LEA Math-Mode Report

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Abstract

Multiparameter extensions of (linear and nonlinear) descent methods have been proposed for the solution of finite dimensional time independent problems; these new methods are based on a different treatment of several blocks of components of the solution, basically via the substitution of a scalar relaxation by a (suitable) matrix relaxation. We introduce and analyze an enhanced stable explicit time scheme for the wave equation. The new scheme is obtained by low-pass like filter with a proper damping operator that applies only on the high frequencies.

In the classical numerical analysis of PDE, the stability conditions traduce the capability of the scheme to capture the fastest oscillations, say the high frequencies. When considering an explicit scheme, the mathematical analysis of its stability (Neumann's-like, energy estimates) is often expressed as a inequality to be satisfied by the parameters (space step, time step, relaxation parameters) from which one infers bounds. On the other hand, a Fourier-like analysis (Fourier, wavelets) enable to reorganize the unknowns into:

- large eddies which contain the mean part of the energy and are associated to the low frequencies of the solution,

- small eddies, which contain the residual part of the energy of the solution and that are associated to the high frequencies.

In this project we propose to build explicit time marching schemes for the wave equation by using a damping that is applied only on the fast wave components; the damping is represented as a nonlocal differential operator and we discuss on its construction in several cases.

Preprint: *Multilevel stabilizations for the wave equation*, J.P. Chehab and L.I. Ignat.

We present here some of the scheme we have analyzed.

Scheme 1. Let N be a fixed integer, sufficiently large. We define now the damping operator $\mathcal{L}$ as

$$\mathcal{L}v = \sum_{k=N+1}^{+\infty} \theta k \widehat{v}_k(t) \sin(k\pi x) \tag{0.1}$$

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For the stabilized scheme each Fourier mode satisfies:

$$\frac{\widehat{u}_k^{(n+1)} - 2\widehat{u}_k^{(n)} + \widehat{u}_k^{(n-1)}}{\Delta t^2} + \pi^2 k^2 \widehat{u}_k^{(n)} + \theta k \mathbf{1}_{\{k \geq N+1\}} \frac{\widehat{u}_k^{(n+1)} - \widehat{u}_k^{(n)}}{\Delta t} = 0, \tag{0.2}$$

or

$$\begin{cases} \widehat{u}_k^{(n+1)} = -\widehat{u}_k^{(n-1)} + (2 - \Delta t^2 \pi^2 k^2)\widehat{u}_k^{(n)}, & k = 1, \cdots, N, \\ \widehat{u}_k^{(n+1)} = \frac{1}{1+\theta k \Delta t}\left(-\widehat{u}_k^{(n-1)} + (2 - \Delta t^2 \pi^2 k^2 + \theta \pi k \Delta t)\widehat{u}_k^{(n)}\right) & k \geq N+1. \end{cases}$$

We recall that the stability condition (CFL) for the scheme without damping is

$$\Delta t \sqrt{\lambda_k} \leq 2, \ \forall k. \tag{0.3}$$

Let us now study the stability of the scheme written in the following form

$$\begin{pmatrix} \widehat{u}_k^{(n+1)} \\ \widehat{u}_k^{(n)} \end{pmatrix} = \begin{pmatrix} \frac{2-\Delta t^2 \lambda_k + \theta \Delta t \sqrt{\lambda_k}}{1+\theta \Delta t \sqrt{\lambda_k}} & -\frac{1}{1+\theta \Delta t \sqrt{\lambda_k}} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \widehat{u}_k^{(n)} \\ \widehat{u}_k^{(n-1)} \end{pmatrix}.$$

We set $\xi = \Delta t \sqrt{\lambda_k}$ and then the eigenvalues ρ of the matrix must verify $|\rho| \leq 1$ with

$$\rho_\pm = \frac{1}{2+2\theta\xi}\left(-2 + \xi^2 - \theta\xi \pm \sqrt{\xi^4 - 2\theta\xi^3 + (\theta^2 - 4)\xi^2}\right).$$

The presence of the parameter θ allows to consider $\xi > 2$ and so to enhance the stability. We plotted hereafter the complex modulus of the roots when ξ varies, for different values of θ.

