Report for 2010 for the project
Control of nonlinear PDE's
LEA Franco-Roumain Math–Mode

February 24, 2011

Visits of members of Cergy's partner to the University of Iaşi

Two visits were carried out in the framework of the project:

1. S. Rodrigues visited the University of Iaşi from 1 to 6 March 2010.
2. H. Nersisyan visited the University of Iaşi from 1 to 5 November 2010.

Visits of members of Iasi's partner to the University of Cergy

1. C. Lefter visited the University of Cergy from 15 to 25 November 2010.
2. C. Popa visited the University of Cergy from 25 November to 5 December 2010.

Talks related to the project

1. S. Rodrigues, *Exponential stabilization to a non-stationary solutuon for Navier–Stokes equations*, University of Iaşi, March 2010.
2. A. Shirikyan, *Exponential stabilisation to a non-stationary solution for Navier–Stokes equations and applications*, Paris–London Analysis Seminar, London, October 2010.
3. H. Nersisyan, *Stabilization of the 3D incompressible Euler system in infinite cylinder*, University of Iaşi, November 2010.

Articles

[1] V. Barbu, S. Rodrigues, A. Shirikyan, *Internal exponential stabilization to a non-stationary solution for 3D Navier–Stokes equations*, SIAM J. Control Optimization, accepted conditionally.

[2] V. Barbu, S. Rodrigues, A. Shirikyan, *Exponential boundary stabilization for linear parabolic equations*, in preparation.

[3] C. Popa, A. Shirikyan, *New observability inequality for the adjoint linearized MHD system*, in preparation.

[4] C. Lefter, A. Shirikyan, *Control of the Navier–Stokes equations with variable density*, in work.

1. Boundary stabilization for linear parabolic equations

The results obtained in paper [1] were described in the report of 2009. Therefore we shall confine ourselves to a brief description of a stabilization result which is the subject of the paper [2].

Let us consider the following linear problem in a bounded domain $D \subset \mathbb{R}^d$ with a smooth boundary ∂D:

$$\partial_t u - \Delta u + a(t,x)u = 0, \tag{1}$$

$$u\big|_{\partial D} = \eta. \tag{2}$$

Here a is a smooth function, bounded together with its first-order derivatives, and η is a finite-dimensional control supported by an open subset $\Gamma \subset \partial D$. Using the fact that the problem has finitely many determining modes and an observation inequality for parabolic equations, we construct a control that exponentially stabilizes the zero solution. Combining this the dynamic programming principle, we establish the existence of a feedback control. The main additional problem compared with the case of an internal stabilization is related to the low regularity of the control function.

2. Observability inequality for the adjoint linearized MHD equations

We (C. Popa and A. Shirikyan) have proposed to establish the following result of exact controllability for the magnetohydrodynamic (MHD) equations: The solution of the MHD system can be driven towards a target solution by acting (on the control parameters) either just in its hydrodynamic part or just in its magnetic part.

Let Ω be a bounded multi-connected open set in $\mathbb{R}^3$ with the boundary $\partial\Omega$ of class C^2 and let $T > 0$. Let us fix an open subset ω of Ω. Consider the following controlled MHD equations:

$$\begin{aligned}
&\frac{\partial y}{\partial t} - \nu\Delta y + (y\cdot\nabla)y - (B\cdot\nabla)B + \nabla p + \nabla\left(\frac{1}{2}B^2\right) \\
&\qquad\qquad\qquad\qquad = f + \chi_\omega u \qquad \text{in } \Omega\times(0,T), \\
&\frac{\partial B}{\partial t} + \eta\ curl(curl\ B) + (y\cdot\nabla)B - (B\cdot\nabla)y \\
&\qquad\qquad\qquad\qquad = P(\chi_\omega v) \qquad \text{in } \Omega\times(0,T), \\
&\text{div } y = 0, \text{div } B = 0 \qquad \text{in } \Omega\times(0,T), \\
&y = 0,\ B\cdot N = 0,\ (curl\ B)\times N = 0 \qquad \text{on } \partial\Omega\times 0, T, \\
&y(\cdot,0) = y_0,\ B(\cdot,0) = B_0 \qquad \text{in } \Omega.
\end{aligned} \tag{3}$$

Here $y : \Omega\times[0,T]\to\mathbb{R}^3$ is the velocity vector field, $p : \Omega\times(0,T)\to\mathbb{R}$ is the pressure, and $B : \Omega\times[0,T]\to\mathbb{R}^3$ is the magnetic field. The control of system (1) is done inside the set ω by means of the vector functions $u : \Omega\times(0,T)\to\mathbb{R}^3$ and $v : \Omega\times(0,T)\to\mathbb{R}^3$ (χ_ω is the characteristic function of ω). The symbol P denotes the Leray projector.

