Synthèses des résultats d'activitée sur le projet LEA Math-Mode intitulé

Équations aux dérivées partielles stochastiques sans viscosité

Bruno Saussereau

Laboratoire de mathématiques de Besançon,
UMR CNRS 6623,
16 route de Gray,
25030 Besançon, France.

e-mail: bruno.saussereau@univ-fcomte.fr

Ion Lucretiu Stoica

Institut de Matematica IMAR,
Bucarest,
Romania.

e-mail: lucretiu.stoica@univ-bucarest.ro

*

Abstract: We study existence and uniqueness of scalar conservation laws with a stochastic force. The qualitative behavior is investigated by means of the characteristics of the conservation laws. In our stochastic framework, the characteristics are given as the unique solution of an ordinary random differential equation.

AMS 2000 subject classifications: 60H06, HJB equation, scalar conservation laws, deterministic control theory.
Keywords and phrases: Scalar conservation laws, random perturbations, variational principle, deterministic control theory, Hamilton Jacobi Bellmann equation.

*This work is supported by the Laboratoire Européen Associé CNRS Franco-Roumain, Math Mode

1. Notations and main results

In this note we study the following scalar conservation law

$$\partial_t u(t,x,\omega) + \mathrm{div}_x \Psi\big(u(t,x,\omega)\big) = \mathrm{div}_x \dot{F}(t,x,\omega)\ . \tag{1}$$

In the above equation, $x \in \mathbb{R}$, $t \geq 0$, $u(t,x,\cdot)$ is a random variable with values in $\mathbb{R}$ and F is a random force. A deterministic initial data $u(t_0,x) = u_0(x)$ is given. We will always assume that $u_0 \in \mathbb{L}^\infty(\mathbb{R})$. As usual the random force will not be differentiable in the time variable, hence $\dot{F}$ denotes its formal time derivative. The sense given to the above equation will be stated below thanks to a weak formulation. Since we will essentially work on each trajectory, our random term F will belong to a large class of processes. Before we give a precise description of the class of random forcing we deal with, we recall that in the pioneer work of [3], the authors proved the existence of an invariant measure for the Burgers equation (this means that $\Psi(u) = u^2/2$ with stochastic forcing given by $F(t,x,\omega) = \sum_{k=1}^\infty F_k(x)\dot{B}_k(t)$ where $(B_k)_{k\geq 1}$ are independent standard Wiener processes on the real line $\mathbb{R}$ ($\dot{B}_k$ is again designates the formal time derivative of this process).

In a way, our work is a generalization of the existence and uniqueness results contained in [3] because we work with a general conservation law depending on the function Ψ and also because we can reach a large class of noise as we describe it now.

We denote $C_b^r(\mathbb{R})$ the space of $r-$times (r is an integer) differentiable bounded functions with bounded derivatives endowed with the norm given by $\|\varphi\|_{C_b^r(\mathbb{R})} = \sum_{i=0}^r \|\varphi^{(i)}\|_\infty$.

For $0 < \lambda \leq 1$ and $-\infty < a < b < +\infty$, $C^\lambda(a,b)$ is the space of λ-Hölder continuous functions $f : [a,b] \to \mathbb{R}$, equipped with the norm $\|f\|_\lambda := \|f\|_{a,b,\infty} + \|f\|_{a,b,\lambda}$, where

$$\|f\|_{a,b,\infty} = \sup_{a\leq r\leq b} |f(r)| \quad \text{and} \quad \|f\|_{a,b,\lambda} = \sup_{a\leq r\leq s\leq b} \frac{|f(s)-f(r)|}{|s-r|^\lambda}\ .$$

In the probabilistic framework of $(\Omega, \mathcal{F}, \mathbb{P})$, we make the following assumption on the stochastic forcing term F.

Hypothesis 1. *For any t,x, the stochastic term F can be decomposed as $F(t,x) = \sum_{k=1}^\infty F_k(x)B_k(t)$ where:*

(i) the sequence $(F_k)_{k\geq 1}$ is such that for any k, the function F_k belongs to $C_b^r(\mathbb{R})$ with an integer $r \geq 3$ and satisfies $\|F_k\|_{C_b^r(\mathbb{R})} \leq C/k^2$.

(ii) there exists $\lambda > 0$ such that the sequence of processes $\big((B_k(t))_{t\in(-\infty,\infty)}\big)_{k\geq 1}$ satisfies $B_k(\cdot) \in C^\lambda(a,b)$ for any $k \geq 1$, $-\infty < a < b < +\infty$. Without loss of generality we impose that $\|B_k\|_\lambda \leq C$.

We remark that the processes B_k are not necessarily. It is quite straightforward that the above noise term covers the one of [3] but it also covers sequences of processes as fractional Brownian motion of any Hurst parameter. One may assume that the Hölder norm of B_k depends on k but in this case one have to impose additionally that $\sum_{k\geq 1} \|B_k\|_\lambda / k^2 < \infty$.

When all the functions F_k are null, Equation (1) is a deterministic scalar conservation law and the is a wide literature on this subject. One can cite (in a non exhaustive way) [2, 14, 8]. In the deteministic case, this kind of equation has a nice qualitative behavior: discontinuities that are related with the creation of shocks, description of the behavior in terms of characteristic. The advantage of the path-wise approach, as it has been done in [3], is to transfert a lot of tools from the deterministic case to the random equation. More precisely, we will be able to state important qualitative properties thanks to the equation satisfied by the characteristic. In the stochastic framework, the Euler-Lagrange equations will be some ordinary random differential equations.

The function Ψ will satisfy the following assumption.

Hypothesis 2. *The function Ψ is convex and satisfies the following super linear growth condition*

$$\frac{\Psi(v)}{|v|} \xrightarrow[|v|\to\infty]{} +\infty.$$

Now we give the precise meaning of (1) (see Definition 2.1 in [3]).

