Scalar conservation laws with fractional stochastic forcing: existence, uniqueness and invariant measure

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Abstract: We study a fractional stochastic perturbation of a first order hyperbolic equation of nonlinear type. Existence and uniqueness of the solution is investigated via a Lax-Oleĭnik formula. The generalized characteristic associated to the variational principle permit us to construct an invariant measure.

Keywords and phrases: Scalar conservation laws, random perturbations, variational principle, deterministic control theory, Hamilton Jacobi Bellmann equation, fractional Brownian motion.

1. Introduction

In this paper we study the following scalar conservation law

$$\partial_t u(t,x,\omega) + \partial_x \Psi\big(u(t,x,\omega)\big) = \partial_x \dot{F}(t,x,\omega)\ . \tag{1}$$

In the above equation, $x \in \mathbb{R}$, $t \geq 0$, $u(t,x,\cdot)$ is a random variable with values in $\mathbb{R}$ and F is a random force. A deterministic initial data $u(t_0,x) = u_0(x)$ is given. We will always assume that $u_0 \in L^\infty(\mathbb{R})$. As usual the random force will not be differentiable in the time variable, hence $\dot{F}$ denotes its formal time derivative.

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The sense given to the above equation will be stated below thanks to a weak formulation.

When all the functions F_k are null, Equation (1) is a deterministic scalar conservation law and there exists wide literature on this subject. Remind that in this deterministic case, the weak solution to such a problem is not unique in general. One needs to introduce the notion of entropy solution in order to discriminate the physical solution. Furthermore the selected solution has a nice qualitative behavior: discontinuities that are related with the creation of shocks, description of the behavior in terms of characteristic (see [1]). One can cite in a non exhaustive way [2, 4, 9, 15] for didactic introduction about this wide area.

Stochastic scalar conservation laws is a topic of growing interest in the few years. Nevertheless, there is only a few number of works on this subject. In the paper of [8] an operator splitting method is proposed to proved the existence of a weak solution to the Cauchy problem in $\mathbb{R}$ $du+\partial_x f(u)dt = g(u)dW_t$. In [11] a method of compensated-compactness is used to prove the existence of a stochastic weak entropy solution to the problem $du+\partial_x f(u)dt = g(t,x)dW_t$, $x \in \mathbb{R}$. the uniqueness s achieved thanks to a Kruzkhov-type method. A notion of strong entropy solution is proposed by [5] in order to extend the above-mentioned result to the problem $du + \operatorname{div} f(u)dt = \sigma(t,u)dW_t$, $x \in \mathbb{R}^d$. A stochastic scalar conservation law in a bounded domain of $\mathbb{R}^d$ is investigated in [18] using measure-valued solution and Kruzkhov's entropy formulation.

Besides this work, let us mention more precisely the work of E, Khanin, Mazel and Sinai [3] which was the starting point of our investigation. This article deals with the Burger's case (that is $\Psi(u) = u^2/2$):

$$\partial_t u(t,x,\omega) + \partial_x \big(u(t,x,\omega)\big)^2 = \partial_x \dot{F}(t,x,\omega)\ ,$$

with a stochastic forcing given by $F(t,x,\omega) = \sum_{k=1}^{\infty} F_k(x)\dot{B}_k(t)$ where $(B_k)_{k\geq 1}$ are independent standard Wiener processes on the real line $\mathbb{R}$ ($\dot{B}_k$ is again designates the formal time derivative of this process). The existence and uniqueness is proved together with the existence of an invariant measure. A parabolic perturbation problem approach is considered, based on the Hopf-Cole transformation.

On the one hand, our work is a generalization of the existence and uniqueness results contained in [3] because we work with a general conservation law depending on the function Ψ and also because we can reach a large class of noise. In one word since we work on each trajectory of the noise, we prove the existence and uniqueness of (1) for any noise having Hölder continuous paths. We will also prove a Lax-Oleĭnik formula using a direct approach via the Hamilton-Jacobi equation that is naturally associated to our problem. The existence and uniqueness result is presented in the next section in Theorem 1.

On the other hand, we generalize the existence of an invariant measure to the case of a fractional noise that is when the sequence of independent Brownian motion is replaced by fractional Brownian motions on the real line. There are serious difficulties to work with fBm. First, unlike the classical Brownian motion, the two-sided (this means defined on the all real line) fBm is not obtained by gluing two independent copies of a one-sided (defined on $\mathbb{R}^+$) fBm together at time $t = 0$. Moreover, when $t \leq 0$, the two-sided fBm is no more a Volterra type process (as it is the case for the classical fBm). One refer to [10] for a more detailed discussion on this fact. In [3], there is roughly speaking only one purely probabilistic property of the noise that is employed. This property is the fact that the Brownian noise is arbitrary small on an infinite number of arbitrary long time intervals. In other words for all $\varepsilon > 0$, $T > 0$, for almost-all ω, there exists a sequence of random time $(t_n(\omega))_{n\geq 1}$, such that $t_n(\omega) \to -\infty$ and

$$\forall\ n\ ,\ \sup_{t_n - T \leq s \leq t_n} \textstyle\sum_{k\geq 1} \Big\{ \|F_k\|_{C_b^2(\mathbb{R})} |B_k(s) - B_k(t_n)| \Big\}\ \leq \varepsilon\ .$$

This result relies on the independence of the increments of a Brownian motion and on Borel-Cantelli lemma. In a fractional Brownian framework the increments are no more independent. So one have to adapt this argument thanks to a conditional version Borel-Cantelli lemma to prove an analogous property for the trajectories of a fBm when the time goes to $-\infty$.

In the following section, we will state our hypothesis and give the main results of our work. Section 3 is devoted to the variational principle which is used to prove the existence and uniqueness. As regards the the calculus of variation problem considered in Section 3, we study in Section 4 a particular class of minimizers of the action appearing in the Lax-Oleĭnik formula. These one-sided minimizers are used to construct a unique solution of (1) defined on the time interval $\mathbb{R}$. In other words the random attractor consists of a single trajectory almost surely. Then we prove easily the existence os an invariant solution. Finally, the proof of the oscillation property (see Theorem 2) of the fractional Noise is given in Section 5.

2. Notations and main results

We will use the following notations:

- $C_b^r(\mathbb{R})$ is the space of $r-$times differentiable bounded functions with bounded derivatives endowed with the norm given by $\|\varphi\|_{C_b^r(\mathbb{R})} = \sum_{i=0}^r \|\varphi^{(i)}\|_\infty$;
- for $0 < \lambda < 1$ and $-\infty < a < b < +\infty$, $C^\lambda(a,b)$ is the space of λ-Hölder continuous functions $f\ :\ [a,b] \to \mathbb{R}$, equipped with the norm $\|f\|_\lambda := \|f\|_{a,b,\infty} + \|f\|_{a,b,\lambda}$, where

$$\|f\|_{a,b,\infty} = \sup_{a\leq r\leq b} |f(r)| \quad \text{and} \quad \|f\|_{a,b,\lambda} = \sup_{a\leq r\leq s\leq b} \frac{|f(s)-f(r)|}{|s-r|^\lambda}\ ;$$

- for two times t_1, t_2, $H^1(t_1,t_2)$ is the Sobolev space of $L^2(t_1,t_2)-$weakly differentiable functions from $[t_1,t_2]$ to $\mathbb{R}$ equipped of the scalar product

$$\langle \xi_1,\xi_2\rangle = \int_{t_1}^{t_2} \xi_1(s)\xi_2(s)ds + \int_{t_1}^{t_2} \dot{\xi}_1(s)\dot{\xi}_2(s)ds\ ;$$

- for a function f from $\mathbb{R} \to \mathbb{R}$, we denote f^* its Legendre transform defined as $f^*(q) = \sup_{p\in\mathbb{R}} \big(pq - f(p)\big)$ for $q \in \mathbb{R}$.

In the probabilistic framework of $(\Omega, \mathcal{F}, \mathbb{P})$, we make the following assumption on the stochastic forcing term F.

Hypothesis I. *For any t, x, the stochastic term F can be decomposed as $F(t,x) = \sum_{k=1}^{\infty} F_k(x) B_k(t)$ where:*

(a) the sequence $(F_k)_{k\geq 1}$ is such that for any k, the function F_k belongs to $C_b^3(\mathbb{R})$ satisfies $\|F_k\|_{C_b^3(\mathbb{R})} \leq C k^{-\frac{2+H}{H}}$.

(b) there exists $\lambda > 0$ such that the sequence of processes $\big((B_k(t))_{t\in(-\infty,\infty)}\big)_{k\geq 1}$ satisfies $B_k(\cdot) \in C^\lambda(a,b)$ for any $k \geq 1$, $-\infty < a < b < +\infty$. Without loss of generality we impose that $\|B_k\|_\lambda \leq C$.

We remark that the processes B_k are not necessarily independent. It is quite straightforward that the above noise term covers the one of [3] but it also covers sequences of processes as fractional Brownian motion of any Hurst parameter. One may assume that the Hölder norm of B_k depends on k but in this case one have to impose additionally that $\sum_{k\geq 1} \|B_k\|_\lambda \, k^{-\frac{2+H}{H}} < \infty$.

The function Ψ will satisfy the following assumption.

Hypothesis II. *The flux Ψ satisfies*

(a) Ψ is uniformly convex: there exists $\theta > 0$ such that $\Psi''(v) \geq \theta$ for all $v \in \mathbb{R}$,

(b) super-linear growth condition: there exists $k_2 > k_1 > 0$ and two constants l_1, l_2 such that $l_1|v|^{k_1} \leq \frac{\Psi(v)}{|v|} \leq l_2|v|^{k_2}$,

(c) there exists L such that $|\Psi'(v) - \Psi'(v')| \leq L|v - v'|$,

(d) there exists a positive function $R \mapsto C(R)$ such that $|\Psi^*(v) - \Psi^*(v')| \leq C(R)|v - v'|$ whenever $\max(|v|, |v'|) \leq R$.

We stress the fact that our assumptions Ψ are clearly true if the flux is the square function as in the Burger's case.

Now we give the precise meaning of (1).

Definition 1. *A random field u defined on $[t_0, +\infty) \times \mathbb{R} \times \Omega$ with real values is a weak solution of (1) with initial condition $u(t_0, \cdot) = u_0(\cdot) \in L^\infty(\mathbb{R})$ if:*

(i) For all $t > t_0$ and $x \in \mathbb{R}$, $u(t, x, \cdot)$ is measurable with respect to $\mathcal{F}_{t_0,t} = \sigma\{B_k(s), t_0 \leq s \leq t, k \geq 1\}$.

(ii) Almost surely, $u(\cdot, \cdot, \omega) \in L^1_{\mathrm{loc}}([t_0, \infty) \times \mathbb{R})$ and $u(t, \cdot, \omega) \in L^\infty(\mathbb{R})$ for any $t \geq t_0$

(iii) For all test function $\varphi \in C_c^2(\mathbb{R}\times\mathbb{R})$ (the set of twice differentiable functions

with compact support) the following equality holds almost-surely

$$\int_{t_0}^{\infty}\int_{\mathbb{R}} \frac{\partial\varphi(t,x)}{\partial t} u(t,x)dxdt + \int_{t_0}^{\infty}\int_{\mathbb{R}} \frac{\partial\varphi(t,x)}{\partial x}\Psi\big(u(t,x)\big)dxdt = \\ -\int_{\mathbb{R}} u_0(x)\varphi(t_0,x)dx \\ -\int_{\mathbb{R}}\sum_{k=1}^{\infty}\left\{F_k(x)\int_{t_0}^{\infty}\frac{\partial^2\varphi(t,x)}{\partial t\partial x}(B_k(t)-B_k(t_0))dt\right\}dx\ . \qquad (2)$$

It is well known that this notion of weak solution is not sufficient to have uniqueness for the solution of (1) in the deterministic case. One have to introduce the notion of admissible solution (or weak-entropy solution).

Definition 2. *We say that a random field u which is a already a weak solution of Equation (1) is an entropy-weak solution if there exists $C > 0$ such that for almost-all $\omega \in \Omega$,*

$$u(t,x+z,\omega) - u(t,x,\omega) \le C\big(1+\tfrac{1}{t-t_0}\big)\ z \qquad (3)$$

for all $(t,x) \in (t_0,\infty)\times\mathbb{R}$ and $z > 0$.

The above entropy condition is the historical *"condition E"* as so called in [14]. This condition will ensure us the uniqueness of bounded weak solution. It follows from (3) that for $t > t_0$ the function $x \mapsto u(t,x) - Cx$ is nonincreasing, and consequently has left and right hand limits at each point. Thus also $x \mapsto u(t,x)$ has left and right hand limits at each point, with $u(t,x-) \ge u(t,x+)$. In particular, this classical form of the entropy condition holds at any point of discontinuity.

First of all, in this paper we are interested in the existence and uniqueness of the entropy-weak solution of (1). In [3], this property is proved for the particular case of the Burger's equation. The authors use the standard mollification of the Brownian noise and then obtain a variational formula as the deteministic Lax-Oleĭnik formula. Our method is slightly different because we do not use any regularization of the noise (see Section 3).

We generalize this result for a general flux and a wide class of noise in the following theorem.