Scheme 2. As we have seen above, the new scheme allows to have an enhanced stability condition, but the high modes are too hardly smoothed. An idea to overcome this difficulty is to tune the stabilization parameter in such a way the local iteration matrix has its spectral radius equal to 1 exactly. For that purpose, we restart from the general stabilized scheme

$$\frac{\widehat{u}_k^{(n+1)} - 2\widehat{u}_k^{(n)} + \widehat{u}_k^{(n-1)}}{\Delta t^2} + \lambda_k \widehat{u}_k^{(n)} + \alpha_k \mathbf{1}_{\{k \geq N+1\}} \frac{\widehat{u}_k^{(n+1)} - \widehat{u}_k^{(n)}}{\Delta t} = 0 \tag{0.4}$$

Here α_k is to be chosen in an optimal way. Now, we set $\xi = \sqrt{\lambda_k}\Delta t$ and $\theta = \alpha_k \Delta t$. We obtain

$$(1+\theta)\widehat{u}_k^{(n+1)} - (2 - \xi^2 + \theta)\widehat{u}_k^{(n)} + \widehat{u}_k^{(n-1)} = 0 \tag{0.5}$$

The associated characteristic equation is

$$(1+\theta)X^2 - (2 - \xi^2 + \theta)X + 1 = 0$$

whose the roots are

$$X_\pm = \frac{(2 - \xi^2 + \theta) \pm \sqrt{(2 - \xi^2 + \theta)^2 - 4(1+\theta)}}{2 + 2\theta}$$

The largest root in modulus is equal to 1 when $\theta = \frac{\xi^2}{2} - 2$, the other root being of modulus less than 1. So, since the scheme is damped only for positive values of θ, we will take