Definition 1. *A random field u defined on $[t_0, +\infty) \times \mathbb{R} \times \Omega$ with real values is a weak solution of (1) if:*

(i) For all $t > t_0$ and $x \in \mathbb{R}$, $u(t, x, \cdot)$ is measurable with respect to $\mathcal{F}_{t_0,t} = \sigma\{B_k(s), t_0 \leq s \leq t, k \geq 1\}$.
(ii) Almost surely, $u(\cdot, \cdot, \omega) \in L^1_{loc}([t_0, \infty) \times \mathbb{R})$
(iii) For all test function $\varphi \in C^2_c(\mathbb{R}\times\mathbb{R})$ (the set of twice differentiable functions with compact support) the following equality holds almost-surely

$$\begin{aligned}\int_{t_0}^\infty \int_{\mathbb{R}} \frac{\partial \varphi(t,x)}{\partial t} u(t,x)dxdt + \int_{t_0}^\infty \int_{\mathbb{R}} \frac{\partial \varphi(t,x)}{\partial x} \Psi\big(u(t,x)\big)dxdt = \\ - \int_{\mathbb{R}} u_0(x)\varphi(t_0,x)dx \\ - \int_{\mathbb{R}} \sum_{k=1}^\infty \left\{ F_k(x) \int_{t_0}^\infty \frac{\partial^2 \varphi(t,x)}{\partial t \partial x} (B_k(t) - B_k(t_0))dt \right\} dx \ . \end{aligned} \tag{2}$$

It is well known that this notion of weak solution is not sufficient to have uniqueness for the solution of (1) in the deterministic case. One have to introduce the notion of admissible solution (or weak-entropy solution).

Definition 2. *We say that a random field u which is a already a weak solution of Equation (1) is an entropy-weak solution if for almost-all $\omega \in \Omega$,*

$$u(t, x+, \omega) \leq u(t, x-, \omega) \tag{3}$$

for all $(t, x) \in (t_0, \infty) \times \mathbb{R}$.

One refers to [4, 17] for other formulation of stochastic entropy solutions.

First of all, in this paper we are interested in the existence and uniqueness of the entropy-weak solution of (1). In [3], this property is proved for the particular case of the Burger's equation. The authors use the standard mollification of the Brownian noise and then obtain a variational formula as the deteministic Lax-Oleĭnik formula. We will also prove a Lax-Oleĭnik formula using a direct approach via the Hamilton-Jacobi equation naturally associated to our problem.

Before stating our existence and uniqueness result one have to introduce further notations.

For two times t_1, t_2, we denote $H^1(t_1,t_2)$ the Sobolev space of $\mathbb{L}^2(t_1,t_2)-$weakly differentiable functions from $[t_1,t_2]$ to $\mathbb{R}$.

For a function f, we denote f^* its Legendre transform defined as

$$f^*(q) = \sup_p \big(pq - f(p)\big) \ .$$

Our first important result is the following.

Theorem 3. *There exists a unique entropy-weak solution to the stochastic scalar conservation law* (1) *such that $u(t_0,x) = u_0(x)$. For $t \geq t_0$, this solution is given by the following Lax-Oleĭnik type formula :*

$$u(t,x,\omega) = \frac{\partial}{\partial_x}\left(\inf_{\substack{\xi \,\in\, H^1(t_0,t)\\ \xi(t)\,=\,x}} \left\{ \mathcal{A}_{t_0,t} + \int_0^{\xi(t_0)} u_0(z)dz \right\} \right) \ , \tag{4}$$

with

$$\begin{aligned} \mathcal{A}_{t_0,t}(\xi) = \int_{t_0}^t &\Big\{ \Psi^*(\dot{\xi}(s)) - \textstyle\sum_{k\geq 1}\big(B_k(s)-B_k(t_0)\big) f_k(\xi(s))\dot{\xi}(s) \Big\} ds \\ &+ \textstyle\sum_{k\geq 1}\big(B_k(t)-B_k(t_0)\big) F_k(\xi(t)) \ . \end{aligned} \tag{5}$$

Stochastic scalar conservation laws is a topic of growing interest in the few years. Nevertheless, there is only a few number of works on this subject. Besides the work of [3], one can refer to Kim [10], Holden-Risebro [7], Nualart-Feng [4] or Vallet-Wittbold [17].

We think that the originality of our contribution is to give a Lax-Oleĭnik formula. Indeed this variational representation is a powerful tool in the study of the singularities (shocks) of the solution via the characteristic equations.

The second contribution of our work is a first step toward the study of the invariant measure for the stochastic conservation law (1) for the particular case of a fractional noise. There is only one paper that deals with invariant measure and this is for the particular case of Burgers equation with a Brownian noise (see [3]).

We shall work with the following particular noise term F.

Hypothesis 3. *The sequence of processes $\big((B_k(t))_{t\in\mathbb{R}}\big)_{k\geq 1}$ is a sequence of independent fractional Brownian motions (fBm in short) with Hurst parameter $H \in (0,1)$. This means that for each k, $(B_k(t))_{t\in\mathbb{R}}$ is a Gaussian process satisfying $B_k(0) = 0$ and $\mathbb{E}(|B_k(t) - B_k(s)|^2) = |t-s|^{2H}$.*

There are serious difficulties to work with this fBm. First, unlike the classical Brownian motion, the two-sided (this means defined on the all real line) fBm is not obtained by gluing two independent copies of a one-sided (defined on $\mathbb{R}^+$) fBm together at time $t = 0$. Moreover, when $t \leq 0$, the two-sided fBm is no more a Volterra type process (as it is the case for the classical fBm). One refer to [9] for a more detailed discussion on this fact.

At last, the increments are not independent. In [3], there is roughly speaking only one probabilistic property of the noise that is employed. Indeed all the work can be written, as we did it, for general noise with Hölder regularity of the trajectories (see Hypothesis 1). In order to prove the existence of an invariant measure, the authors used that the increments os the Brownian noise are arbitrary small on an infinite number of arbitrary long time intervals. In other words for all $\varepsilon > 0$, $T > 0$, for almost-all ω, there exists a sequence of random time $(t_n(\omega))_{n\geq 1}$, such that $t_n(\omega) \to -\infty$ and

$$\forall\ n\ ,\ \sup_{t_n - T \leq s \leq t_n} \textstyle\sum_{k\geq 1} \Big\{ \|F_k\|_{C_b^2(\mathbb{R})} |B_k(s) - B_k(t_n)| \Big\}\ \leq \varepsilon\ .$$

This result relies on the independence of the increments of a Brownian motion and on the Borel-Cantelli lemma. In a fractional Brownian framework, one have to adapt this argument to prove an analogous property for the trajectories of a fBm when the time goes to $-\infty$.

Despite these difficulties, one can state the following results concerning the increments of a fractional Brownian motion defined on the real line. This result is new and is one of the first result on this topic;

Theorem 4. *For all $\varepsilon > 0$, $T > 0$, for almost-all ω, there exists a sequence of random time $(t_n(\omega))_{n\geq 1}$, such that $t_n(\omega) \to -\infty$ and*

$$\forall\ n\ ,\ \textstyle\sum_{k\geq 1} \Big\{ \|F_k\|_{C_b^2(\mathbb{R})} \displaystyle\sup_{t_n - T \leq s \leq r \leq t_n} |B_k(r) - B_k(s)| \Big\}\ \leq \varepsilon\ .$$

The proof of this result is given in Section 4.