Theorem 1. *We assume Hypotheses I and II. Let $u_0 \in L^\infty(\mathbb{R})$. There exists a unique entropy-weak solution to the stochastic scalar conservation law (1) such that $u(t_0,x) = u_0(x)$. For $t \ge t_0$, this solution is given by the following Lax-Oleĭnik type formula :*

$$u(t,x,\omega) = \frac{\partial}{\partial_x}\left(\inf_{\substack{\xi\,\in\,H^1(t_0,t)\\ \xi(t)\,=\,x}}\left\{\mathcal{A}_{t_0,t} + \int_0^{\xi(t_0)} u_0(z)dz\right\}\right)\ , \qquad (4)$$

with

$$\mathcal{A}_{t_0,t}(\xi) = \int_{t_0}^{t} \Big\{ \Psi^*(\dot{\xi}(s)) - \textstyle\sum_{k\geq 1} \big(B_k(s) - B_k(t_0)\big) f_k(\xi(s))\dot{\xi}(s) \Big\} ds$$
$$+ \textstyle\sum_{k\geq 1} \big(B_k(t) - B_k(t_0)\big) F_k(\xi(t)) \ . \qquad (5)$$

The second (and certainly the most important) contribution of our work is the study of the invariant measure for the stochastic conservation law (1) for the particular case of a fractional noise. There is only the work of E, Khanin, Mazel and Sinai that deals with invariant measure for such equation and this is in the particular case of the Burger's equation with a Brownian noise. In order to state the results concerning the invariant measure, we work with the following particular noise term F.

Hypothesis III. *The stochastic term F has again the decomposition $F(t,x) = \sum_{k=1}^{\infty} F_k(x) B_k(t)$ with the property stated in Hypothesis I(a). Moreover the sequence of processes $\big((B_k(t))_{t\in\mathbb{R}}\big)_{k\geq 1}$ is a sequence of independent fractional Brownian motions (fBm in short) with Hurst parameter $H \in (0,1)$. This means that for each k, $(B_k(t))_{t\in\mathbb{R}}$ is a Gaussian process satisfying $B_k(0) = 0$ and $\mathbb{E}(|B_k(t) - B_k(s)|^2) = |t-s|^{2H}$.*

The probabilistic property of the noise that is employed to construct the invariant measure is the fact that it has periods of arbitrary length and arbitrary small amplitude oscillation as time goes to $-\infty$. The result, which is interesting in itself and new to our knowledge, is the following.

Theorem 2. *For all $\varepsilon > 0$, $T > 0$, for almost-all ω, there exists a sequence of random time $(t_n(\omega))_{n\geq 1}$, such that $t_n(\omega) \to -\infty$ and*

$$\forall\ n\ ,\ \textstyle\sum_{k\geq 1} \Big\{ \|F_k\|_{C_b^2(\mathbb{R})} \sup\limits_{t_n - T \leq s \leq r \leq t_n} |B_k(r) - B_k(s)| \Big\} \ \leq \varepsilon\ . \qquad (6)$$

In the Brownian case, this property is easy to prove thanks to the independence of the increments and the classical Borel-Cantelli lemma. In the framework of the fBm, the increments are no more independent and we naturally employ a conditional version of Borel-Cantelli lemma to prove this path-property of the fBm. We will additionally make use of the Garsia-Rodemich-Rumsey inequality and Talagrand's small ball estimate (see the proof given in Section 5).

Despite these difficulties, one can state the following results concerning the invariant measure for the stochastic scalar conservation law with fractional forcing. Let us introduce what is the precise formulation of the result.

We denote $\mathbb{D}$ the Skorohod space consisting of functions from $\mathbb{R}$ to $\mathbb{R}$ having discontinuities of the first kind. It is endowed with the metric

$$d(f,g) = \sum_{n\geq 1} 2^{-n} \big(1 \vee d_n(f,g)\big)$$

where d_n is the usual distance of Skorohod on $[-n,n]$. Hence $(\mathbb{D},\mathcal{D})$ is a measurable space with $\mathcal{D}$ the sigma-algebra of Borel sets on $\mathbb{D}$.

In order to construct an invariant measure, we will construct an invariant solution. To this aim we show that for almost-all ω, there exists a solution $(t,x) \mapsto u^\sharp(t,x,\omega)$ starting from $u_0 \equiv 0$ at $t_0 = -\infty$. This solution will be build via minimizers of the action $\mathcal{A}_{t_0,0}$ when $t_0 \to -\infty$ (see Section 4).

More precisely we will prove that there exists $u^\sharp$ from $\mathbb{R} \times \mathbb{R} \times \Omega$ to $\mathbb{R}$ such that:

(i) almost-surely, $u^\sharp(t,\cdot,\omega) \in L^\infty(\mathbb{R})$ for any t;
(ii) almost-surely, $u^\sharp(t,\cdot,\omega) \in \mathbb{D}$ for any t;
(iii) given t, the mapping $\omega \mapsto u^\sharp(t,\cdot,\omega)$ is measurable from $(\Omega,\mathcal{F})$ to $(\mathbb{D},\mathcal{D})$;
(iv) on any finite time interval $[t_1,t_2]$, for almost-all ω, $(t,x) \mapsto u^\sharp(t,x,\omega)$ is a weak solution of (1) with initial data $u_0(x) = u^\sharp(t_1,x,\omega)$.

On the canonical space $\Omega = C_0(\mathbb{R},\mathbb{R})$ the space of continuous functions vanishing at 0, we denote θ^τ the shift operator on Ω with increment τ defined by $\theta^\tau(\omega) = \omega(\cdot+\tau)$ for any $\omega \in \Omega$. The solution operator S^τ_ω is defined for $v \in L^\infty(\mathbb{R})$ by $S^\tau_\omega(v)$ as the solution of (1) at time τ, with initial condition v at time $t_0 = 0$ when the realization of the noise is ω.

Now we can state the most important result of this work.

Theorem 3. *On $(\Omega \times \mathbb{D}\ ;\ \mathcal{F} \otimes \mathcal{D})$, the measure μ defined by*

$$\mu(d\omega,dv) = \delta_{u^\sharp(0,\cdot,\omega)}(dv)\ \mathbb{P}(d\omega) \tag{7}$$

is the unique measure that leaves invariant the (skew-product) transformation

$$\begin{array}{rcl} \Omega \times \mathbb{D} & \longrightarrow & \Omega \times \mathbb{D} \\ (\omega, v) & \longrightarrow & (\theta^t\omega, S^t_\omega(v)) \end{array}$$

with given projection $\mathbb{P}$ on $(\Omega,\mathcal{F})$.

The proof of this result is given at the end of Section 4.

3. Variational principle

First we give a detail discussion to introduce the variational principle.

3.1. The Burger's case

We begin with the particular case of Burgers equation when the flux is $\Psi(u) = u^2/2$. We recall that if we consider the one dimensional (inviscid) Burgers equation

$$\partial_t u + \partial_x \left(\frac{u^2}{2}\right) = \frac{\partial}{\partial x} G(t,x) \quad t > 0\ ,\ x \in \mathbb{R}$$

then for an initial condition u_0 having discontinuities of the first kind (i.e. u_0 belongs to the Skorohod space $\mathbb{D}$) there unique entropy-weak solution u is given by

$$u(t,x) = \frac{\partial}{\partial_x}\left(\inf_{\substack{\xi \in C^1(0,t) \\ \xi(t)=x}} \left\{ \mathcal{A}_{0,t} + \int_0^{\xi(0)} u_0(z)dz \right\} \right) ,$$

where

$$\mathcal{A}_{0,t}(\xi) = \int_0^t \left(\frac{1}{2}\dot{\xi}(s)^2 + G(t,\xi(s)) \right) ds \ . \tag{8}$$

For two times t_1, t_2, we have denoted $C^1(t_1,t_2)$ the space of continuously differentiable functions from $[t_1,t_2]$ to $\mathbb{R}$.

This relation between the Burgers's equation and the minimization problem is known as Lax-Oleĭnik formula (see [12, 14]) (and Hopf-Lax formula in its original context of Hamilton-Jacobi equations). It will be fully exploited in the study of scalar conservation law with stochastic forcing as we will see it right now.

In the above equation we have intuitively assumed that G is a deterministic regular force. Now the source term in the action $\mathcal{A}_{\tau,t}$ is $\int_\tau^t \sum_{k\geq 1} F_k(\xi(s))dB_k(s)$ where the above integral is not a stochastic integral but a path-wise integral. Indeed, since the trajectories $\omega \to B_k(t)(\omega)$ are $\varepsilon-$Hölder continuous and ξ is differentiable, $\int_\tau^t F_k(\xi(s))dB_k(s)$ exists as a Riemann-Stieltjes integral thanks to a result of Young [19]. Nevertheless, to avoid the use of such integrals we use integration by parts formula : one have with $g(\cdot) := F_k(\xi(\cdot))$

$$\int_\tau^t g(s)dB_k(s) = \lim_{\Delta\to 0} \sum_{i=0}^n g(t_i)(B_k(t_{i+1}) - B_k(t_i))$$

where the convergence holds uniformly in all finite partitions $\mathcal{P}_\Delta := \{\tau = t_0 \leq t_1 \leq ...t_{n+1} = t\}$ with $\max_i |t_{i+1} - t_i| < \Delta$. With $\bar{B}(s) := B_k(s) - B(\tau)$ one writes

$$\begin{aligned}
\sum_{i=0}^n g(t_i)(B_k(t_{i+1}) - B_k(t_i)) &= \sum_{i=0}^n g(t_i)(\bar{B}(t_{i+1}) - \bar{B}(t_i)) \\
&= -\sum_{i=0}^n \bar{B}(t_{i+1})\big(g(t_{i+1}) - g(t_i)\big) \\
&\quad + \sum_{i=0}^n \big\{\bar{B}(t_i)\big(g(t_{i+1}) - g(t_i)\big) + \big(\bar{B}(t_{i+1}) - \bar{B}(t_i)\big)g(t_{i+1})\big\} \\
&= -\sum_{i=0}^n \bar{B}(t_{i+1})\big(g(t_{i+1}) - g(t_i)\big) + \bar{B}(t)g(t) - \bar{B}(\tau)g(\tau) \ .
\end{aligned}$$

Consequently

$$\int_\tau^t g(s)dB_k(s) = -\int_t^\tau \big(B_k(s) - B_k(\tau)\big)\dot{g}(s)ds + \big(B_k(t) - B_k(\tau)\big)g(t)$$

and we rewrite the stochastic term of the action as

$$\int_\tau^t \sum_{k\geq 1} F_k(\xi(s))dB_k(s) = -\int_\tau^t \sum_{k\geq 1}\big(B_k(s)-B_k(\tau)\big)f_k(\xi(s))\dot{\xi}(s)ds + \sum_{k\geq 1}\big(B_k(t)-B_k(\tau)\big)F_k(\xi(t)) \tag{9}$$

where $f_k = F_k'$. If $\xi(t)$ is fixed to be x, then the second term in the above equality is independent on ξ, hence as in [3] the action is redefined as for $\xi \in C^1(\tau,t)$ as

$$\mathcal{A}_{\tau,t}(\xi) = \int_\tau^t \left(\frac{1}{2}\dot{\xi}(s)^2 - \sum_{k\geq 1}\big(B_k(s)-B_k(\tau)\big)f_k(\xi(s))\dot{\xi}(s)\right)ds + \sum_{k\geq 1}\big(B_k(t)-B_k(\tau)\big)F_k(\xi(t)) \ .$$

Remark. *Since the action is defined path-wisely it depends on ω hence should be denoted $\mathcal{A}^\omega_{\tau,t}$. We will not do for brevity of notations.*

Remark. *We strength the fact that (9) is a true integration by parts that allows us to rewrite the stochastic term and not a formal one as it was mentioned in [3].*

The Burgers case is particular because the Legendre transform of the flux $\Psi(p) = p^2/2$ that appears in (8) with the term $\frac{1}{2}\dot{\xi}^2$ is again the half of the function square. This is no more the case when the flux is another convex function. This term still be the Legendre transform of Ψ and these remarks motivate the Lax-Oleĭnik formula (4) with the action defined in (5).

There is another way of thinking in order to introduce the optimization problem: one can make a kind of change of variable in the variational formulation (2) and introduce an Hamilton-Jacobi-Bellman equation (HJB equation in short). Thus it is well known that these partial differential equation is related to a variational principle. This is briefly discussed in the following subsection.

3.2. Redefining the action via HJB equation

Let us develop the following non rigorous arguments. Let φ a test function in $C_c^2(\mathbb{R}\times\mathbb{R})$, thanks to an integration by parts one rewrites (2) as

$$\int_{t_0}^\infty \int_{\mathbb{R}} \partial_t\varphi(t,x)u(t,x)dxdt + \int_{t_0}^\infty \int_{\mathbb{R}} \partial_x\varphi(t,x)\Psi\big(u(t,x)\big)dxdt = -\int_{\mathbb{R}} u_0(x)\varphi(t_0,x)dx + \int_{\mathbb{R}}\int_{t_0}^\infty \partial_t\varphi(t,x)v(t,x)dtdx \tag{10}$$

with $F_k' = f_k$ and

$$v(t,x) = \sum_{k=1}^\infty f_k(x)(B_k(t)-B_k(t_0)) \ . \tag{11}$$

Consequently

$$\int_{t_0}^{\infty}\int_{\mathbb{R}} \partial_t\varphi(t,x)\big[u(t,x)-v(t,x)\big]dxdt + \int_{t_0}^{\infty}\int_{\mathbb{R}} \partial_x\varphi(t,x)\Psi\big(u(t,x)\big)dxdt$$
$$= -\int_{\mathbb{R}} u_0(x)\varphi(t_0,x)dx$$

and if W is such that $\partial_x W = w$ with $w = u+v$ we obtain

$$\int_{t_0}^{\infty}\int_{\mathbb{R}} \partial_t\varphi(t,x)w(t,x)dxdt + \int_{t_0}^{\infty}\int_{\mathbb{R}} \partial_x\varphi(t,x)\Psi\big(w(t,x)+v(t,x)\big)dxdt$$
$$= -\int_{\mathbb{R}} u_0(x)\varphi(t_0,x)dx \ .$$

Hence w is a solution of the stochastic scalar conservation law

$$\partial_t w + \mathrm{div}_x \Psi(w+v) = 0$$

and if we integrate with respect to the space variable x this equation, we derive the HJB equation

$$\partial_t W + \Psi(\partial_x W + v) = 0 \ .$$

This HJB is related to an optimization problem with an action involving the Legendre transform of $p \mapsto \Psi(p+v)$. Thanks to the behavior under translation of the Legendre transformation, one have $\big(\Psi(\cdot+v)\big)^*(q) = \Psi^*(q) - vq$ and we obtain the same king of action that in (5).

