset functor is non-zero and when the above bilinear form is of maximal rank.

# 2 Relation with representation theory and finite group theory

A first step in modular representation theory is to reduce to the study of modules of the blocks of the group algebra $k[G]$, where $G$ is a finite group and $k$ is an algebraically closed field of prime characteristic $p$. Alperin and Broué [AB] associated to each block a fusion system on the defect group of the block. Crucial conjectures in representation theory can be rephrased in terms of centric linking systems associated to a block. This is the case for Alperin's weight conjecture [A]. This conjecture is a numerical statement on the number $l(kG)$ of isomorphism classes of simple modules over the group algebra $kG$, in terms of invariants of normalizers of non-trivial $p$-subgroups of $G$. Knörr and Robinson [KR] reformulate this conjecture by counting the number of ordinary irreducible characters of a $p$-block in terms of alternating sums indexed by $G$-conjugacy classes of chains of $p$-subgroups, suggesting there should be a complex behind these sums. Boltje [Bol] observed the existence of complexes whose Euler characteristic yields those alternating sums. Linckelmann [L1] constructed these complexes for a general fusion system of blocks, in the case when one is able to 'glue together' a certain family of 2-cocycles of the automorphism groups of the underlying fusion system of a block, i.e. find a 2-cocycle whose restrictions to the automorphism groups are the corresponding 2-cocycles in the family. This cocycle is trivial for the principal block or, if the defect group is cyclic or a Klein four group. Park [P] proved that this gluing conjecture [L2, Conjecture 4.2.] has multiple solutions in some cases. There is hope that a general way to construct this 2-cocyle for a fusion system of a block will bring information on the representation theory of the groups algebra.

The fusion system of a group at some prime $p$ is the fusion system of the principal block of this group. There is no known example of a fusion system of block that is not the fusion system of a group. Under a slight different angle, Kessar [K] and, Kessar and Stancu [KS] showed that the exotic fusion systems of Solomon (in [K]) and of Ruiz and Viruel (in [KS]) cannot occur as the fusion system of a block. The question of the 'gap' between the fusion systems of groups and the fusion systems of blocks, or, further more the fusion systems of saturated triples [AKO, IV.3], is a difficult one. A first step would be to try showing that other exotic examples cannot occur as the fusion systems of saturated triples.

# 3 Aims

- further investigate the relation between the idempotents in $\mathbb{Z}_{(p)}B_{\mathrm{fr}}(S,S)$ satisfying Frobenius reciprocity and the saturated fusion systems over $S$. For example, if $\mathcal{F}$ is such fusion system, it is interesting how can one construct the characteristic idempotent of the centralizer or normalizer of a subgroup of $S$ in $\mathcal{F}$ starting from the characteristic idempotent of $\mathcal{F}$.

- understand the relation between the idempotents in $\mathbb{Z}_{(p)}B_{\mathrm{fr}}(S,S)$ satisfying Frobenius reciprocity and those in $\mathbb{Q}B_{fr}(S,S)$ with the same property.

- find the structure of the radical of the algebra $kB(G,G)$ for a field of characteristic 0, in particular what are the simple $kB(G,G)$ modules that can occur. More ambitious, study the representation theory of $\mathbb{Z}_{(p)}B(G,G)$.

- Linckelmann's Gluing Conjecture for saturated fusion systems.

- study the gap between fusion systems of groups and fusion systems of saturated triples.

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