2. Variational principle

We give a detail discussion to introduce the variational principle.

2.1. The Burger's case

We first begin with the particular case of Burgers equation when the flux is $\Psi(u) = u^2/2$. We recall that if we consider the one dimensional (inviscid) Burgers equation

$$\partial_t u + \partial_x \left(\frac{u^2}{2} \right) = \frac{\partial}{\partial x} G(t,x) \quad t > 0\ ,\ x \in \mathbb{R} \tag{6}$$

then for an initial condition u_0 having discontinuities of the first kind (i.e. u_0 belongs to the Skorohod space $\mathbb{D}$) there unique entropy-weak solution u is given by

$$u(t,x) = \frac{\partial}{\partial_x}\left(\inf_{\substack{\xi \in C^1(0,t) \\ \xi(t)=x}} \left\{ \mathcal{A}_{0,t} + \int_0^{\xi(0)} u_0(z)dz \right\} \right) ,$$

where

$$\mathcal{A}_{0,t}(\xi) = \int_0^t \left(\frac{1}{2}\dot{\xi}(s)^2 + G(t,\xi(s)) \right) ds \ . \tag{7}$$

For two times t_1, t_2, we have denoted $C^1(t_1,t_2)$ the space of continuously differentiable functions from $[t_1,t_2]$ to $\mathbb{R}$.

This relation between the Burgers's equation and the minimization problem is known as Lax-Oleĭnik formula (see [11, 13]) (and Hopf-Lax formula in its original context of Hamilton-Jacobi equations). It will be fully exploited in the study of scalar conservation law with stochastic forcing as we will see it right now.

In the above equation we have intuitively assumed that G is a deterministic regular force. Now the source term in the action $\mathcal{A}_{\tau,t}$ is $\int_\tau^t \sum_{k\geq 1} F_k(\xi(s))dB_k(s)$ where the above integral is not a stochastic integral but a path-wise integral. Indeed, since the trajectories $\omega \to B_k(t)(\omega)$ are $\varepsilon-$Hölder continuous and ξ is differentiable, $\int_\tau^t F_k(\xi(s))dB_k(s)$ exists as a Riemann-Stieltjes integral thanks to a result of Young [18]. Nevertheless, to avoid the use of such integrals we use integration by parts formula : one have with $g(\cdot) := F_k(\xi(\cdot))$

$$\int_\tau^t g(s)dB_k(s) = \lim_{\Delta\to 0}\sum_{i=0}^n g(t_i)(B_k(t_{i+1}) - B_k(t_i))$$

where the convergence holds uniformly in all finite partitions $\mathcal{P}_\Delta := \{\tau = t_0 \leq t_1 \leq ...t_{n+1} = t\}$ with $\max_i |t_{i+1} - t_i| < \Delta$. With $\bar{B}(s) := B_k(s) - B(\tau)$ one writes

$$\begin{aligned}
&\sum_{i=0}^n g(t_i)(B_k(t_{i+1}) - B_k(t_i)) \\
&\qquad = \sum_{i=0}^n g(t_i)(\bar{B}(t_{i+1}) - \bar{B}(t_i)) \\
&\qquad = -\sum_{i=0}^n \bar{B}(t_{i+1})\big(g(t_{i+1}) - g(t_i)\big) \\
&\qquad\qquad + \sum_{i=0}^n \big\{\bar{B}(t_i)\big(g(t_{i+1}) - g(t_i)\big) + \big(\bar{B}(t_{i+1}) - \bar{B}(t_i)\big)g(t_{i+1})\big\} \\
&\qquad = -\sum_{i=0}^n \bar{B}(t_{i+1})\big(g(t_{i+1}) - g(t_i)\big) + \bar{B}(t)g(t) - \bar{B}(\tau)g(\tau) \ .
\end{aligned}$$

Consequently

$$\int_\tau^t g(s)dB_k(s) = -\int_t^\tau \big(B_k(s)-B_k(\tau)\big)\dot g(s)ds + \big(B_k(t)-B_k(\tau)\big)g(t)$$

and we rewrite the stochastic term of the action as

$$\int_\tau^t \textstyle\sum_{k\geq 1} F_k(\xi(s))dB_k(s) = -\displaystyle\int_\tau^t \textstyle\sum_{k\geq 1}\big(B_k(s)-B_k(\tau)\big)f_k(\xi(s))\dot\xi(s)ds \\ + \textstyle\sum_{k\geq 1}\big(B_k(t)-B_k(\tau)\big)F_k(\xi(t)) \qquad (8)$$

where $f_k = F_k'$. If $\xi(t)$ is fixed to be x, then the second term in the above equality is independent on ξ, hence as in [3] the action is redefined as for $\xi \in C^1(\tau,t)$ as

$$\mathcal{A}_{\tau,t}(\xi) = \int_\tau^t \left(\frac{1}{2}\dot\xi(s)^2 - \textstyle\sum_{k\geq 1}\big(B_k(s)-B_k(\tau)\big)f_k(\xi(s))\dot\xi(s)\right)ds \\ + \textstyle\sum_{k\geq 1}\big(B_k(t)-B_k(\tau)\big)F_k(\xi(t))\ .$$

Remark 5. *Since the action is defined path-wisely it depends on ω hence should be denoted $\mathcal{A}^\omega_{\tau,t}$. We will not do for brevity of notations.*

Remark 6. *We strength the fact that it is a true integration by parts that allows us to rewrite the stochastic term and not a formal one as it was mentioned in [3].*

The Burgers case is particular because the Legendre transform of the flux $\Psi(p) = p^2/2$ that appears in (7) with the term $\frac{1}{2}\dot\xi^2$ is again the half of the function square. This is no more the case when the flux is another convex function. This term still be the Legendre transform of Ψ and these remarks motivate the Lax-Oleĭnik formula (4) with the action defined in (5).

There is another way of thinking in order to introduce the optimization problem: one can make a kind of change of variable in the variational formulation (2) and introduce an Hamilton-Jacobi-Bellman equation (HJB equation in short). Thus it is well known that these partial differential equation is related to a variational principle. This is briefly discussed in the following subsection.