The above remarks are now made rigorous in the following subsection.

3.3. Dynamic programming equation

First we express the action $\mathcal{A}_{t_0,t}$ as

$$\mathcal{A}_{t_0,t}(\xi) = \underbrace{\int_{t_0}^{t} L(s,\xi(s),\dot{\xi}(s))ds}_{\widetilde{\mathcal{A}}_{t_0,t}(\xi)} + V(t,\xi(t)) \quad \text{with}$$
$$L(s,x,p) = \big(\Psi(\cdot + v(s,x))\big)^*(p)$$
$$= \Psi^*(p) - \textstyle\sum_{k\geq 1}\big(B_k(s)-B_k(t_0)\big)f_k(x)\times p \quad \text{and}$$
$$V(t,x) = \textstyle\sum_{k\geq 1}\big(B_k(t)-B_k(t_0)\big)F_k(x) \ .$$

With U_0 such that $\partial_x U_0 = u_0$, we define

$$W(t,x) = \inf_{\substack{\xi \in H^1(t_0,t) \\ \xi(t)=x}} \Big\{\widetilde{\mathcal{A}}_{t_0,t}(\xi) + U_0(\xi(t_0))\Big\} \ . \tag{12}$$

We remark that $U_0(\xi(t_0)) = \int_0^{\xi(t_0)} u_0(z)dz$ and

$$W(t,x) = \inf_{\substack{\xi \in H^1(t_0,t) \\ \xi(t)=x}} \Big\{ \mathcal{A}_{t_0,t}(\xi) \Big\} - V(t,x) \ .$$

The function W will be the unique solution of an Hamilton-Jacobi-Bellman equation. In classical calculus of variations, the left end point is fixed. This minor modification is not difficult and do not imply any changes except in the expression of the Hamiltonian that becomes in our case the Legendre transform of $p \mapsto L(t,x,p)$. Since we do not know any precise reference where these changes are discussed, we shortly prove that there exists a minimizer of the action $\widetilde{A}_{t_0,t}$. We recall the definition:

Definition 3. *On the interval* $[t_1,t_2]$*, we say that* $\xi \in H^1(t_1,t_2)$ *is a minimizer of the action* $\widetilde{\mathcal{A}}_{t_1,t_2}$ *if for any* $\gamma \in H^1(t_1,t_2)$ *with* $\gamma(t_1)=\xi(t_1)$ *and* $\gamma(t_2)=\xi(t_2)$ *we have* $\widetilde{\mathcal{A}}_{t_1,t_2}(\xi) \le \widetilde{\mathcal{A}}_{t_1,t_2}(\gamma)$.

We prove in the following proposition that the function W solves an Hamilton-Jacobi-Bellman equation.

Proposition 4. *The function* $(t,x) \mapsto W(t,x)$ *is Lipschitz continuous and satisfies for almost-all* t, x *the Hamilton-Jacobi-Bellman equation*

$$\partial_t W(t,x) + \Psi\Big(\partial_x W(t,x) + \textstyle\sum_{k\ge1} f_k(x)(B_k(t)-B_k(t_0))\Big) = 0 \ . \tag{13}$$

Proof. We denote

$$B_R(t_1,t_2) = \Big\{ \xi \in H^1(t_1,t_2) \ ; \ |\xi(t_1)| + \textstyle\int_{t_1}^{t_2} |\dot{\xi}(s)|^2 ds \le R \Big\}$$

which is clearly a closed and bounded subset of $H^1(t_1,t_2)$, hence weakly compact. Now we prove that there exists on $B_R(t_0,t)$ one minimizer of $\xi \mapsto F(\xi) := \widetilde{A}_{t_0,t}(\xi) + U_0(\xi(t_0))$. By the weak compactness of $B_R(t_0,t)$ it is sufficient that $\xi \mapsto F(\xi)$ is lower semi-continuous. Following [6], Theorem I.9.1 we just have to check the lower semi-continuity of the stochastic part

$$S(\xi) = -\textstyle\sum_{k\ge1} \displaystyle\int_{t_0}^{t} (B_k(s)-B_k(t_0)) f_k(\xi(s))\dot{\xi}(s) ds \ .$$

Let $(\xi_n)_{n\ge1}$ a sequence of $B_R(t_0,t)$ converging to ξ weakly. The weak convergence on $B_R(t_0,t)$ implies the uniform convergence on $[t_0,t]$. Writing $S(\xi) - S(\xi_n) = S_n^1 + S_n^2$ with

$$S_n^1 = \textstyle\sum_{k\ge1} \displaystyle\int_{t_0}^{t} (B_k(s)-B_k(t_0)) \big[f_k(\xi_n(s)) - f_k(\xi(s)) \big] \dot{\xi_n}(s) ds$$

$$S_n^2 = \textstyle\sum_{k\ge1} \displaystyle\int_{t_0}^{t} (B_k(s)-B_k(t_0)) f_k(\xi(s)) \big[\dot{\xi_n}(s) - \dot{\xi}(s) \big] ds \ ,$$

and by uniform convergence, $\lim_n S_n^1 = 0$. The weak convergence and the fact that $s \mapsto \sum_{k\geq 1}(B_k(s) - B_k(t_0))f_k(\xi(s))$ belongs to $L^2(t_0,t)$ yield $\lim_n S_n^2 = 0$.

Hence we have the lower semi-continuity and then there exists a minimizer $\xi_{\min} \in B_R(t_0,t)$ of $\xi \mapsto \widetilde{A}_{t_0,t}(\xi) + U_0(\xi(t_0))$. So for every t, x, there exists a minimizer $\xi_{\min} \in H^1(t_0,t)$ with $\xi_{\min}(t) = x$ such that

$$\begin{aligned} W(t,x) &= \inf_{\substack{\xi \in H^1(t_0,t) \\ \xi(t) = x}} \left\{ \widetilde{\mathcal{A}}_{t_0,t}(\xi) + U_0(\xi(t_0)) \right\} \\ &= \int_{t_0}^{t} L(s, \xi_{\min}(s), \dot{\xi}_{\min}(s))ds + U_0(\xi_{\min}(t_0)) \ . \end{aligned} \tag{14}$$

Working with the right end-point condition $\xi(t) = x$ in the calculus of variations will not affect theorems I.9.2, I.9.3 and I.9.4 of [6]. Then there exists M such that for any (t, x) and (t', x') in $\mathbb{R} \times \mathbb{R}$,

$$|W(t,x) - W(t',x')| \leq M\big(|t - t'| + |x - x'|\big) \ . \tag{15}$$

The equation satisfied by W will be obtained thanks to the following version of the dynamic programming principle. Indeed we can observe that for any $t_0 \leq r \leq t$,

$$W(t,x) = \inf_{\substack{\xi \in H^1(t_0,t) \\ \xi(t) = x}} \left(\int_r^t L(s, \xi(s), \dot{\xi}(s))ds + W(r, \xi(r)) \right) \ .$$

Now let $0 < h < t - t_0$ and take $r = t - h$ in the above identity. We substract $W(t,x)$ from both sides and we get

$$\inf_{\substack{\xi \in H^1(t_0,t) \\ \xi(t) = x}} \left(\frac{1}{h} \int_{t-h}^{t} L(s, \xi(s), \dot{\xi}(s))ds + \frac{1}{h}\big(W(t-h, \xi(t-h)) - W(t,x)\big) \right) = 0 \, .$$

When $h \downarrow 0$, we obtain

$$\begin{aligned} &- \frac{\partial W}{\partial t}(t,x) + \inf_{\substack{\xi \in H^1(t_0,t) \\ \xi(t) = x}} \left(L(t,x,\dot{\xi}(t)) - \frac{\partial W}{\partial x}(t,x) \times \dot{\xi}(t) \right) = 0 \\ &+ \frac{\partial W}{\partial t}(t,x) - \inf_{q\in\mathbb{R}} \left(-q \times \frac{\partial W}{\partial x}(t,x) + L(t,x,q) \right) = 0 \\ &+ \frac{\partial W}{\partial t}(t,x) + \sup_{q\in\mathbb{R}} \left(+q \times \frac{\partial W}{\partial x}(t,x) - L(t,x,q) \right) = 0 \\ &+ \frac{\partial W}{\partial t}(t,x) + H\left(t, x, \frac{\partial W}{\partial x}(t,x) \right) = 0 \end{aligned}$$

where $p \mapsto H(t,x,p)$ is the Legendre transform of $q \mapsto L(t,x,q)$. Using the behavior under translation of the Legendre transform, we have $H(t,x,p) = \Psi(p + v(t,x))$ where v is defined in (11). In other words, for al t, x W satisfies Hamilton-Jacobi-Bellman equation (13) (also refer in the literature as the dynamic programming equation). ∎

We will also need the following property.

Proposition 5. *For any t, the function $x \mapsto W(t,x)$ is semi concave. More precisely there exists a constant K such that $x \mapsto W(t,x) - K(1 - \frac{1}{t-t_0})x^2$ is concave.*

Proof. The concavity of $x \mapsto W(t,x) - Kx^2$ is concave is equivalent to

$$W(t,x) \geq \frac{1}{2}\big(W(t,x+h) + W(t,x-h)\big) - K\left(1 + \tfrac{1}{t-t_0}\right) \times h^2, \ \forall\, x, h. \tag{16}$$

Let $\xi_{\min}$ be the minimizer of the action such that W satisfies (14) (we recall that $\xi_{\min}(t) = x$). We introduce γ_{x+h} and γ_{x-h} in $H^1(t_0,t)$ defined by

$$\gamma_{x\pm h}(s) = \xi_{\min}(s) \pm \frac{s-t_0}{t-t_0}\ h\ ,$$

then satisfying $\gamma_{x\pm h}(t) = x \pm h$ and $\gamma_{x\pm h}(t_0) = \xi_{\min}(t_0)$. We calculate

$$\begin{aligned}
\Delta^1_{x,h} &= W(t,x+h) + W(t,x-h) \\
&\leq \int_{t_0}^{t} \big(L(s,\gamma_{x+h}(s),\dot\gamma_{x+h}(s)) + L(s,\gamma_{x-h}(s),\dot\gamma_{x-h}(s))\big)ds \\
&\qquad + U_0(\gamma_{x+h}(t_0)) + U_0(\gamma_{x-h}(t_0)) \\
&\leq \int_{t_0}^{t} \big(L(s,\xi_{\min}(s),\dot\gamma_{x+h}(s)) + L(s,\xi_{\min}(s),\dot\gamma_{x-h}(s))\big)ds \\
&\qquad + \int_{t_0}^{t} \big(L(s,\gamma_{x+h}(s),\dot\gamma_{x+h}(s)) - L(s,\xi_{\min}(s),\dot\gamma_{x+h}(s))\big)ds \\
&\qquad + \int_{t_0}^{t} \big(L(s,\gamma_{x-h}(s),\dot\gamma_{x-h}(s)) - L(s,\xi_{\min}(s),\dot\gamma_{x-h}(s))\big)ds \\
&\qquad + 2U_0(\xi_{\min}(t_0)) \\
&\leq \delta^1_{x,h} + \delta^2_{x,h} + \delta^3_{x,h} + 2U_0(\xi_{\min}(t_0))\ ,
\end{aligned}$$

with obvious notations. First we evaluate the term $\delta^1_{x,h}$ we recall that since Ψ is uniformly convex, for any real q we have $\Psi''(q) \geq \theta$. Then the Legendre transform $L(s,x,p) = \Psi(\cdot,\gamma_{x+h}(s))^*(p)$ satisfies (see [4, page 131])

$$\frac{1}{2}L(s,x,p_1) + \frac{1}{2}L(s,x,p_2) \leq L\big(s,x,(p_1+p_2)/2\big) + \frac{1}{8\theta}|p_1 - p_2|^2\ .$$

Using the identities $\dot\gamma_{x+h} + \dot\gamma_{x-h} = 2\dot\xi_{\min}$ and $\dot\gamma_{x+h} - \dot\gamma_{x-h} = 2h/(t-t_0)$, we deduce that

$$\begin{aligned}
\delta^1_{x,h} &\leq 2\int_{t_0}^{t} \left\{L\big(s,\xi_{\min}(s),(\dot\gamma_{x+h}(s)+\dot\gamma_{x-h}(s))/2\big) + C|\dot\gamma_{x+h}(s) - \dot\gamma_{x-h}(s)|^2\right\} ds \\
&\leq 2\int_{t_0}^{t} L(s,\xi_{\min}(s),\dot\xi_{\min}(s))ds + C\frac{h^2}{t-t_0}\ .
\end{aligned}$$