$$\theta = \left(\frac{\xi^2}{2} - 2\right)\mathbf{1}_{\{|\xi| \geq 2\}},$$

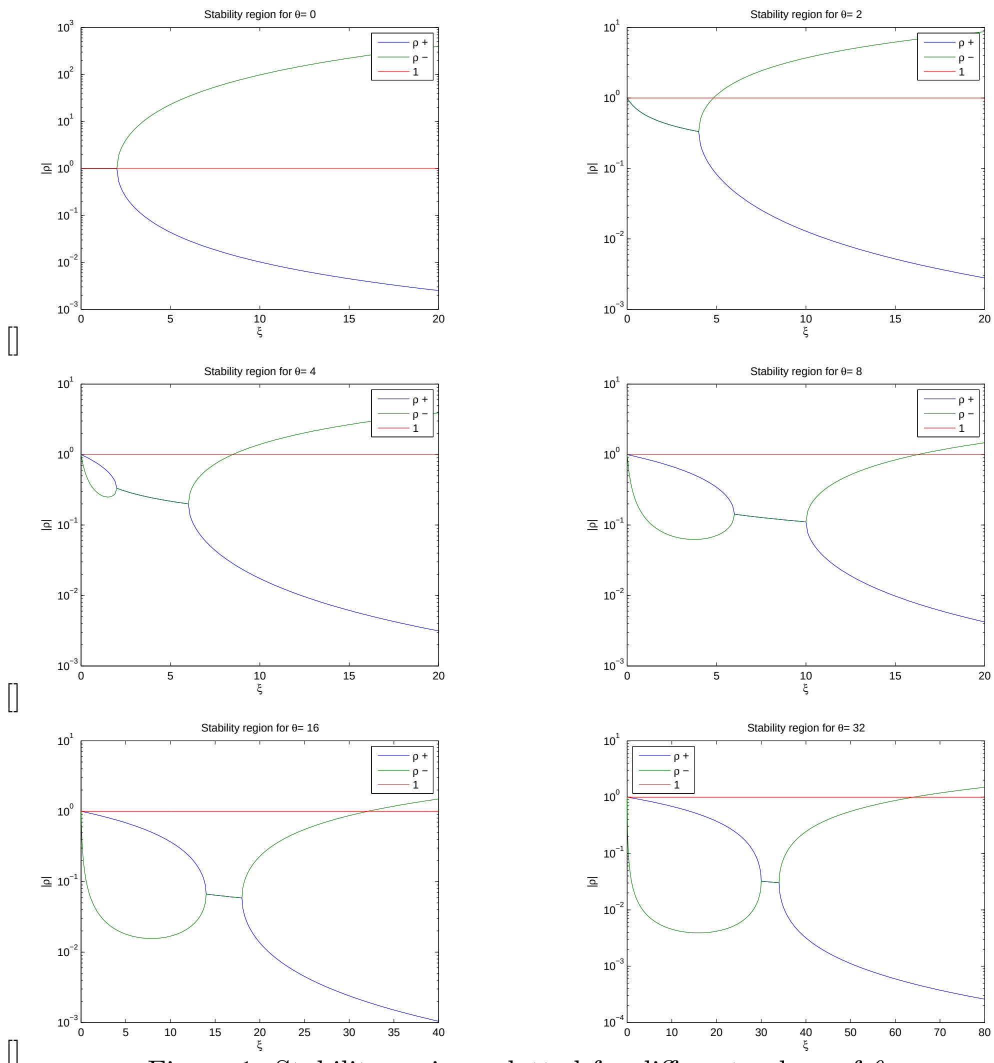

Figure 1: Stability regions plotted for different values of θ

so that the low mode are not damped, and the high one are damped in one direction and preserved in the other. Finally, we obtain the scheme

$$\frac{\widehat{u}_k^{(n+1)} - 2\widehat{u}_k^{(n)} + \widehat{u}_k^{(n-1)}}{\Delta t^2} + \lambda_k \widehat{u}_k^{(n)} + (\frac{\lambda_k \Delta t^2}{2} - 2)\mathbf{1}_{\lambda_k \leq \frac{2}{\Delta t}} \frac{\widehat{u}_k^{(n+1)} - \widehat{u}_k^{(n)}}{\Delta t} = 0.$$

At this point we make an important remark: the stabilized scheme is consistent in time for a fixed maximum number N of frequencies. Indeed, as Δt goes to 0, α_k goes also to 0: it becomes identically null, once $\Delta t \leq \frac{2}{\sqrt{\lambda_k}}$.

Scheme 3. Multilevel version of the above schemes has been also analyzed

$$\widehat{u}_k^{(n+1)} = -\widehat{u}_k^{(n-1)} + (2 - \Delta t^2 \lambda_k)\widehat{u}_k^{(n)}, \qquad k = 1, \cdots, N_0$$

For $l = 0, \cdots, m-1$

$$\widehat{u}_k^{(n+1)} = \frac{1}{1 + \theta_m k \Delta t \sqrt{\lambda_k}} \left(-\widehat{u}_k^{(n-1)} + (2 - \Delta t^2 \lambda_k + \theta_m \sqrt{\lambda_k} \Delta t)\widehat{u}_k^{(n)} \right), \quad k = N_l + 1, \cdots, N_{l+1}.$$

A important feature is to chose θ_m constant per level of frequencies: given a decomposition of the frequency range $[1, N] = \bigcup_{k=1}^{m} [N_{k-1}, N_k]$ with

$$1 = N_0 < N_1 < \cdots < N_m = N.$$

This will be of importance when considering multigrid like decompositions.

We now would like to adapt the scheme presented above when another discretisation is used, such as finite differences, finite differences, wavelets. The key of the stabilization is on an appropriate numerical treatment of different set of unknowns, the first ones containing the mean part of the energy of the solution and capturing only slow oscillations, the other one composed of small components and that capture the high frequencies. In a Fourier-like decomposition these scales appear naturally, in the other situations, one has to enforce it. To this effect, one considers embbebed grids, take unchanged the values of the coarse grid and replace those of the successive complementary grids by proper increments to the local coarse values, making in a same time a separation of the scales in space but also in frequencies.