2.2. Redefining the action via HJB equation

Let us develop the following non rigorous arguments. Let φ a test function in C_c^2, thanks to an integration by parts one rewrites (2) as

$$\int_{t_0}^\infty \int_{\mathbb{R}} \partial_t\varphi(t,x)u(t,x)dxdt + \int_{t_0}^\infty \int_{\mathbb{R}} \partial_x\varphi(t,x)\Psi\big(u(t,x)\big)dxdt = \\ -\int_{\mathbb{R}} u_0(x)\varphi(t_0,x)dx + \int_{\mathbb{R}}\int_{t_0}^\infty \partial_t\varphi(t,x)v(t,x)dtdx \qquad (9)$$

with $F_k' = f_k$ and

$$v(t,x) = \sum_{k=1}^{\infty} f_k(x)(B_k(t) - B_k(t_0)) \ . \tag{10}$$

Consequently

$$\int_{t_0}^{\infty}\int_{\mathbb{R}} \partial_t\varphi(t,x)\big[u(t,x)-v(t,x)\big]dxdt + \int_{t_0}^{\infty}\int_{\mathbb{R}} \partial_x\varphi(t,x)\Psi\big(u(t,x)\big)dxdt$$
$$= -\int_{\mathbb{R}} u_0(x)\varphi(t_0,x)dx$$

and if W is such that $\partial_x W = w$ with $w = u + v$ we obtain

$$\int_{t_0}^{\infty}\int_{\mathbb{R}} \partial_t\varphi(t,x)w(t,x)dxdt + \int_{t_0}^{\infty}\int_{\mathbb{R}} \partial_x\varphi(t,x)\Psi\big(w(t,x)+v(t,x)\big)dxdt$$
$$= -\int_{\mathbb{R}} u_0(x)\varphi(t_0,x)dx \ .$$

Hence w is a solution of the stochastic scalar conservation law

$$\partial_t w + \mathrm{div}_x \Psi(w+v) = 0$$

and if we integrate with respect to the space variable x this equation, we derive the HJB equation

$$\partial_t W + \Psi(\partial_x W + v) = 0 \ .$$

This HJB is related to an optimization problem with an action in which the Legendre transform of $p \mapsto \Psi(p+w)$ is involved. Thanks to the behavior under translation of the Legendre transformation, one have $\big(\Psi(\cdot + v)\big)^*(q) = \Psi^*(q) - vq$ and we obtain the same king of action that in (5).

The above remarks are now made rigorous in the following subsection.

2.3. Dynamic programming equation

First we express the action $\mathcal{A}_{t_0,t}$ as

$$\begin{aligned}
\mathcal{A}_{t_0,t}(\xi) &= \widetilde{\mathcal{A}}_{t_0,t}(\xi) + V(t,\xi(t)) \quad \text{with} \\
\widetilde{\mathcal{A}}_{t_0,t}(\xi) &= \int_{t_0}^{t} L(s,\xi(s),\dot{\xi}(s))ds \ , \\
L(s,x,p) &= \Psi^*(p) - \textstyle\sum_{k\geq 1}\big(B_k(s)-B_k(t_0)\big)f_k(x)\times p \quad \text{and} \\
V(t,x) &= \textstyle\sum_{k\geq 1}\big(B_k(t)-B_k(t_0)\big)F_k(x) \ .
\end{aligned}$$

With U_0 defined by $\partial_x U_0 = u_0$ we define

$$W(t,x) = \inf_{\substack{\xi \in H^1(t_0,t) \\ \xi(t) = x}} \Big\{\widetilde{\mathcal{A}}_{t_0,t}(\xi) + U_0(\xi(t_0))\Big\} \ . \tag{11}$$

We remark that $U_0(\xi(t_0)) = \int_0^{\xi(t_0)} u_0(z)dz$ and

$$W(t,x) = \inf_{\substack{\xi \in H^1(t_0,t) \\ \xi(t)=x}} \{\mathcal{A}_{t_0,t}(\xi)\} - V(t,x) \ .$$

The function W will be the unique solution of an Hamilton-Jacobi-Bellman equation. In classical calculus of variations, the left end point is fixed and the functional L is assumed to be regular (three times differentiable in space and time if one refers to [5]). These minor modifications are not difficult and do not imply any changes except in the expression of the Hamiltonian that becomes in our case the Legendre transform of $p \mapsto L(t,x,p)$ instead of $q \mapsto \sup_p(-pq - L(t,x,p))$. Since we do not know any precise reference where these changes are discussed, we shortly prove the following facts.

Proposition 7. *The function $(t,x) \mapsto W(t,x)$ is Lipschitz continuous and satisfies for almost-all t, x the Hamilton-Jacobi-Bellman equation*

$$\partial_t W(t,x) + \Psi\left(\partial_x W(t,x) + \textstyle\sum_{k\geq 1} f_k(x)(B_k(t)-B_k(t_0))\right) = 0 \ . \tag{12}$$

The Sobolev space $H^1(t_1,t_2)$ is equipped of the scalar product

$$\langle \xi_1,\xi_2\rangle = \int_{t_1}^{t_2} \xi_1(s)\xi_2(s)ds + \int_{t_1}^{t_2} \dot{\xi}_1(s)\dot{\xi}_2(s)ds \ .$$

We have the following lemma.

Lemma 8. *For any $\xi \in H^1(t_0,t)$, there exists a constant c such that*

$$\widetilde{A}_{t_0,t}(\xi) \geq \frac{c}{2}\int_{t_0}^{t} |\dot{\xi}(s)|^{2\wedge\alpha} ds - c \ . \tag{13}$$

Proof. If Ψ is superlinear, it is also the case of its Legendre transform Ψ^*. So there exits c_1 and c_2 such that $\Psi^*(v) \geq c_1|v|^{2\wedge\alpha} - c_2$. Then we have

$$\widetilde{A}_{t_0,t}(\xi) \geq c_1\int_{t_0}^{t} |\dot{\xi}(s)|^{2\wedge\alpha} ds - c_2(t-t_0) - \textstyle\sum_{k\geq 1}\|f_k\|_\infty \displaystyle\int_{t_0}^{t} |B_k(s)-B_k(t_0)||\dot{\xi}(s)|ds$$

By Young's inequality $ab \leq \dfrac{a^{2\wedge\alpha}}{\varepsilon} + c_\varepsilon b^\beta$ with $\varepsilon = c_1/(2\sum_{k\geq 1}\|f_k\|_\infty)$ we obtain

$$\widetilde{A}_{t_0,t}(\xi) \geq \frac{c_1}{2}\int_{t_0}^{t} |\dot{\xi}(s)|^{2\wedge\alpha} ds - c_2(t-t_0) - c\textstyle\sum_{k\geq 1}\|f_k\|_\infty \|B_k\|_{t_0,t,\lambda}|t-t_0|^{\beta\lambda+1}$$

and we have (**??**). □

Our aim is to prove that there exists a minimizer of the action $\widetilde{A}_{t_0,t}$. We recall the definition