We finally obtain that

$$\delta^1_{x,h} + 2U_0(\xi_{\min}(t_0)) \leq 2W(t,x) + C\frac{h^2}{t-t_0} \ .$$

Now we write

$$\begin{aligned}
\delta^2_{x,h} &= \int_{t_0}^t \textstyle\sum_{k\geq 1}(B_k(s)-B_k(t_0))\big[f_k(\gamma_{x+h}(s)) - f_k(\xi_{\min}(s))\big]\dot\gamma_{x+h}(s)ds \\
&= \int_{t_0}^t \Big\{\textstyle\sum_{k\geq 1}(B_k(s)-B_k(t_0))\dot\gamma_{x+h}(s) \\
&\qquad\qquad \left(\displaystyle\int_0^1 \partial_x f_k\left((1-\nu)\gamma_{x+h}(s) - \nu\xi_{\min}(s)\right)(\gamma_{x+h}(s)-\xi_{\min}(s))d\nu\right)\Big\}ds \\
&= \int_{t_0}^t \Big\{\textstyle\sum_{k\geq 1}(B_k(s)-B_k(t_0))\left(\int_0^1 \partial_x f_k\left(\xi_{\min}(s) + (1-\nu)\frac{s-t_0}{t-t_0}h\right)d\nu\right) \\
&\qquad \left(\dot\xi_{\min}(s) + \frac{h}{t-t_0}\right)\frac{s-t_0}{t-t_0}h\Big\}ds
\end{aligned}$$

and analogously it holds that

$$\begin{aligned}
\delta^3_{x,h} = \int_{t_0}^t &\Big\{\textstyle\sum_{k\geq 1}(B_k(s)-B_k(t_0))\left(\int_0^1 \partial_x f_k\left(\xi_{\min}(s) - (1-\nu)\frac{s-t_0}{t-t_0}h\right)d\nu\right) \\
&\qquad \left(\dot\xi_{\min}(s) - \frac{h}{t-t_0}\right)\frac{s-t_0}{t-t_0}(-h)\Big\}ds \ .
\end{aligned}$$

We compute the sum

$$\begin{aligned}
&\delta^2_{x,h} + \delta^3_{x,h} \\
&= \int_{t_0}^t \Big\{\textstyle\sum_{k\geq 1}(B_k(s)-B_k(t_0))\frac{h^2(s-t_0)}{t-t_0} \\
&\quad \left(\int_0^1\left[\partial_x f_k\left(\xi_{\min}(s) + (1-\nu)\frac{s-t_0}{t-t_0}h\right) + \partial_x f_k\left(\xi_{\min}(s) - (1-\nu)\frac{s-t_0}{t-t_0}h\right)\right]d\nu\right)\Big\}ds \\
&+ \int_{t_0}^t \Big\{\textstyle\sum_{k\geq 1}(B_k(s)-B_k(t_0))\frac{h(s-t_0)}{t-t_0}\dot\xi_{\min}(s) \\
&\quad \left(\int_0^1\left[\partial_x f_k\left(\xi_{\min}(s) + (1-\nu)\frac{s-t_0}{t-t_0}h\right) - \partial_x f_k\left(\xi_{\min}(s) - (1-\nu)\frac{s-t_0}{t-t_0}h\right)\right]d\nu\right)\Big\}ds
\end{aligned}$$

and using hypothesis I and the identity

$$\begin{aligned}
&\int_0^1 \left[\partial_x f_k\left(\xi_{\min}(s) + (1-\nu)\frac{s-t_0}{t-t_0}h\right) - \partial_x f_k\left(\xi_{\min}(s) - (1-\nu)\frac{s-t_0}{t-t_0}h\right)\right]d\nu \\
&\quad = \int_0^1\int_0^1 \partial_x^2 f_k\left(\xi_{\min}(s) + (1-2\mu)(1-\nu)\frac{s-t_0}{t-t_0}h\right)d\mu\ 2(1-\nu)\frac{s-t_0}{t-t_0}\ h\ d\mu
\end{aligned}$$

we deduce that

$$\begin{aligned}\delta^2_{x,h} + \delta^3_{x,h} &\leq 2(t-t_0)^{\lambda+1} \sum_{k\geq 1} \|\partial_x f_k\|_\infty \|B_k\|_{t_0,t,\lambda} \times h^2 \\ &\qquad + (t-t_0)^{\lambda} \sum_{k\geq 1} \|\partial^2_x f_k\|_\infty \|B_k\|_{t_0,t,\lambda} \times h^2 \times \|\xi_{\min}\|_{H^1(t_0,t)} \\ &\leq C \times h^2 \ .\end{aligned}$$

As a conclusion we obtain (16). □

Remark. *By Alexandrov's theorem (see Appendix E in [6]), $x \mapsto W(t,x)$ is almost everywhere twice differentiable.*

3.4. Proof of Theorem 1

Now the proof of existence and uniqueness of the solution of (1).

Existence: Our candidate is $u = \partial_x W + v$ with W defined in (12) and v defined by (11). It is clearly adapted. Hypothesis I implies that $v(t,\cdot) \in L^\infty(\mathbb{R})$ and the Lipschitz property (15) for W imply that (ii) in definition 1 holds true.

We prove the variational formulation. Let φ be a test function in $C^2_c(\mathbb{R}\times\mathbb{R})$. We integrate the HJB equation (13) against $\partial_x\varphi$ and we integrate by parts in order to obtain :

$$\begin{aligned} - \int_{t_0}^{\infty} \int_{\mathbb{R}} \Psi\big(\partial_x W(t,x) + v(t,x)\big)\partial_x\varphi(t,x)dxdt &= \int_{t_0}^{\infty} \int_{\mathbb{R}} \partial_s W(t,x)\partial_x\varphi(t,x)dxdt \\ &= - \int_{\mathbb{R}} W_0(x)\partial_x\varphi(t_0,x)dx + \int_{t_0}^{\infty} \int_{\mathbb{R}} \partial_x W(t,x)\partial_t\varphi(t,x)dxdt \ .\end{aligned}$$

We have $\partial_x W_0(x) = \partial_x W(t_0,x) = u(t_0,x) + v(t_0,x) = u_0(x)$. By another integration by parts one obtains (10) that is an equivalent form of (2).

The entropy condition (3) is a consequence of the semi concavity of W (see proposition 5). Indeed, the concavity of $x \mapsto W(t,x) - Kx^2$ implies that its derivative is is a decreasing function. Then for any $z > 0$,

$$\partial_x W(t,x+z) - 2K(x+z) \leq \partial_x W(t,x) - 2Kx \ .$$

Moreover it holds that $\|\partial_x v(t,.)\|_\infty \leq (t-t_0)^\lambda \sum_{k\geq 1} \|\partial_x f_k\|_\infty \|B_k\|_{t,t_0,\lambda} := C$ and consequently $x \mapsto v(t,x) - 2C\, x$ is a decreasing function and for any $z > 0$,

$$v(t,x+z) - 2C(x+z) \leq v(t,x) - 2Cx \ .$$

The two above inequalities imply that $u = \partial_x W + v$ satisfies Oleĭnik's entropy condition (3).

Uniqueness: Since the random force in Equation (1) does not depend on u, the uniqueness is given by classical arguments as in Theorem 3 in [4].

4. Action minimizers and generalized characteristics

In order to construct an invariant measure for the stochastic scalar conservation law (1), we will construct an invariant solution. To do this we will use minimizers of the action $\mathcal{A}_{\tau,t}$ defined for a piecewise regular curve ξ with $\xi(t) = x$ as:

$$\mathcal{A}_{\tau,t}(\xi) = \int_\tau^t \Psi^*(\dot{\xi}(s)) - \sum_{k\geq 1}(B_k(s) - B_k(\tau))f_k(\xi(s))\dot{\xi}(s)ds \\ + \sum_{k\geq 1}(B_k(t) - B_k(\tau))F_k(\xi(t))$$

with $t_0 \to -\infty$. Using (9), of any path $\eta \in C^1(s,t)$ can be expressed as

$$\mathcal{A}_{s,t}(\eta) = \int_s^t \Psi^*(\dot{\eta}(r))dr + \int_s^t \sum_{k\geq 1} F_k(\eta(r))dB_k(r).$$

Hence the action is additive with respect to C^1 curves.

As in [3], the fundamental object is the one-sided minimizer defined as follows.

Definition 4. *Let $t \in \mathbb{R}$. A piecewise C^1 curve ξ: $]-\infty,t] \to \mathbb{R}$ is a one-sided minimizer if*

(i) for any $\tilde{\xi} \in H^1(-\infty,t)$ such that $\tilde{\xi}(t) = \xi(t)$ and $\tilde{\xi} = \xi$ on $]-\infty,\tau]$ for some $\tau < t$, it holds that $\mathcal{A}_{s,t}(\xi) \leq \mathcal{A}_{s,t}(\tilde{\xi})$ for any $s \leq \tau$;
(ii) for any $s \leq t$, $|\xi(s) - \xi(t)| \leq 1$.

Most of the properties of these one-sided minimizers are quite basic facts proves in [3]. Nevertheless we will give precisions as regard to the fact that we work with a general convex flux instead of the square function used in the Burger's case. We strength the fact that we choose a slightly different definition of one sided-minimizer (we impose the boundedness when the value $\xi(t)$ is fixed) because we do not work on the torus as in [3] but on $\mathbb{R}$.

4.1. Euler-Lagrange equations and properties of the action minimizers

We begin our study on a finite time interval $[t_1,t_2]$. We prove that

- a minimizer of the action satisfies an Euler-Lagrange equation and is a regular curve (Lemma 6),
- there exists effectively a unique solution to such an equation (Lemma 7),
- we give estimation on the velocities of such a minimizer (Lemma 8).

All these facts are true for all $\omega \in \Omega$ or in other words, we still work on each trajectories of the noise term. For any times t_1, t_2 and any $x_1, x_2 \in \mathbb{R}$, we denote

$$\mathcal{H}_{x_1,x_2}^{t_1,t_2} = \left\{ \xi \in H^1(t_1,t_2) \ ; \ \xi(t_1) = x_1 \ , \ \xi(t_2) = x_2 \right\}.$$

We have the following lemma in which we give the Euler-Lagrange equations satisfied by the minimizers.

Lemma 6. *If γ is a minimizer of $\mathcal{A}$ on $[t_1,t_2]$, that is*

$$\mathcal{A}_{t_1,t_2}(\gamma) = \inf_{\xi \in \mathcal{H}^{t_1,t_2}_{x_1,x_2}} \left\{ \int_{t_1}^{t_2} \Psi^*(\dot{\xi}(s)) - \textstyle\sum_{k\geq 1}(B_k(s) - B_k(t_1)) f_k(\xi(s))\dot{\xi}(s) ds \right.$$
$$\left. + \textstyle\sum_{k\geq 1}(B_k(t_2) - B_k(t_1)) F_k(\xi(t_2)) \right\}$$

then $\dot{\gamma} \in C^1(t_1,t_2)$ satisfies for $t_1 \leq r \leq s \leq t_2$

$$(\Psi^*)'(\dot{\gamma}(s)) - (\Psi^*)'(\dot{\gamma}(r)) = \int_r^s \textstyle\sum_{k\geq 1} f_k(\gamma(\tau)) dB_k(\tau) \ . \tag{17}$$

Proof. Since γ minimizes the functional $\mathcal{A}_{t_1,t_2}$, we have for any $\xi \in \mathcal{H}^{t_1,t_2}_{x_1,x_2}$, $\varepsilon \mapsto \frac{d}{d\varepsilon}\mathcal{A}_{t_1,t_2}(\gamma + \varepsilon\xi)$ equals 0 in $\varepsilon = 0$. This yields

$$0 = \int_{t_1}^{t_2} \Big[(\Psi^*)'(\dot{\gamma})(s)\dot{\xi}(s) - \textstyle\sum_{k\geq 1}(B_k(s) - B_k(t_1))\big(f_k'(\gamma)\dot{\gamma}\xi + f_k(\gamma)\dot{\xi}\big)(s) \Big] ds$$
$$+ \textstyle\sum_{k\geq 1}(B_k(t_2) - B_k(t_1)) f_k(\gamma(t_2))\xi(t_2) \ .$$

For $t_1 < \tau_1 \leq \tau_2 < t_2$, we write this identity with ξ_n defined as

$$\xi_n(s) = 0 \times 1\!\!1_{[t_1,\tau_1]\cup[\tau_2,t_2]}(s) + n\big(s - (\tau_1 - 1/n)\big) 1\!\!1_{[\tau_1 - 1/n, \tau_1]}(s)$$
$$+ 1\!\!1_{[\tau_1,\tau_2]}(s) + n\big(- s + (\tau_2 + 1/n)\big) 1\!\!1_{[\tau_2, \tau_2 + 1/n]}(s).$$

We obtain

$$\begin{aligned}
\int_{\tau_2}^{\tau_2+1/n} n(\Psi^*)'(\dot{\gamma}(s)) ds - \int_{\tau_1-1/n}^{\tau_1} n(\Psi^*)'(\dot{\gamma}(s)) ds = &\\
& - \int_{\tau_1}^{\tau_2} \textstyle\sum_{k\geq 1}(B_k(s) - B_k(t_1)) f_k'(\gamma(s))\dot{\gamma}(s) ds \\
& - \int_{\tau_1-1/n}^{\tau_1} \textstyle\sum_{k\geq 1}(B_k(s) - B_k(t_1))(f_k'(\gamma)\dot{\gamma}\xi_n)(s) ds \\
& - \int_{\tau_1-1/n}^{\tau_1} n\textstyle\sum_{k\geq 1}(B_k(s) - B_k(t_1)) f_k(\gamma(s)) ds \\
& - \int_{\tau_2}^{\tau_2+1/n} \textstyle\sum_{k\geq 1}(B_k(s) - B_k(t_1))(f_k'(\gamma)\dot{\gamma}\xi_n)(s) ds \\
& + \int_{\tau_2}^{\tau_2+1/n} n\textstyle\sum_{k\geq 1}(B_k(s) - B_k(t_1)) f_k(\gamma(s)) ds \ .
\end{aligned}$$