As target we take a solution $(\widetilde{y},\widetilde{B})$ of the uncontrolled version of system (1); that is, $(\widetilde{y},\widetilde{B})$ satisfies (1) with $u\equiv 0$ and $v\equiv 0$. In the case when the target solution $(\widetilde{y},\widetilde{B})$ is a strong solution, the exact controllability od system (1) was obtained by V. Barbu, T. Havârneanu, C. Popa, and S. S. Sritharan in 2003 and by T. Havârneanu, C. Popa, and S. S. Sritharan in 2006 for lower regularity for the target solution.

We have proposed to get the exact controllability of system (1) without any action in its magnetic part, that is, with $v\equiv 0$. To be able to so this, we need an observability inequality for the adjoint of the linearization of system (1), where appropriate weighted $L^2(\Omega\times(0,T))$ norms of the "velocity" and "magnetic" components of solutions are estimated in terms of a weighted $L^2(\omega\times(0,T))$ norm of just the "velocity". We have succeeded (paper [3]) in obtaining such an observability inequality in the first stage of our research. The removal of the magnetic–like component from the right–hand side of the inequality was quite delicate and required the combination of a global Carleman-type estimate for the adjoint linearized system with a suitable interior estimate for the same system. Our arguments work only under certain hypotheses on the magnetic part $\widetilde{B}$ of the target solution in ω. For instance, a possible condition is the following:

$$\det(\nabla\widetilde{B}) \neq 0 \text{ in } \omega.$$

In the next stage we intend to use the observability inequality for the adjoint linearized system in establishing the exact controllability of the MHD equations with control action only in their hydrodynamic part.

Let us describe the strategy we plan to follow. We can reformulate the local exact controllability of the MHD system as the surjectivity property of a certain nonlinear map (acting between suitable function spaces). According to an infinite-dimensional variant of the local inversion theorem, the local invertibility of the nonlinear map around the target solution follows if its differential evaluated at the target solution is an epimorphism. But the latter can be expressed as the global controllability property for the linearized equations around the target solution. Such a global controllability result for the linearized system can be derived by employing the observability inequality for the adjoint system in an essential way.

3. Control of the Navier-Stokes equations with variable density

In last 20 years many results were obtained in the field of control of fluid dynamics, mainly Navier-Stokes and Euler equations.

The first result we mention is the controllability of Euler equations in 2-d, which was obtained by J.-M. Coron (C.R. Acad. Sci. Paris 317(1993)). In this paper the author introduces the so called return method. This is essentially the linearization method but along a specified trajectory, which needs to have special properties such that the linearized system is controllable. The 3-d case was studied by O. Glass (C.R. Acad. Sci. Paris 325(1997)).

The second result we discuss is that of O.Yu. Imanuvilov (ESAIM Control. Optim. Calc. Var., 3 (1998), 97-131) who obtained observability (Carleman) inequalities for the linearized Navier-Stokes equations (Stokes-Oseen operators) and, by a local surjectivity result based on the corresponding statement for the linearization, local exact controllability, with controllers distributed in subdomains, for Navier-Stokes, is deduced.

The third thing which interests us in our research is the paper of A.V. Fursikov and O.Yu. Imanuvilov, Exact boundary controllability of the Navier-Stokes and Boussinesq equations (Russian Math. Surveys 54(1999)). The authors prove a result of approximate controllability, based on the return method of Coron and then couple this with a local exact controllability result based on Carleman inequalities.

The research we started (C.Lefter and A.Shirikyan) concerns the Navier-Stokes system with variable density:

$$\begin{cases} \dfrac{\partial(\rho y)}{\partial t} + \nabla \cdot (\rho y y) - \nabla \cdot (\nu(\nabla u + \nabla^t u)) + \nabla p = \rho f \\ \dfrac{\partial \rho}{\partial t} + \nabla \cdot (\rho y) = 0 \\ \nabla \cdot y = 0 \end{cases} \tag{4}$$

Here y is the velocity field and ρ is the density. Various control problems were proposed. One of them is the boundary controllability but acting only on y. Things are quite difficult because one may not expect to have controllability also for ρ, which satisfies a transport equation.

The results we expect to obtain (paper [4]), combining the above techniques, seem to be important also because they need an approach which would be useful also for other coupled systems of parabolic-hyperbolic type.