Definition 9. *On the interval* $[t_1,t_2]$*, we say that* $\xi \in H^1(t_1,t_2)$ *is a minimizer of the action* $\widetilde{\mathcal{A}}_{t_1,t_2}$ *if for any* $\gamma \in H^1(t_1,t_2)$ *with* $\gamma(t_1)=\xi(t_1)$ *and* $\gamma(t_2)=\xi(t_2)$ *we have* $\widetilde{\mathcal{A}}_{t_1,t_2}(\xi) \leq \widetilde{\mathcal{A}}_{t_1,t_2}(\gamma)$*.*

We denote

$$B_R(t_1,t_2)=\Big\{\xi \in H^1(t_1,t_2)\ ;\ |\xi(t_1)|+\textstyle\int_{t_1}^{t_2}|\dot\xi(s)|^2 ds \leq R\Big\}$$

which is clearly a closed and bounded subset of $H^1(t_1,t_2)$, hence weakly compact. Now we prove that there exists on $B_R(t_0,t)$ one minimizer of $\xi \mapsto F(\xi) := \widetilde{A}_{t_0,t}(\xi)+U_0(\xi(t_0))$. By the weak compactness of $B_R(t_0,t)$ it is sufficient that $\xi \mapsto F(\xi)$ is lower semi-continuous. Following [5], Theorem I.9.1 we just have to check the lower semi-continuity of the stochastic part

$$S(\xi)=-\textstyle\sum_{k\geq 1}\displaystyle\int_{t_0}^{t}(B_k(s)-B_k(t_0))f_k(\xi(s))\dot\xi(s)ds\ .$$

Let $(\xi_n)_{n\geq 1}$ a sequence of $B_R(t_0,t)$ converging to ξ weakly. The weak convergence on $B_R(t_0,t)$ implies the uniform convergence on $[t_0,t]$. Writing $S(\xi)-S(\xi_n)=S_n^1+S_n^2$ with

$$S_n^1=\textstyle\sum_{k\geq 1}\displaystyle\int_{t_0}^{t}(B_k(s)-B_k(t_0))\big[f_k(\xi_n(s))-f_k(\xi(s))\big]\dot{\xi_n}(s)ds$$
$$S_n^2=\textstyle\sum_{k\geq 1}\displaystyle\int_{t_0}^{t}(B_k(s)-B_k(t_0))f_k(\xi(s))\big[\dot{\xi_n}(s)-\dot\xi(s)\big]ds\ ,$$

and by uniform convergence, $\lim_n S_n^1=0$. The weak convergence and the fact that $s\mapsto\sum_{k\geq 1}(B_k(s)-B_k(t_0))f_k(\xi(s))$ belongs to $\mathbb{L}^2(t_0,t)$ yield $\lim_n S_n^2=0$.

Hence we have the lower semi-continuity and then there exists a minimizer $\xi_{\min}\in B_R(t_0,t)$ of $\xi\mapsto\widetilde{A}_{t_0,t}(\xi)+U_0(\xi(t_0))$.

Working with the right end-point condition $\xi(t)=x$ in the calculus of variations will not affect theorems I.9.2, I.9.3 and I.9.4 of [5]. Thus for every t,x, there exists a minimizer $\xi_{\min}\in H^1(t_0,t)$ with $\xi_{\min}(t)=x$ such that

$$\begin{aligned}W(t,x)&=\inf_{\substack{\xi\,\in\,H^1(t_0,\,t)\\ \xi(t)\,=\,x}}\Big\{\widetilde{\mathcal{A}}_{t_0,t}(\xi)+U_0(\xi(t_0))\Big\}\\ &=\int_{t_0}^{t}L(s,\xi_{\min}(s),\dot\xi_{\min}(s))ds+U_0(\xi_{\min}(t_0))\ .\end{aligned}\tag{14}$$

Moreover there exists M such that for any (t,x) and (t',x') in $\mathbb{R}\times\mathbb{R}$,

$$|W(t,x)-W(t',x')|\leq M\big(|t-t'|+|x-x'|\big)\ .\tag{15}$$

The equation satisfied by W will be obtained thanks to the following version of the dynamic programming principle. Indeed we can observe that for any

$t_0 \leq r \leq t$,

$$W(t,x) = \inf_{\substack{\xi \in H^1(t_0,t) \\ \xi(t)=x}} \left(\int_r^t L(s,\xi(s),\dot{\xi}(s))ds + W(r,\xi(r)) \right) . \tag{16}$$

Now let $0 < h < t - t_0$ and take $r = t - h$ in the dynamical programming principle (16). We substract $W(t,x)$ from both sides and we get

$$\inf_{\substack{\xi \in H^1(t_0,t) \\ \xi(t)=x}} \left(\frac{1}{h} \int_{t-h}^t L(s,\xi(s),\dot{\xi}(s))ds + \frac{1}{h}\big(W(t-h,\xi(t-h)) - W(t,x)\big) \right) = 0 \,.$$

When $h \downarrow 0$, we obtain

$$\begin{aligned}
&-\frac{\partial W}{\partial t}(t,x) + \inf_{\substack{\xi \in H^1(t_0,t) \\ \xi(t)=x}} \left(L(t,x,\dot{\xi}(t)) - \frac{\partial W}{\partial x}(t,x) \times \dot{\xi}(t) \right) = 0 \\
&+\frac{\partial W}{\partial t}(t,x) - \inf_{q\in\mathbb{R}} \left(-q \times \frac{\partial W}{\partial x}(t,x) + L(t,x,q) \right) = 0 \\
&+\frac{\partial W}{\partial t}(t,x) + \sup_{q\in\mathbb{R}} \left(+q \times \frac{\partial W}{\partial x}(t,x) - L(t,x,q) \right) = 0 \\
&+\frac{\partial W}{\partial t}(t,x) + H\left(t,x,\frac{\partial W}{\partial x}(t,x) \right) = 0
\end{aligned}$$

where $p \mapsto H(t,x,p)$ is the Legendre transform of $q \mapsto L(t,x,q)$. Using the behavior under translation of the Legendre transform, we have $H(t,x,p) = \Psi(p + v(t,x))$ where v is defined in (10). In other words, for al t,x W satisfies Hamilton-Jacobi-Bellman equation (12) (also refer in the literature as the dynamic programming equation).

We will also need the following property.