We remark that $\sup_n \|\xi_n\|_\infty \leq c$ and easy arguments allow us to let n goes to infinity. Hence

$$(\Psi^*)'(\dot{\gamma}(\tau_2)) - (\Psi^*)'(\dot{\gamma}(\tau_1)) = - \int_{\tau_1}^{\tau_2} \textstyle\sum_{k\geq 1}(B_k(s) - B_k(t_1)) f_k'(\gamma(s))\dot{\gamma}(s) ds$$
$$+ \textstyle\sum_{k\geq 1}(B_k(\tau_2) - B_k(t_1)) f_k(\gamma(\tau_2)) - \textstyle\sum_{k\geq 1}(B_k(\tau_1) - B_k(t_1)) f_k(\gamma(\tau_1))$$

that implies that $\tau \mapsto (\Psi^*)'(\dot\gamma(\tau))$ is continuous and since $(\Psi^*)' = (\Psi')^{-1}$, $\tau \mapsto \dot\gamma(\tau)$ is also continuous. Consequently, and with $g(s) = \sum_{k\geq 1}(B_k(s) - B_k(t_1))f_k'(\gamma(s))\dot\gamma(s)$ and the integration by parts formula 9 one may write

$$
\begin{aligned}
&(\Psi^*)'(\dot\gamma(\tau_2)) - (\Psi^*)'(\dot\gamma(\tau_1)) \\
&\qquad = -\int_{t_1}^{\tau_2} g(s)ds + g(\tau_2) - g(t_1) - \Big(-\int_{t_1}^{\tau_1} g(s)ds + g(\tau_1) - g(t_1)\Big) \\
&\qquad = \int_{t_1}^{\tau_2} \textstyle\sum_{k\geq 1} f_k(\gamma(s))dB_k(s) - \displaystyle\int_{t_1}^{\tau_1} \textstyle\sum_{k\geq 1} f_k(\gamma(s))dB_k(s) \\
&\qquad = \int_{\tau_1}^{\tau_2} \textstyle\sum_{k\geq 1} f_k(\gamma(s))dB_k(s)\ .
\end{aligned}
$$

By continuity of $\tau \mapsto \int_{t_1}^{\tau} \sum_{k\geq 1} f_k(\gamma(s))dB_k(s)$ (see Prop. 4.4.1 in [20]), the above formula is also true for $\tau_1 = t_1$ and $\tau_2 = t_2$. Then the formula (17) is true and $\gamma \in C^1(t_1,t_2)$. □

Remark. *Any action minimizer γ satisfies the following Euler-Lagrange equation:*

$$
\left\{\begin{array}{ll} \dot\gamma(s) & = \Psi'(v(s)) \\ dv(s) & = \sum_{k\geq 1} f_k(\gamma(\tau))dB_k(\tau) \end{array}\right. \tag{18}
$$

This Euler-Lagrange equation can be formally deduced from the following computation. If we want to find two curves γ and v such that $v(t) = u(t,\gamma(t))$, then

$$dv(t) = \partial_t u(t,\gamma(t)) + \partial_x u(t,\gamma(t))\dot\gamma(t).$$

With $\dot\gamma(t) = \Psi'(u(t,\gamma(t)))$ (or equivalently $v(t) = (\Psi')^{-1}(\dot\gamma(t))$), together with (1) one writes

$$dv(t) = \partial_t u(t,\gamma(t)) + \partial_x \Psi(u(t,\gamma(t))),$$

and we obtain (18). The curve γ is a generalized characteristic in the sense of Dafermos (see [1]).

Remark. *The equation (18) is a generalization of the Euler-Lagrange equation (2.3) in [3] obtained for $\Psi(z) = z^2/2$:*

$$
\left\{\begin{array}{ll} \dot\gamma(s) & = v(s) \\ dv(s) & = \sum_{k\geq 1} f_k(\gamma(\tau))dB_k(\tau). \end{array}\right.
$$

For sake of completeness we state in the following Lemma that there exists a unique solution to the Euler-Lagrange system of equations.

Lemma 7. *Let two times $T < \tau_0$ be fixed and ξ_0 and v_0 are two given real numbers. There exists a unique solution $\xi \in C^1(T,\tau_0)$ to the Euler-Lagrange equation (18)*

$$
\begin{aligned}
&\dot\xi(s) = \Psi'(v(s)) \\
&v(s) = v(\tau_0) + \int_s^{\tau_0} \textstyle\sum_{k\geq 1} f_k(\xi(r))dB_k(r) \qquad T \leq s \leq \tau_0
\end{aligned} \tag{19}
$$

such that $\xi(\tau_0) = \xi_0$ *and* $\dot{\xi}(\tau_0) = \Psi'(v_0)$ *with the initial condition* $(\xi(\tau_0),\dot{\xi}(\tau_0)) = (\xi_0,\Psi'(v_0))$.

Proof. Let $T < t < \tau_0$. We denote

$$K_{T,\tau_0}^{B,F} = \sum_{k\geq 1} \|F_k\|_{C^3} \left\{ \sup_{T\leq t_1\leq t_2\leq\tau_0} |B_k(t_1) - B_k(t_2)| \right\}.$$

The operator $L\ : C^1(t,\tau_0) \to C^1(t,\tau_0)$ is defined by

$$\begin{cases} \dot{L(\xi)} &= \Psi'(v) \\ v(s) &= v(\tau_0) - \int_s^{\tau_0} \sum_{k\geq 1} \big(B_k(r) - B_k(s)\big) f_k'(\xi(r))\dot{\xi}(r)dr \\ & \qquad\qquad\qquad + \sum_{k\geq 1} \big(B_k(\tau_0) - B_k(s)\big) f_k(\xi(\tau_0)) \end{cases}$$

with $L(\xi)(\tau_0) = \xi(\tau_0)$ and $\dot{L(\xi)}(\tau_0) = \Psi'(v_0) = \dot{\xi}(\tau_0)$. We have

$$\begin{aligned} \|\dot{L(\xi)}\|_{t,\tau_0,\infty} &\leq |\Psi'(v(\tau_0))| + \|\Psi'(v) - \Psi'(v(\tau_0))\|_{t,\tau_0,\infty} \\ &\leq |\dot{\xi}(\tau_0)| + L\|v - v(\tau_0)\|_{t,\tau_0,\infty} \\ &\leq |\dot{\xi}(\tau_0)| + L(\tau_0 - T)^\lambda K_{T,\tau_0}^{B,F}\big((\tau_0 - T)\|\dot{\xi}\|_{t,\tau_0,\infty} + 1\big) \end{aligned}$$

and since $L(\xi)(s) = \xi(\tau_0) + \int_s^{\tau_0} \dot{L(\xi)}(r)dr$ we may write

$$\|L(\xi)\|_{t,\tau_0,\infty} \leq |\xi(\tau_0)| + C_{T,\tau_0}(\tau_0 - t)\big(1 + \|\dot{\xi}\|_{t,\tau_0,\infty}\big)\ .$$

Consequently $\|L(\xi)\|_{C^1(t,\tau_0)} \leq |\xi_0| + |\Psi'(v_0)| + C(\tau_0 - t)(1 + \|\dot{\xi}\|_{C^1(t,\tau_0)})$ and the operator L satisfies $L(\mathcal{B}_0) \subseteq \mathcal{B}_0$ with

$$\mathcal{B}_0 = \{\xi \in C^1(t,\tau_0)\ :\ \|\xi\|_{C^1(t,\tau_0)} \leq 2(1 + |\xi_0| + |\Psi'(v_0)|)\}$$

provided that t is small enough to ensure that $C(t - \tau_0) \leq 1/2$. Let $\xi_1, \xi_2 \in \mathcal{B}_0$ and $v_i = (\Psi')^{-1}(\dot{\xi}_i)$ for $i = 1, 2$. Thanks to the following identity

$$\begin{aligned} v_1(s) - v_2(s) = &- \int_s^{\tau_0} \textstyle\sum_{k\geq 1} \big(B_k(r) - B_k(s)\big) f_k'(\xi_1(r))\big[\dot{\xi}_1(r) - \dot{\xi}_2(r)\big]dr \\ &- \int_s^{\tau_0} \textstyle\sum_{k\geq 1} \big(B_k(r) - B_k(s)\big)\big[f_k'(\xi_1(r)) - f_k'(\xi_2(r))\big]\dot{\xi}_2(r) \end{aligned}$$

We can easily prove that

$$\|L(\xi_1) - L(\xi_2)\|_{C^1(t,\tau_0)} \leq C(\tau_0 - t)\|\xi_1 - \xi_2\|_{C^1(t,\tau_0)}\ .$$

Hence L is a contraction on $\mathcal{B}_0$ (with t eventually smaller) and there exists $\xi \in \mathcal{B}_0$ such that $L(\xi) = \xi$ and then there exists a unique solution in $C^1(t,\tau_0)$ to the euler-Lagrange equations (19) for short time. By a concatenation argument, the existence and uniqueness is extended to $C^1(T,\tau_0)$ for any $T < \tau_0$. □

The following lemma gives a key estimation on the velocities of the characteristics. This will play a central role in our further investigations.

We recall that Ψ' is Lipschitz (Hypothesis II(a)) and the Legendre transform of Ψ satisfies also the linear growth condition $c_1|v|^{1+\alpha} \leq |\Psi^*(v)| \leq c_2|v|^{1+\beta}$ with $\alpha = 1/k_2$, $\beta = 1/k_1$ and and two positive constants c_3 and c_4 different from those in Hypothsesis II(b)).

Lemma 8. *If γ is a minimizer of the action $\mathcal{A}$ on the time interval $[t_1,t_2]$ with $\gamma(t_1) = x_1$, $\gamma(t_2) = x_2$ and $t_2 - t_1 \geq 1$, then there exists a constant c such that*

$$\begin{aligned}\|\dot\gamma\|_{t_1,t_2,\infty} &\leq c\ C_{t_1,t_2}\\ &\quad + \left((t_2-t_1)^{\frac{-1}{1+\alpha}} + C_{t_1,t_2}(t_2-t_1)^{\frac{\alpha}{1+\alpha}}\right)(x_2-x_1)^{\frac{1+\beta}{1+\alpha}}\\ &\quad + \left((t_2-t_1)^{\frac{-1}{1+\alpha}} + C_{t_1,t_2}(t_2-t_1)^{\frac{\alpha}{1+\alpha}}\right)\left(C_{t_1,t_2}^{\frac{1}{1+\alpha}} + C_{t_1,t_2}^{\frac{1}{\alpha}}(t_2-t_1)^{\frac{1}{1+\alpha}}\right)\end{aligned} \tag{20}$$

with $C_{t_1,t_2} = \sum_{k\geq 1}\|F_k\|_{C^2}\left\{\sup_{t_1\leq r\leq r'\leq t_2}|B_k(r)-B_k(r')|\right\}$.

Proof. Let $t_1 \leq t \leq t_2$ and s be such that $|\dot\gamma(s)| = \inf_{r\in[t_1,t_2]}|\dot\gamma(r)|$. Writing $\dot\gamma(t) = (\Psi'\circ(\Psi')^{-1})(\dot\gamma(t)) - (\Psi'\circ(\Psi')^{-1})(\dot\gamma(s)) + \dot\gamma(s)$, we have

$$\begin{aligned}|\dot\gamma(t)| &\leq L\left|(\Psi')^{-1}(\dot\gamma(t)) - (\Psi')^{-1}(\dot\gamma(s))\right| + |\dot\gamma(s)|\\ &\leq L\times\Delta_{s,t} + \frac{\|\dot\gamma\|_{L^1(t_1,t_2)}}{t_2-t_1}\ ,\end{aligned}$$

with

$$\begin{aligned}\Delta_{s,t} &= \left|(\Psi')^{-1}(\dot\gamma(t)) - (\Psi')^{-1}(\dot\gamma(s))\right|\\ &= -\int_s^t \textstyle\sum_{k\geq1}\big(B_k(r)-B_k(s)\big)f_k'(\gamma(r))\dot\gamma(r)dr\\ &\qquad\qquad + \textstyle\sum_{k\geq1}\big(B_k(t)-B_k(s)\big)f_k(\gamma(t))\\ &\leq C_{t_1,t_2} + C_{t_1,t_2}\|\dot\gamma\|_{L^1(t_1,t_2)}\ .\end{aligned}$$

Consequently,

$$|\dot\gamma(t)| \leq C_{t_1,t_2}L + \left(C_{t_1,t_2}L + 1/(t_2-t_1)\right)\|\dot\gamma\|_{L^1(t_1,t_2)}\ . \tag{21}$$

Now we estimate the L^1 norm of $\dot\gamma$. We recall that $c_1|v|^{1+\alpha} \leq |\Psi^*(v)|$. By Young's inequality $ab \leq (c_1/2)\ a^{1+\alpha} + c\ b^{(1+\alpha)/\alpha}$ and Jensen's inequality we obtain

$$\begin{aligned}&\int_{t_1}^{t_2}\textstyle\sum_{k\geq1}\big(B_k(r)-B_k(s)\big)f_k'(\gamma(r))\dot\gamma(r)dr\\ &\qquad\qquad\leq c\ (t_2-t_1)C_{t_1,t_2}^{(1+\alpha)/\alpha} + \frac{c_1}{2}\int_{t_1}^{t_2}|\dot\gamma(s)|^{1+\alpha}ds\ .\end{aligned}$$