Proposition 10. *For any t, the function $x \mapsto W(t,x)$ is semi concave.*

Proof. We fix t. We must find a constant K such that the function g defined by $g(x) = W(t,x) - Kx^2$ is concave. This is equivalent to

$$W(t,x) \geq \frac{1}{2}\big(W(t,x+h) + W(t,x-h)\big) - Kh^2, \ \forall\, x,h. \tag{17}$$

Let $\gamma_x = \xi_{\min}$ be the minimizer of the action such that W satisfies (14). Remind that $\gamma_x(t) = x$, so if we introduce $\gamma_{x+h} \in H^1(t_0,t)$ defined by

$$\gamma_{x+h}(s) = x + h + \int_s^t \dot{\gamma}_x(s)ds = h + \gamma_x(s),$$

it satisfies $\gamma_{x+h}(t) = x+h$ and $\dot\gamma_{x+h} = \dot\gamma_x$. We compute the first order difference

$$\begin{aligned}
\Delta^1_{x,h} &= \widetilde{\mathcal{A}}_{t_0,t}(\gamma_{x+h}) - \widetilde{\mathcal{A}}_{t_0,t}(\gamma_x) + U_0(\gamma_{x+h}(t_0)) - U_0(\gamma_x(t_0)) \\
&= \delta^1_{x,h} + \delta^2_{x,h} + \delta^3_{x,h} \text{ with,} \\
\delta^1_{x,h} &= \int_{t_0}^t \big(\Psi^*(\dot\gamma_{x+h}(s)) - \Psi^*(\dot\gamma_x(s))\big)ds = 0 \\
\delta^2_{x,h} &= \int_{t_0}^t \textstyle\sum_{k\geq 1}(B_k(s) - B_k(t_0))\big[f_k(\gamma_x(s)) - f_k(\gamma_{x+h}(s))\big]\dot\gamma_x(s)ds \\
\delta^3_{x,h} &= U_0(\gamma_{x+h}(t_0)) - U_0(\gamma_x(t_0)) \ ,
\end{aligned}$$

and we have

$$\begin{aligned}
\delta^2_{x,h} = &\int_{t_0}^t \Big\{ \textstyle\sum_{k\geq 1}(B_k(s) - B_k(t_0))\dot\gamma_x(s) \\
&\qquad \left(\int_0^1 \partial_x f_k\left((1-\nu)\gamma_x(s) - \nu\gamma_{x+h}(s)\right)(\gamma_x(s) - \gamma_{x+h}(s))hd\nu \right) \Big\} ds \\
= &\int_{t_0}^t \Big\{ \textstyle\sum_{k\geq 1}(B_k(s) - B_k(t_0))\dot\gamma_x(s) \\
&\qquad \left(\int_0^1 \partial_x f_k\left((1-\nu)\gamma_x(s) - \nu\gamma_{x+h}(s)\right) h^2 d\nu \right) \Big\} ds \ .
\end{aligned}$$

Consequently

$$\delta^2_{x,h} \leq \left(\|\xi_{\min}\|_{H^1(t_0,t)}(t_0 - t)^{\lambda+1/2} \textstyle\sum_{k\geq 1} \|\partial_x f_k\|_\infty \|B_k\|_{t_0,t,\lambda} \right) \times h^2 \ ,$$

and analogously $\delta^3_{x,h} \leq \|\partial_x U_0\|_\infty \times h^2$. Finally there exists K such that $\Delta^1_{x,h} \leq Kh^2$. Now since γ_x is a minimizer,

$$\begin{aligned}
W(t, x+h) &\leq \widetilde{\mathcal{A}}_{t_0,t}(\gamma_{x+h}) - U_0(\gamma_{x+h}(t_0)) \\
&\leq \widetilde{\mathcal{A}}_{t_0,t}(\gamma_x) + U_0(\gamma_x(t_0)) + Kh^2 \\
&\leq W(t,x) + Kh^2
\end{aligned}$$

and this inequality implies (17). □

Remark 11. *By Alexandrov's theorem (see Appendix E in [5]), $w \mapsto W(t,x)$ is almost everywhere twice differentiable.*

2.4. Proof of Theorem 3

Now the proof of existence and uniqueness of the solution of (1) is simple.

The Lipschitz property (15) for W implies that (ii) in definition 1 holds true.

We prove the variational formulation. Let φ be a test function in $C_c^2(\mathbb{R}\times\mathbb{R})$. We integrate the HJB equation (12) against $\partial_x\varphi$ and we integrate by parts in order to obtain :

$$\begin{aligned}-\int_{t_0}^{\infty}\int_{\mathbb{R}}\Psi\big(\partial_x W(t,x)+v(t,x)\big)\partial_x\varphi(t,x)dxdt &= \int_{t_0}^{\infty}\int_{\mathbb{R}}\partial_s W(t,x)\partial_x\varphi(t,x)dxdt\\ &= -\int_{\mathbb{R}} W_0(x)\partial_x\varphi(t_0,x)dx+\int_{t_0}^{\infty}\int_{\mathbb{R}}\partial_x W(t,x)\partial_t\varphi(t,x)dxdt\ .\end{aligned}$$

We have $\partial_x W_0(x) = \partial_x W(t_0,x) = u(t_0,x)+v(t_0,x) = u_0(x)$. As we did it before, we let $u=\partial_x W+v$. By another integration by parts one obtains (9) that is an equivalent form of (2).

The entropy condition (3) is a consequence of the semi concavity of W (see proposition 10). Indeed, $x\mapsto W(t,x)-Kx^2$ concave implies that its derivative is is a decreasing function. Then $\partial_x W(t,x-)\geq\partial_x W(t,x+)$ and since $u=\partial_x W+v$ with v continuous, we obtain (3).

3. Action minimizers and generalized characteristics

We recall that in order to construct an invariant measure for the stochastic scalar conservation law (1), we will construct an invariant solution. To do this we will use minimizers of the action $\mathcal{A}_{t_0,t}$ with $t_0\to-\infty$.

For any times t_1, t_2 and any $x_1,x_2\in\mathbb{R}$, we denote

$$\mathcal{H}_{x_1,x_2}^{t_1,t_2}=\left\{\xi\in H^1(t_1,t_2)\ ;\ \xi(t_1)=x_1\ ,\ \xi(t_2)=x_2\right\}.$$

We have the following lemma in which we give the Euler-Lagrange equations satisfied by the minimizers.

Lemma 12. *If γ is a minimizer of $\mathcal{A}$ on $[t_1,t_2]$, that is*

$$\begin{aligned}\mathcal{A}_{t_1,t_2}(\gamma)=\inf_{\xi\in\mathcal{H}_{x_1,x_2}^{t_1,t_2}}\Bigg\{&\int_{t_1}^{t_2}\Psi^*(\dot\xi(s))-\textstyle\sum_{k\geq1}(B_k(s)-B_k(t_1))f_k(\xi(s))\dot\xi(s)ds\\ &+U_0(\xi(t_1))+\textstyle\sum_{k\geq1}(B_k(t_2)-B_k(t_1))F_k(\xi(t_2))\Bigg\}\end{aligned}$$

then $\dot\gamma$ satisfies $t_1\leq r\leq s\leq t_2$

$$(\Psi^*)'(\dot\gamma(s))-(\Psi^*)'(\dot\gamma(r))=\int_r^s\textstyle\sum_{k\geq1}f_k(\gamma(\tau))dB_k(\tau)\ . \tag{18}$$

Proof. Since γ minimizes the functional $\mathcal{A}_{t_1,t_2}$, we have for any $\xi\in\mathcal{H}_{x_1,x_2}^{t_1,t_2}$, $\varepsilon\mapsto\frac{d}{d\varepsilon}\mathcal{A}_{t_1,t_2}(\gamma+\varepsilon\xi)$ equals 0 in $\varepsilon=0$. This yields

$$\begin{aligned}0=\int_{t_1}^{t_2}\Big[&(\Psi^*)'(\dot\gamma)(s)\dot\xi(s)-\textstyle\sum_{k\geq1}(B_k(s)-B_k(t_1))\big(f_k'(\gamma)\dot\gamma\xi+f_k(\gamma)\dot\xi\big)(s)\Big]ds\\ &+U_0(\gamma(t_1))\xi(t_1)+\textstyle\sum_{k\geq1}(B_k(t_2)-B_k(t_1))f_k(\gamma(t_2))\xi(t_2)\ .\end{aligned}$$

For $t_1 < \tau_1 \leq \tau_2 < t_2$, we write this identity with ξ_n defined as

$$\xi_n(s) = 0 \times 1\!\!1_{[t_1,\tau_1]\cup[\tau_2,t_2]}(s) + n\big(s - (\tau_1 - 1/n)\big)1\!\!1_{[\tau_1-1/n,\tau_1]}(s) \\ +1\!\!1_{[\tau_1,\tau_2]}(s) + n\big(-s + (\tau_2 + 1/n)\big)1\!\!1_{[\tau_2,\tau_2+1/n]}(s).$$

We obtain

$$\begin{aligned}\int_{\tau_2}^{\tau_2+1/n} n(\Psi^*)'(\dot\gamma(s))ds - \int_{\tau_1-1/n}^{\tau_1} n(\Psi^*)'(\dot\gamma(s))ds = \\ -\int_{\tau_1}^{\tau_2} \textstyle\sum_{k\geq 1}(B_k(s) - B_k(t_1))f_k'(\gamma(s))\dot\gamma(s)ds \\ -\int_{\tau_1-1/n}^{\tau_1} \textstyle\sum_{k\geq 1}(B_k(s) - B_k(t_1))(f_k'(\gamma)\dot\gamma\xi_n)(s)ds \\ -\int_{\tau_1-1/n}^{\tau_1} n\textstyle\sum_{k\geq 1}(B_k(s) - B_k(t_1))f_k(\gamma(s))ds \\ -\int_{\tau_2}^{\tau_2+1/n} \textstyle\sum_{k\geq 1}(B_k(s) - B_k(t_1))(f_k'(\gamma)\dot\gamma\xi_n)(s)ds \\ +\int_{\tau_2}^{\tau_2+1/n} n\textstyle\sum_{k\geq 1}(B_k(s) - B_k(t_1))f_k(\gamma(s))ds\ .\end{aligned}$$

We remark that $\sup_n \|\xi_n\|_\infty \leq c$ and easy arguments allow us to let n goes to infinity. Hence

$$(\Psi^*)'(\dot\gamma(\tau_2)) - (\Psi^*)'(\dot\gamma(\tau_1)) = -\int_{\tau_1}^{\tau_2} \textstyle\sum_{k\geq 1}(B_k(s) - B_k(t_1))f_k'(\gamma(s))\dot\gamma(s)ds \\ + \textstyle\sum_{k\geq 1}(B_k(\tau_2) - B_k(t_1))f_k(\gamma(\tau_2)) - \sum_{k\geq 1}(B_k(\tau_1) - B_k(t_1))f_k(\gamma(\tau_1))$$

and with $g(s) = \sum_{k\geq 1}(B_k(s) - B_k(t_1))f_k'(\gamma(s))\dot\gamma(s)$ one may write

$$\begin{aligned}&(\Psi^*)'(\dot\gamma(\tau_2)) - (\Psi^*)'(\dot\gamma(\tau_1)) \\ &\quad= -\int_{t_1}^{\tau_2} g(s)ds + g(\tau_2) - g(t_1) - \Big(-\int_{t_1}^{\tau_1} g(s)ds + g(\tau_1) - g(t_1)\Big) \\ &\quad= \int_{t_1}^{\tau_2} \textstyle\sum_{k\geq 1}f_k(\gamma(s))dB_k(s) - \int_{t_1}^{\tau_1} \sum_{k\geq 1}f_k(\gamma(s))dB_k(s) \\ &\quad= \int_{\tau_1}^{\tau_2} \textstyle\sum_{k\geq 1}f_k(\gamma(s))dB_k(s)\ ,\end{aligned}$$

where we have used (8). By continuity of $\tau \mapsto \int_{t_1}^{\tau} \sum_{k\geq 1}f_k(\gamma(s))dB_k(s)$ (see Prop. 4.4.1 in [19]), the above formula is also true for $\tau_1 = t_1$ and $\tau_2 = t_2$. Then the formula (18) is true. $\square$

We stress the fact that $(\Psi^*)' = (\Psi')^{-1}$.

Remark 13. *Any action minimizer γ satisfies the following Euler-Lagrange equation:*

$$\left\{\begin{array}{ll}\dot\gamma(s) & = \Psi'(v(s))\\ dv(s) & = \sum_{k\geq 1} f_k(\gamma(\tau))dB_k(\tau)\end{array}\right. \Longleftrightarrow \left\{\begin{array}{ll} v(s) & = (\Psi')^{-1}(\dot\gamma(s))\\ dv(s) & = \sum_{k\geq 1} f_k(\gamma(\tau))dB_k(\tau).\end{array}\right. \tag{19}$$

Indeed, the Euler-Lagrange equation can be formally deduced from the following computation. If we want to find two curves γ and v such that $v(t) = u(t,\gamma(t))$, then

$$dv(t) = \partial_t u(t,\gamma(t)) + \partial_x u(t,\gamma(t))\dot\gamma(t).$$

With $\dot\gamma(t) = \Psi'(u(t,\gamma(t)))$ (or equivalently $v(t) = (\Psi')^{-1}(\dot\gamma(t))$), together with (1) one writes

$$dv(t) = \partial_t u(t,\gamma(t)) + \partial_x \Psi(u(t,\gamma(t))),$$

and we obtain (19).