Since γ is a minimizer,

$$\mathcal{A}_{t_1,t_2}(\gamma) = \int_{t_1}^{t_2} \Psi^*(\dot{\gamma}(s)) - \textstyle\sum_{k\geq 1}(B_k(s) - B_k(t_1)) f_k(\gamma(s))\dot{\gamma}(s) ds \\ + \textstyle\sum_{k\geq 1}(B_k(t_2) - B_k(t_1)) F_k(\gamma(t_2))$$

and

$$\frac{c_1}{2}\int_{t_1}^{t_2} |\dot{\gamma}(s)|^{1+\alpha} ds \leq \mathcal{A}_{t_1,t_2}(\gamma) + c\ (t_2 - t_1)\ C_{t_1,t_2}^{(1+\alpha)/\alpha} + C_{t_1,t_2}\ . \tag{22}$$

By the minimization property of γ, $\mathcal{A}_{t_1,t_2}(\gamma) \leq \mathcal{A}_{t_1,t_2}(\xi)$ with the curve ξ defined by $\xi(s) = x_1 + (s - t_1)/(t_2 - t_1) \times (x_2 - x_1)$. Using $|\Psi^*(v)| \leq c_2 |v|^{1+\beta}$ we may write

$$\mathcal{A}_{t_1,t_2}(\gamma) \leq c\ C_{t_1,t_2} + c\ \frac{(x_2 - x_1)^{1+\beta}}{(t_2 - t_1)^{\beta}} \leq c\ ((x_2 - x_1)^{1+\beta} + C_{t_1,t_2})$$

where we used the fact that $t_2 - t_1 \geq 1$. We report the above inequality in (22) and we get that

$$\int_{t_1}^{t_2} |\dot{\gamma}(s)|^{1+\alpha} ds \leq c\left((x_2 - x_1)^{1+\beta} + C_{t_1,t_2} + C_{t_1,t_2}^{(1+\alpha)/\alpha}(t_2 - t_1)\right)\ .$$

Since $\|\dot{\gamma}\|_{L^1(t_1,t_2)} \leq (t_2 - t_1)^{\alpha/(1+\alpha)} \|\dot{\gamma}\|_{L^{1+\alpha}(t_1,t_2)}$, with (21) we obtain

$$|\dot{\gamma}(t)| \leq c\ C_{t_1,t_2} + c\ \left(C_{t_1,t_2} + \frac{1}{t_2 - t_1}\right) (t_2 - t_1)^{\alpha/(1+\alpha)} \\ \times \left((x_2 - x_1)^{1+\beta} + C_{t_1,t_2} + C_{t_1,t_2}^{(1+\alpha)/\alpha}(t_2 - t_1)\right)^{1/(1+\alpha)}$$

and using the inequality $(1 + x)^a \leq 1 + x^a$ when $a < 1$ and $x \geq 0$ we obtain (20). $\square$

4.2. Existence and uniqueness of one-sided minimizers

The following proposition establishes the existence of a one-sided minimizer. It is a short rewriting of the one contained in [3] that takes care of the fact that we do not work on the torus.

Proposition 9. *For every $x \in \mathbb{R}$ and $t \in \mathbb{R}$, there exists a one-sided minimizer γ such that $\gamma(t) = x$.*

Proof. Let n be an integer such that $-n < t$ and γ_n a minimizer of $\mathcal{A}_{-n,t}$ satisfying $\gamma_n(t) = x$, $\gamma_n(-n) = x+1$ and $\sup_{-n\leq s\leq t} |\gamma_n(s) - x| \leq 1$. As regards to the proof of Proposition 4 such a γ_n exists. For $-n < s < t$ we have $\|\dot{\gamma}_n\|_{s,t,\infty} \leq K$ by Lemma 8, where K depends on s and t but do not depend on x. Hence, up to a subsequence, there exists $\gamma \in H^1(s,t)$ such that $\lim_{n\to\infty} \gamma_n = \gamma$ in $C(s,t)$

and $\lim_{n\to\infty}\dot\gamma_n = \dot\gamma$ weakly in $L^2(s,t)$. From the Euler-Lagrange equation (19) it follows that $\lim_{n\to\infty}\gamma_n = \gamma$ in $C^1(s,t)$ (after a new extraction of a subsequence). A diagonal process implies that there exists $\gamma \in C^1(-\infty,t)$ such that $\lim_{n\to\infty}\gamma_n = \gamma$ for the C^1 convergence on any compact of $]-\infty,t]$.

It remains to prove that γ is a one-sided minimizer. By construction, (ii) in Definition 4 is satisfied. Let a curve $\xi \in H^1(-\infty,t)$ with $\xi(t) = x$ and $\xi = \gamma$ on $]-\infty,\tau]$ for some τ. Without loss of generality we can take $\xi \in C^1(-\infty,t)$ because the action can be strictly decreased by smoothing a curve containing corners (see Fact 2 page 885 in [3]). Fix $s \le \tau$ and let $(\xi_n)_{n\ge1}$ a sequence in $C^1(s,t)$ such that $\xi_n(s) = \gamma_n(s)$, $\xi_n(t) = x$ and $\lim_{n\to\infty}\xi_n = \xi$ in $C^1(s,t)$. We have $\lim_{n\to\infty}\xi_n(s) = \lim_{n\to\infty}\gamma_n(s) = \gamma(s) = \xi(s)$. Using Hypothesis I(d) we obtain

$$\begin{aligned}
\big|\mathcal{A}_{s,t}(\xi) - \mathcal{A}_{s,t}(\xi_n)\big| \le & \int_s^t |\Psi^*(\dot\xi(r)) - \Psi^*(\dot\xi_n(r))|dr \\
& + \int_s^t \Big|\textstyle\sum_{k\ge1}(B_k(r)-B_k(s))f_k(\xi_n(r))\big(\dot\xi(r)-\dot\xi_n(r)\big)\Big|\,dr \\
& + \int_s^t \Big|\textstyle\sum_{k\ge1}(B_k(r)-B_k(s))\big(f_k(\xi(r))-f_k(\xi_n(r))\big)\dot\xi(r)\Big|\,dr \\
& + \Big|\textstyle\sum_{k\ge1}(B_k(t)-B_k(s))\big(F_k(\xi_n(t))-F_k(\xi(t))\big)\Big| \\
\le & \,(C(R)+C_{s,t})\ \|\dot\xi-\dot\xi_n\|_{L^1(s,t)} + C_{s,t}\ \|\dot\xi\|_{L^2(s,t)}\ \|\xi-\xi_n\|_{L^2(s,t)}
\end{aligned}$$

with $C_{s,t} = \sum_{k\ge1}\|F_k\|_{C^2}\left\{\sup_{t_1\le r\le r'\le t_2}|B_k(r)-B_k(r')|\right\}$ is defined as in Lemma 8 and R is such that $\|\dot\xi\|_{s,t,\infty} \vee \left(\sup_{n\ge1}\|\dot\xi_n\|_{s,t,\infty}\right) \le R$. The above estimation implies $\lim_{n\to\infty}\mathcal{A}_{s,t}(\xi_n) = \mathcal{A}_{s,t}(\xi)$. Moreover

$$|\mathcal{A}_{s,t}(\gamma) - \mathcal{A}_{s,t}(\gamma_n)| \le C\ \|\gamma-\gamma_n\|_{C^1(s,t)} \xrightarrow[n\to\infty]{} 0\ ,$$

with C depending on $\|\gamma\|_{C^1(s,t)}$ and for $-n \le s$, $\mathcal{A}_{s,t}(\gamma_n) \le \mathcal{A}_{s,t}(\xi_n)$ because γ_n is a minimizer of $\mathcal{A}_{-n,t}$. Therefore

$$\mathcal{A}_{s,t}(\gamma) = \lim_{n\to\infty}\mathcal{A}_{s,t}(\gamma_n) \le \lim_{n\to\infty}\mathcal{A}_{s,t}(\xi_n) = \mathcal{A}_{s,t}(\xi)\ .$$

We conclude that γ is a one-sided minimizer. □

4.3. Intersection of one-sided minimizers

It is a classical fact that two different one-sided minimizers $\gamma_1 \in C^1(-\infty,t_1)$ and $\gamma_2 \in C^1(-\infty,t_2)$ with the same end $\gamma_1(t_1) = \gamma_2(t_2)$ cannot intersect each other more than once (see [3, Lemma 3.2]). So if two one-sided minimizers intersect more than once, they coincide on their common interval of definition.

Now we will use for the first time the randomness of the force. More precisely since our force is random, it can be proved that two minimizers have an effective intersection at $-\infty$. We will use Theorem 2 stating that the fractional Brownian noise is arbitrary small on an infinite number of arbitrary long time intervals. In other words for all $\varepsilon > 0$, $T > 0$, for almost-all ω, there exists a sequence of random time $(t_n(\omega))_{n\geq 1}$, such that $t_n(\omega) \to -\infty$ and

$$\forall\ n\ ,\ \sup_{t_n-T\leq s\leq t_n} \textstyle\sum_{k\geq 1}\Big\{\|F_k\|_{C_b^2(\mathbb{R})}|B_k(s)-B_k(t_n)|\Big\}\ \leq \varepsilon\ .$$

We can state the following proposition which is contained in [3].

Proposition 10. *For almost-all ω, for any distinct one-sided minimizers γ_1 and γ_2 on $]-\infty, t_1]$ and $]-\infty, t_2]$ respectively the following result holds. Assume that γ_1 and γ_2 intersect at time t in a point x, then $t_1 = t_2 = t$ and $\gamma_1(t_1) = \gamma_2(t_2) = x$.*

The proof of this result is exactly the same that the proof of Theorem 3.2 in [3] so we do not repeat it. Nevertheless, it is based on [3, Lemma 3.3] that we recall and briefly prove because there are minor modification due to our fractional noise.

Lemma 11. *Almost-surely, for any $\varepsilon > 0$ and any two one-sided minimizers $\gamma_1 \in C^1(-\infty, t_1)$ and $\gamma_2 \in C^1(-\infty, t_2)$, there exists $T = T(\varepsilon)$ and a sequence of random times $t_n = t_n(\omega, \varepsilon) \to -\infty$ such that*

$$|\mathcal{A}_{t_n-T,t_n}(\gamma_i) - \mathcal{A}_{t_n-T,t_n}(\zeta)| < \varepsilon \quad , \quad \textit{for } i = 1,2 \textit{ and } \zeta \in \{\widehat{\gamma_1\gamma_2}, \widehat{\gamma_2\gamma_1}\}$$

where $\widehat{\gamma_1\gamma_2}$ and $\widehat{\gamma_2\gamma_1}$ are reconnecting curves defined by

$$\begin{aligned}\widehat{\gamma_1\gamma_2}(s) &= \tfrac{t_n-s}{T}\gamma_1(s) - \tfrac{t_n-T-s}{T}\gamma_2(s)\\ \widehat{\gamma_2\gamma_1}(s) &= \tfrac{t_n-s}{T}\gamma_2(s) - \tfrac{t_n-T-s}{T}\gamma_1(s)\ .\end{aligned}$$

Proof. For T sufficiently large, we use (6) (which is recalled above) in order to find a sequence of random time $(t_n)_{n\geq 1}$ such that $\lim_{n\to\infty} t_n = -\infty$ and

$$\forall\ n\ ,\ C_{t_n-T,t_n} = \sup_{t_n-T\leq s\leq t_n} \textstyle\sum_{k\geq 1}\Big\{\|F_k\|_{C_b^2(\mathbb{R})}|B_k(s)-B_k(t_n)|\Big\}\ \leq \dfrac{1}{T}\ , \quad (23)$$

where the notation C_{t_n-T,t_n} comes from (20) of Lemma 8.

Now we make the following remark. If a curve γ minimizes the action on the interval $[s,t]$ in the sense of Lemma 6, then for any $s < r < t$, its restriction on $[s,r]$ will minimize the action with respect to curves in $H^1(s,r)$ having the same ends as γ at s and r. Indeed suppose that there a minimizer $\xi \neq \gamma$ on $[s,r]$ that such that $\mathcal{A}_{s,r}(\xi) = \mathcal{A}_{s,r}(\gamma) - \varepsilon$. Using (9), the action can be written using a true pathwise integral with respect to the noise, so the action of any path $\eta \in C^1(s,t)$ is expressed as

$$\mathcal{A}_{s,t}(\eta) = \int_s^t \Psi^*(\dot\eta(s))ds + \int_s^t \textstyle\sum_{k\geq 1} F_k(\eta(s))dB_k(s)$$

so the action is additive with respect to C^1 curves. Considering the curve $\widehat{\xi\gamma_{r,t}}$ obtained by gluing the path ξ to the restriction of γ on $[r,t]$, we observe that

$$\mathcal{A}_{s,t}(\widehat{\xi\gamma_{r,t}}) = \mathcal{A}_{s,r}(\gamma) - \varepsilon + \mathcal{A}_{r,t}(\gamma) = \mathcal{A}_{s,t}(\gamma) - \varepsilon < \mathcal{A}_{s,t}(\gamma)$$

that contradicts the fact that γ is a minimizer on $[s,t]$.

Therefore, the one-sided minimizers γ_i are minimizers on each time interval $[t_n - T, t_n]$ and we may apply Lemma 8 in order to obtain thanks to (23) that for any n

$$\begin{aligned} \sup_{t_n - T \le s \le t_n} |\dot\gamma_i(s)| &\le \frac{c}{T} + 2\,c\left(T^{-\frac{1}{1+\alpha}} + T^{-\frac{2}{1+\alpha}} + T^{-\frac{1}{\alpha}}\right) \\ &\le \frac{\tilde c}{T^{1/(1+\alpha)}} \ . \end{aligned}$$

Consequently, for $\zeta \in \{\widehat{\gamma_1\gamma_2}, \widehat{\gamma_2\gamma_1}\}$ we have $\|\dot\zeta\|_{t_n - T, t_n, \infty} \le \frac{2\tilde c}{T^{1/(1+\alpha)}}$. Using $|\Psi^*(v)| \le c_2 |v|^{1+\beta}$ and (23) we then compute

$$\begin{aligned} &|\mathcal{A}_{t_n - T, t_n}(\gamma_i) - \mathcal{A}_{t_n - T, t_n}(\zeta)| \\ &\quad \le \int_{t_n - T}^{t_n} \Big| \Psi^*(\dot\gamma_i(s)) - \Psi^*(\dot\zeta(s)) \Big| \, ds \\ &\quad + \int_{t_n - T}^{t_n} \Big| \textstyle\sum_{k \ge 1} (B_k(s) - B_k(t_n - T)) f_k(\zeta) \big(\dot\gamma_i(s) - \dot\zeta(s)\big) \Big| \, ds \\ &\quad + \int_{t_n - T}^{t_n} \Big| \textstyle\sum_{k \ge 1} (B_k(s) - B_k(t_n - T)) \big(f_k(\gamma_i(s)) - f_k(\zeta(s))\big) \dot\gamma_i(s) \Big| \, ds \\ &\quad + \Big| \textstyle\sum_{k \ge 1} (B_k(t_n) - B_k(t_n - T)) \big(F_k(\zeta(t_n)) - F_k(\gamma_i(t_n))\big) \Big| \\ &\quad \le C \left(\frac{2\,T}{T^{\frac{1+\beta}{1+\alpha}}} + T\,\frac{4\,C_{t_n - T, t_n}}{T^{1/(1+\alpha)}} + 2\,C_{t_n - T, t_n} \right) \\ &\quad \le C \left(\frac{2}{T^{\frac{\beta - \alpha}{1+\alpha}}} + \frac{4}{T^{1/(1+\alpha)}} + \frac{2}{T} \right) \ . \end{aligned} \tag{24}$$

where C is a numerical constant. The result follows by choosing T such that the right hand side of (24) is less than ε. $\square$

4.4. *Invariant measure: existence and uniqueness (proof of Theorem 3)*

In this subsection, we prove Theorem 3. First we construct the invariant solution $u^\sharp$. We denote $\mathcal{M}_{t,x}$ the family of all one-sided minimizers with end x at time t. We define

$$u^\sharp(t, x, \omega) = \inf_{\gamma \in \mathcal{M}_{t,x}} \dot\gamma(t) \ . \tag{25}$$

Proposition 10 implies an important property of one-sided minimizers. To any $x \in \mathbb{R}$ such that the cardinal of $\mathcal{M}_{t,x}$ is at least 2 (this means that more than one one-sided minimizer comes to x at time t), there corresponds a non-trivial segment $I(x) = [\gamma_1(t - T), \gamma_2(t - T)]$, where $\gamma_1 < \gamma_2$ on $]-\infty, t]$ because two different one-sided minimizers can not intersect each other more than once. Then the segments $I(x)$ are mutually disjoint. Consequently, for almost-all ω, the set of $x \in \mathbb{R}$ with more than one one-sided minimizer is coming to x at time t is at most countable. This is the key point to prove that $u^\sharp(t,\cdot,\omega) \in \mathbb{D}$ (see [3, Lemma 3.8]).

The fact that $u^\sharp \in L^\infty(\mathbb{R})$ is a trivial consequence of Lemma 8. The measurability issues can be treated as in [3, Lemma 3.9].

The fact that on any finite time interval $[t_1,t_2]$, for almost-all ω, $(t,x) \mapsto u^\sharp(t,x,\omega)$ is a weak solution of (1) with initial data $u_0(x) = u^\sharp(t_1,x,\omega)$ is obtained by construction of $u^\sharp$. Hence $S^t_\omega \delta_{u^\sharp(0,\cdot,\omega)} = \delta_{u^\sharp(0,\cdot,\theta^t\omega)} = \delta_{u^\sharp(t,\cdot,\omega)}$. Thus the measure μ defined in Theorem 3 is invariant.

It only remains to prove the uniqueness. This is also done in the proof of the Theorem 4.2 in [3]. Let us give few details. for λ another invariant measure, we denote λ_ω its projection on Ω in such a way that we may write that $\lambda(d\omega,dv) = \int_\Omega \lambda_\omega(dv)\mathbb{P}(d\omega)$. The invariance of λ implies that there exists a subset D of $\mathbb{D}$ such that $\lambda(D^c) = 0$ and with the property that for any $v \in D$ and any $n \in \mathbb{N}$, there exists v_n such that $H_{-n}(v_n) = v$ where the operator H_t maps the solution of (1) at a negative time t to this solution at time 0. By repeating the end of the proof of Proposition 9, one can prove that if a solution of (1) can be extended to arbitrary negative times, this solution coincides with $u^\sharp$ at time $t = 0$ for almost-all x (because the set of $x \in \mathbb{R}$ with more than one one-sided minimizer is coming to x at time 0 is at most countable). Hence $v(x) = u^\sharp(0,x)$ almost-everywhere and $\lambda_\omega(dv) = \delta_{u^\sharp(0,\cdot,\omega)}(dv)$ so we have uniqueness.

5. An asymptotic property of fBm: proof of Theorem 2

The paper [3] essentially uses the fact that the Brownian motion has periods of arbitrary length and arbitrary small amplitude oscillation as time goes to $-\infty$. In this section, we will prove that a similar property holds for the fBm defined on the all real line $(-\infty,+\infty)$. The result stated in Theorem 2 is recalled below.

Theorem 2: For all $\varepsilon > 0$, $T > 0$, for almost-all ω, there exists a sequence of random time $(t_n(\omega))_{n\geq 1}$, such that $t_n(\omega) \to -\infty$ and

$$\forall\ n\ ,\ \sum_{k\geq 1}\Big\{\|F_k\|_{C^2_b(\mathbb{R})} \sup_{t_n-T\leq s\leq r\leq t_n} |B_k(r)-B_k(s)|\Big\}\ \leq \varepsilon\ .$$

Before proving this theorem, we will recall and prove some basic facts about the fBm defined on the real line $\mathbb{R}$. We first give the moving average representation of the fBm $(B(t))_{t\in\mathbb{R}}$. For $s,t \in \mathbb{R}$, we define

$$f_t(s) = c_H\left((t-s)_+^{H-\frac{1}{2}} - (-s)_+^{H-\frac{1}{2}}\right)$$

with

$$c_H = \left(\int_0^\infty \left((1+s)^{H-\frac{1}{2}} - s^{\frac{1}{2}}\right)^2 ds + \frac{1}{2H}\right)^{-\frac{1}{2}}\ .$$

Notice that $\int_{\mathbb{R}} f_t^2(s)ds < \infty$ and more precisely, if $H \neq 1/2$, $s \mapsto f_t(s)$ behaves like $(-s)^{H-3/2}$ when $s \to -\infty$ which is square integrable at $-\infty$. With these notations, the fBm can be written as

$$B(t) = \int_{\mathbb{R}} f_t(s) dW_s$$

where the process $(W_t)_{t\in\mathbb{R}}$ is a two sided classical Brownian motion which is obtained by gluing two independent copies of one sided Brownian motions together at time $t = 0$.

Since we are interested in the oscillations of the fBm, we express its increments for $t < t' < 0$ as

$$\begin{aligned} B(t) - B(t') &= \int_{\mathbb{R}} c_H \left\{ (t-r)_+^{H-\frac{1}{2}} - (t'-r)_+^{H-\frac{1}{2}} \right\} dW_r \\ &= \int_{-\infty}^{t} c_H \left\{ (t-r)^{H-\frac{1}{2}} - (t'-r)^{H-\frac{1}{2}} \right\} dW_r + \int_t^{t'} c_H (t'-r)^{H-\frac{1}{2}} dW_r \\ &= \int_{\mathbb{R}} g_{t,t'}(r) dW_r \end{aligned}$$

where

$$g_{t,t'}(r) = c_H \left\{ (t-r)^{H-\frac{1}{2}} - (t'-r)^{H-\frac{1}{2}} \right\} 1\!\!1_{]-\infty,t]}(r) + c_H (t'-r)^{H-\frac{1}{2}} 1\!\!1_{[t,t']}(r) \ .$$

Let $\mathcal{F}_s$ the sigma-algebra generated by the family of random variables $\{B(r); -\infty < r \leq s\}$. We remark that for $s \leq 0$, $\mathcal{F}_s \subseteq \sigma\{W_r; -\infty < r \leq s\} := \mathcal{F}^W_{-\infty,s}$. Then we deduce the following expression: for any $-\infty < s \leq t \leq t' \leq 0$

$$\mathbb{E}(B(t) - B(t')|\mathcal{F}_s) = \mathbb{E}\left[\int_{-\infty}^{s} c_H \left\{ (t-r)^{H-\frac{1}{2}} - (t'-r)^{H-\frac{1}{2}} \right\} dW_r \middle| \mathcal{F}_s \right] \ . \tag{26}$$

The proof of Theorem 2 is based on the following reversed conditional Borel-Cantelli's lemma.

Lemma 12. *Let $(\mathcal{F}_n)_{n\geq 1}$ be a decreasing sequence of $\sigma-$fields and $(A_n)_{n\geq 1}$ a sequence of events such that $A_n \in \mathcal{F}_n$. Then the events*

$$\left\{ \sum_{k\geq 1} 1\!\!1_{A_k} < \infty \right\} \qquad \textit{and} \qquad \left\{ \sum_{k\geq 1} \mathbb{E}\left(1_{A_k}|\mathcal{F}_{k+1}\right) < \infty \right\}$$

are almost-surely equal.

Proof. Let $M_n = 1\!\!1_{A_n} - \mathbb{E}(1\!\!1_{A_n}|\mathcal{F}_{n+1})$. We have $\mathbb{E}(M_n|\mathcal{F}_{n+1}) = 0$ so $(M_n)_{n\geq 1}$ is a reversed martingale difference sequence. Thus $\sum_{k\geq 1} \mathbb{E}(M_k^2|\mathcal{F}_{k+1}) < \infty$ implies that $\sum_{k\geq 1} M_k$ is convergent almost-surely (see Stout [16, Theorem 2.8.7]). We have

$$\begin{aligned}\mathbb{E}(M_k^2|\mathcal{F}_{k+1}) &= \mathbb{E}(\mathbb{1}_{A_k}^2|\mathcal{F}_{k+1}) - (\mathbb{E}(\mathbb{1}_{A_k}|\mathcal{F}_{k+1}))^2 \\ &= \mathbb{E}(\mathbb{1}_{A_k}|\mathcal{F}_{k+1})\Big[1 - \mathbb{E}(\mathbb{1}_{A_k}|\mathcal{F}_{k+1})\Big] \\ &\leq \mathbb{E}(\mathbb{1}_{A_k}|\mathcal{F}_{k+1}) \ .\end{aligned}$$

Hence $\sum_{k\geq 1} M_k$ is almost surely convergent and since $\sum_{k\geq 1} M_k = \sum_{k\geq 1} \mathbb{1}_{A_k} - \sum_{k\geq 1} \mathbb{E}(\mathbb{1}_{A_k}|\mathcal{F}_{k+1})$, we deduce that

$$\left\{\sum_{k\geq 1} \mathbb{1}_{A_k} < \infty\right\} \supset \left\{\sum_{k\geq 1} \mathbb{E}\left(1_{A_k}|\mathcal{F}_{k+1}\right) < \infty\right\} \ .$$

It is clear that if $\sum_{k\geq 1} \mathbb{1}_{A_k} < \infty$ then $\sum_{k\geq 1} \mathbb{E}(1_{A_k}|\mathcal{F}_{k+1})$ is integrable and consequently almost-surely finite. So we have the equality of the two events. □

Now we prove Theorem 2.

Proof. Let $\varepsilon > 0$ and $T > 0$ be fixed. Let $(t_n)_{n\geq 1}$ be a decreasing sequence of negative real numbers such that

$$\begin{cases} \lim_{n\to\infty} t_n = -\infty \ ; \\ t_{n+1} < t_n - T \text{ and} \\ \sum_{n\geq 1} (t_n - t_{n+1})^{H-1} < \infty. \end{cases}$$

Step 1:

First we prove the result for a single fBm $(B(t))_{t\in\mathbb{R}}$. Our goal is to show that

$$\lim\inf_{n\to\infty} \sup_{t_n - T\leq t,s\leq t_n} |B_t - B_s| \leq \varepsilon \ .$$

We denote $\mathcal{F}_{t_n} = \sigma\{B(r); -\infty < r \leq t_n\}$. For $t \geq t_{n+1}$ we set

$$\begin{aligned} B^{n+1}(t) &= \mathbb{E}(B(t)|\mathcal{F}_{t_{n+1}}) \\ \overline{B}^{n+1}(t) &= B(t) - B^{n+1}(t) \ . \end{aligned}$$

By the gaussian properties of the fBm it follows that $\overline{B}^{n+1}(t)$ is independent of $\mathcal{F}_{t_{n+1}}$. We set

$$\begin{aligned} A_n(\varepsilon) &= \left\{ \sup_{t_n - T\leq t,s\leq t_n} |B(t) - B(s)| \leq \varepsilon \right\}, \\ \widetilde{A}_n(\varepsilon) &= \left\{ \sup_{t_n - T\leq t,s\leq t_n} \left|B^{n+1}(t) - B^{n+1}(s)\right| \leq \varepsilon \right\}, \\ \overline{A}_n(\varepsilon) &= \left\{ \sup_{t_n - T\leq t,s\leq t_n} \left|\overline{B}^{n+1}(t) - \overline{B}^{n+1}(s)\right| \leq \varepsilon \right\}. \end{aligned}$$

Then obviously one has $\overline{A}_n(\varepsilon/2) \subset A_n(\varepsilon) \cup (\widetilde{A}_n(\varepsilon/2))^c$. This implies

$$\mathbb{1}_{A_n(\varepsilon)} + \mathbb{1}_{(\widetilde{A}_n(\varepsilon/2))^c} \geq \mathbb{1}_{\overline{A}_n(\varepsilon/2)} \ .$$

We take the conditional expectation with respect to $\mathcal{F}_{t_{n+1}}$ and we deduce that

$$\mathbb{E}\left(\mathbb{1}_{A_n(\varepsilon)}|\mathcal{F}_{t_{n+1}}\right) \geq \mathbb{P}\left(\overline{A}_n(\varepsilon/2)\right) - \mathbb{1}_{(\widetilde{A}_n(\varepsilon/2))^c}$$

because $\overline{A}_n(\varepsilon/2)$ is independent of $\mathcal{F}_{t_{n+1}}$, while $\widetilde{A}_n(\varepsilon/2)$ belongs to $\mathcal{F}_{t_{n+1}}$. Arguing as above we also obtain

$$\mathbb{P}(\overline{A}_n(\varepsilon/2)) + \mathbb{P}((\widetilde{A}_n(\varepsilon/4))^c) \geq \mathbb{P}(A_n(\varepsilon/4)) \ .$$

We add these inequalities and we get

$$\mathbb{E}\left(\mathbb{1}_{A_n(\varepsilon)}|\mathcal{F}_{t_{n+1}}\right) \geq \mathbb{P}(A_n(\varepsilon/4)) - \mathbb{P}((\widetilde{A}_n(\varepsilon/4))^c) - \mathbb{1}_{(\widetilde{A}_n(\varepsilon/2))^c}. \tag{27}$$

We will show hereafter that one has

$$\sum_{n\geq 1} \mathbb{P}((\widetilde{A}_n(\varepsilon))^c) < \infty \ , \tag{28}$$

while

$$\mathbb{P}(A_n(\varepsilon)) \geq \exp\left(\frac{-cT}{\varepsilon^H}\right). \tag{29}$$

Assume for a moment that these inequalities hold true. Then from (27) we deduce that $\sum_{n\geq 1} \mathbb{E}(\mathbb{1}_{A_n(\varepsilon)}|\mathcal{F}_{t_{n+1}}) = \infty$ a.s. and by Lemma 12 we obtain $\sum_{n\geq 1} \mathbb{1}_{A_n(\varepsilon)} = \infty$ a.s., which implies

$$\lim\inf_{n\to\infty} \sup_{t_n - T\leq t,s\leq t_n} |B_t - B_s| \leq \varepsilon.$$

Proof of (28)

Let $t_n - T \leq s \leq t \leq t_n$. By (26) we have

$$B^{n+1}(t) - B^{n+1}(s) = \mathbb{E}\left[\int_{-\infty}^{t_{n+1}} c_H \left\{(s-r)^{H-\frac{1}{2}} - (t-r)^{H-\frac{1}{2}}\right\} dW_r \middle| \mathcal{F}_{t_{n+1}}\right]$$

and for $p \geq 1$ we obtain

$$\mathbb{E}\left(|B^{n+1}(t) - B^{n+1}(s)|^{2p}\right) \leq c\left(\int_{-\infty}^{t_{n+1}} \left|(s-r)^{H-\frac{1}{2}} - (t-r)^{H-\frac{1}{2}}\right|^2 dr\right)^p \ .$$

In the above integral we make successively the changes of variables $v = r - s$ and $u = v/(t-s)$. This yield

$$\begin{aligned}\left(\mathbb{E}\left(|B^{n+1}(t) - B^{n+1}(s)|^{2p}\right)\right)^{\frac{1}{p}} &\leq c(t-s)^{2H} \int_{-\infty}^{\frac{t_{n+1}-s}{t-s}} \left|(-u)^{H-\frac{1}{2}} - (1-u)^{H-\frac{1}{2}}\right|^2 du \\ &\leq c(t-s)^{2H} \int_{-\infty}^{\frac{t_{n+1}-s}{t-s}} (-u)^{2H-3} du\end{aligned}$$

where we have used the fact that for $-u$ sufficiently big (and positive), $|(-u)^{H-\frac{1}{2}} - (1-u)^{H-\frac{1}{2}}| \le c(-u)^{H-\frac{3}{2}}$. The above inequality is then true for sufficiently large n. Finally we obtain that

$$\mathbb{E}\left(|B^{n+1}(t) - B^{n+1}(s)|^{2p}\right) \le c\left((t-s)(t_n - t_{n+1})^{H-1}\right)^{2p} \ . \tag{30}$$

Now we use the Garsia-Rodemich-Rumsey inequality (see [7]): let f be a continuous function, ρ and g two continuous stricly increasing functions on $[0,\infty)$ with $\rho(0) = g(0) = 0$ and $\lim_{x\to\infty} \rho(x) = \infty$. Then it holds

$$|f(t) - f(s)| \le 8 \int_0^{t-s} \rho^{-1}\left(\frac{4C_{s,t}}{u^2}\right) dg(u)$$
$$\text{with } \ C_{s,t} = \int_s^t \int_s^t \rho\left(\frac{|f(t') - f(s')|}{g(|t'-s'|)}\right) ds'dt' \ .$$

We apply the above inequality with $\rho(u) = u^4$ and $g(u) = u$. Thus there exists a constant c and a random variable δ_n such that

$$|B^{n+1}(t) - B^{n+1}(s)| \le \delta_n \times |t-s|^{1/2} \qquad \text{with}$$
$$\delta_n = c\left(\int_{t_n-T}^{t_n}\int_{t_n-T}^{t_n}\left(\frac{|B^{n+1}(t') - B^{n+1}(s')|}{|t'-s'|}\right)^4 ds'dt'\right)^{1/4} \ .$$

By (30) and the Jensen inequality, it is clear that

$$\mathbb{E}(|\delta_n|^{2p}) \le cT^p(t_n - t_{n+1})^{2p(H-1)} \ ,$$

and we obtain

$$\sup_{t_n-T\le t,s\le t_n} \left|B^{n+1}(t) - B^{n+1}(s)\right| \le c\, T^{1/2}\ \delta_n \ . \tag{31}$$

Now we write that

$$\begin{aligned}
\mathbb{P}((\widetilde{A}_n(\varepsilon))^c) &\le \frac{1}{\epsilon}\ \mathbb{E}\left(\sup_{t_n-T\le t,s\le t_n}\left|B^{n+1}(t) - B^{n+1}(s)\right|\right)\\
&\le c\ \frac{T^{1/2}}{\epsilon}\ \mathbb{E}(\delta_n) \le c\ \frac{T^{1/2}}{\epsilon}\ \left(\mathbb{E}(\delta_n^2)\right)^{1/2}\\
&\le c\ \frac{T}{\epsilon}\ (t_n - t_{n+1})^{H-1}
\end{aligned}$$

and since $\sum_{n\ge1}(t_n - t_{n+1})^{H-1} < \infty$, we obtain (28).

Proof of (29)

This inequality is a consequence of Talagrand's small ball estimate (see [17] or [13, Theorem 3.8]). Indeed, one needs al least $T\varepsilon^{-H}$ balls of radius ε under the Dudley metric $d(s,t) = \left(\mathbb{E}|B(t) - B(s)|^2\right)^{1/2}$ to cover the time interval $[t_n - T, t_n]$. It follows that that there exists a constant c such that

$$\log \mathbb{P}\left(\sup_{t_n - T \le t, s \le t_n} |B(t) - B(s)| \le \varepsilon\right) \ge -c \, \frac{T}{\varepsilon^H}$$

and we deduce (29). This achieves our first step.

Step 2:

We prove Theorem 2 for the noise $F(t,x) = \sum_{k\ge1} F_k(x) B_k(t)$. We denote $\mathbf{B}(t) = \sum_{k\ge1} c_k B_k(t)$ with $c_k = \|F_k\|_{C_b^2(\mathbb{R})}$, $\mathcal{F}_{t_n} = \sigma\{B_k(r); -\infty < r \le t_n; k \ge 1\}$ and for $t \ge t_{n+1}$ we set

$$\begin{aligned} \mathbf{B}^{n+1}(t) &= \mathbb{E}(\mathbf{B}(t)|\mathcal{F}_{t_{n+1}}) \\ \overline{\mathbf{B}}^{n+1}(t) &= \mathbf{B}(t) - \mathbf{B}^{n+1}(t) \ . \end{aligned}$$

Replacing B by $\mathbf{B}$ in the events $A_n(\varepsilon)$, $\widetilde{A}_n(\varepsilon)$ and $\overline{A}_n(\varepsilon)$, we define the events $\mathbf{A}_n(\varepsilon)$, $\widetilde{\mathbf{A}}_n(\varepsilon)$ and $\overline{\mathbf{A}}_n(\varepsilon)$ by

$$\begin{aligned} \mathbf{A}_n(\varepsilon) &= \left\{ \sum_{k\ge1} c_k \sup_{t_n - T \le t, s \le t_n} |B_k(t) - B_k(s)| \le \varepsilon \right\}, \\ \widetilde{\mathbf{A}}_n(\varepsilon) &= \left\{ \sum_{k\ge1} c_k \sup_{t_n - T \le t, s \le t_n} \left|B_k^{n+1}(t) - B_k^{n+1}(s)\right| \le \varepsilon \right\}, \\ \overline{\mathbf{A}}_n(\varepsilon) &= \left\{ \sum_{k\ge1} c_k \sup_{t_n - T \le t, s \le t_n} \left|\overline{B}_k^{n+1}(t) - \overline{B}_k^{n+1}(s)\right| \le \varepsilon \right\}. \end{aligned}$$

Clearly (6) will be proved as soon as the inequalities (28) and (29) will be replaced by

$$\sum_{n\ge1} \mathbb{P}((\widetilde{\mathbf{A}}_n(\varepsilon))^c) < \infty \qquad \text{and} \tag{32}$$

$$\mathbb{P}(\mathbf{A}_n(\varepsilon)) \ge \exp\left(\frac{-cT}{\varepsilon^H}\right). \tag{33}$$

The inequality (31) is valid for any of the fractional Brownian motion B_k we may write that for any $k \ge 1$

$$\sup_{t_n - T \le t, s \le t_n} \left|B^{n+1}(t) - B^{n+1}(s)\right| \le c \, T^{1/2} \, \delta_n$$

and we deduce that

$$\begin{aligned}\mathbb{P}\big((\widetilde{\mathbf{A}}_n(\varepsilon))^c\big) &\leq \frac{1}{\varepsilon}\, \mathbb{E}\left(\sum_{k\geq 1} c_k \sup_{t_n-T\leq t,s\leq t_n} \left|B_k^{n+1}(t)-B_k^{n+1}(s)\right|\right)\\ &\leq c\, T^{1/2}\left(\sum_{k\geq 1} c_k\right)\, \mathbb{E}(\delta_n)\\ &\leq c\, T\, (t_n-t_{n+1})^{H-1}\left(\sum_{k\geq 1} c_k\right)\,.\end{aligned}$$

Using Hypothesis I, $\sum_{k\geq 1} c_k \leq C \sum_{k\geq 1} k^{-\frac{2+H}{H}} < \infty$ and consequently (32) holds true.

Now we prove (33). Repeating the arguments of the proof of (29) We have for any $k, n \geq 1$

$$\mathbb{P}\left(\sup_{t_n-T\leq t,s\leq t_n} |B_k(t)-B_k(s)| \leq \frac{\varepsilon k^{\alpha/H}}{C_0}\right) \geq \exp\left\{-c\,\frac{TC_0^H}{\varepsilon^H k^\alpha}\right\}\,,$$

with $2+H > \alpha > 1$ and C_0 will be precised later. For each n the events $A_{n,k}(\varepsilon) = \{\sup_{t_n-T\leq t,s\leq t_n} |B_k(t)-B_k(s)| \leq \frac{\varepsilon k^{\alpha/H}}{C_0}\}$ are independant and $\cap_{k\geq 1} A_{n,k}(\varepsilon) \subseteq \mathbf{A}_n(\varepsilon)$ if we choose $C_0 = \sum_{k\geq 1} c_k k^{\alpha/H}$ which is a convergent sum thanks to Hypothesis I. Since $\sum_{k\geq 1}\frac{1}{k^\alpha} < \infty$ it holds

$$\mathbb{P}(\mathbf{A}_n(\varepsilon)) \geq \prod_{k\geq 1}\mathbb{P}\left(A_{n,k}\left(\frac{\varepsilon k^{\alpha/H}}{C_0}\right)\right) \geq \exp\left\{-c\,\frac{TC_0^H}{\varepsilon^H}\sum_{k\geq 1}\frac{1}{k^\alpha}\right\} > 0$$

and (33) is proved. This completes our proof. □

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