The curve γ is a generalized characteristic in the sense of Dafermos (see [1]).

Remark 14. *With $\Psi(z) = z^2/2$, $(\Psi^*)'(z) = z$ for any $z \in \mathbb{R}$. Hence the equation (19) genaralises the Euler-Lagrange equation (2.3) in [3].*

By (19), v is continuous. Since $\dot\gamma = \Psi' \circ v$, we deduce that $\dot\gamma$ is also continuous. Consequently, any action minimizer of $\mathcal{A}$ on $[t_1,t_2]$ is in $C^1(t_1,t_2)$.

One can easily prove that there exists a unique solution to the Euler-Lagrange system of equations (19):

Lemma 15. *Let ξ_0 and v_0 be two given real numbers. There exists a unique solution $\xi \in C^1(-\infty,t_0)$ to the Euler-Lagrange equation (19)*

$$\begin{aligned}&\dot\xi(s) = \Psi'(v(s))\\ &v(s) = v(t_0) - \int_s^{t_0} \textstyle\sum_{k\geq 1}\big(B_k(r) - B_k(s)\big) f_k'(\xi(r))\dot\xi(r)dr\\ &\qquad\qquad + \textstyle\sum_{k\geq 1}\big(B_k(t_0) - B_k(s)\big) f_k(\xi(t_0)) \qquad \forall s \leq t_0\end{aligned}$$

such that $\xi(t_0) = \xi_0$ and $\dot\xi(t_0) = \Psi'(v_0)$.

The following lemma gives a key estimation on the velocities of the characteristics. This will play a central role in our further investigations. We make the following additional assumption on Ψ that is clearly true if the flux is the square function.

Hypothesis 4. *The derivative Ψ' of the flux is Lipschitz and its Legendre transform satisfies $c_3|v|^\alpha \leq |\Psi^*(v)| \leq c_4|v|^\beta$ with $\beta > \alpha > 1$ and two positive constants c_3 and c_4.*

Lemma 16. *With the notation of Lemma 12, if γ is a minimizer of the action $\mathcal{A}$ on the time interval $[t_1,t_2]$ with $t_2 - t_1 \geq 1$, then there exists a constant c*

such that

$$\|\dot\gamma\|_{\infty,t_1,t_2} \le c\left[1+\mathcal{K}^{B,F}_{t_1,t_2}+(\mathcal{K}^{B,F}_{t_1,t_2})^{\frac{\alpha}{\alpha-1}}\right]\left\{(t_2-t_1)^{-\frac{1}{\alpha}}+(t_2-t_1)^{\frac{\alpha-1}{\alpha}}\mathcal{K}^{B,F}_{t_1,t_2}\right\} + c\mathcal{K}^{B,F}_{t_1,t_2} \qquad (20)$$

with

$$\mathcal{K}^{B,F}_{t_1,t_2} = \sum_{k\ge 1}\|F_k\|_{C^2}\left\{\sup_{t_1\le r\le r'\le t_2}|B_k(r)-B_k(r')|\right\} . \qquad (21)$$

Proof. We fix $t_1 \le t \le t_2$ such that $t_2 - t_1 \ge 1$. Let s be such that $|\dot\gamma(s)| = \inf_{r\in[t_1,t_2]}|\dot\gamma(r)|$. Writing $\dot\gamma(t) = (\Psi'\circ(\Psi')^{-1})(\dot\gamma(t)) - (\Psi'\circ(\Psi')^{-1})(\dot\gamma(s)) + \dot\gamma(s)$, we have

$$\begin{aligned}|\dot\gamma(t)| &\le \|\Psi'\|_{\mathrm{Lip}}\Big|(\Psi')^{-1})(\dot\gamma(t)) - (\Psi')^{-1})(\dot\gamma(s))\Big| + |\dot\gamma(s)| \\ &\le \|\Psi'\|_{\mathrm{Lip}}\times\Delta_{s,t} + \frac{\|\dot\gamma\|_{\mathbb{L}^1(t_1,t_2)}}{t_2-t_1} ,\end{aligned}$$

with

$$\begin{aligned}\Delta_{s,t} &= \Big|(\Psi')^{-1})(\dot\gamma(t)) - (\Psi')^{-1})(\dot\gamma(s))\Big| \\ &= -\int_s^t \textstyle\sum_{k\ge1}\big(B_k(r)-B_k(s)\big)f_k'(\gamma(r))\dot\gamma(r)dr \\ &\qquad\qquad + \textstyle\sum_{k\ge1}\big(B_k(t)-B_k(s)\big)f_k(\gamma(t)) \\ &\le \mathcal{K}^{B,F}_{t_1,t_2} + \mathcal{K}^{B,F}_{t_1,t_2}\|\dot\gamma\|_{\mathbb{L}^1(t_1,t_2)} .\end{aligned}$$

Consequently,

$$|\dot\gamma(t)| \le \mathcal{K}^{B,F}_{t_1,t_2}\|\Psi'\|_{\mathrm{Lip}} + \Big(\mathcal{K}^{B,F}_{t_1,t_2}\|\Psi'\|_{\mathrm{Lip}} + 1/(t_2-t_1)\Big)\|\dot\gamma\|_{\mathbb{L}^1(t_1,t_2)} . \qquad (22)$$

Now we estimate the L^1 norm of $\dot\gamma$. We proceed as in the proof of (**??**) in Lemma 8. We use Hypothesis 4 together with the Young inequality $ab \le a^p/p + b^q/q$ with $p=\alpha$ yield

$$\frac{c_1}{2}\int_{t_1}^{t_2}|\dot\gamma(s)|^\alpha ds \le \mathcal{A}_{t_1,t_2}(\gamma) + (\mathcal{K}^{B,F}_{t_1,t_2})^{\frac{\alpha}{\alpha-1}} + \mathcal{K}^{B,F}_{t_1,t_2} - U_0(x_1) . \qquad (23)$$

Since γ is a minimizer of the action $\mathcal{A}_{t_1,t_2}$, we have $\mathcal{A}_{t_1,t_2}(\gamma) \le \mathcal{A}_{t_1,t_2}(\xi)$ with the curve ξ defined by $\xi(s) = x_1 + (s-t_1)/(t_2-t_1)\times(x_2-x_1)$. Using Hypothesis 4 one obtains that

$$\mathcal{A}_{t_1,t_2}(\gamma) \le U_0(x_1) + \mathcal{K}^{B,F}_{t_1,t_2} + c(t_2-t_1)^{1-\beta} .$$

We report the above inequality in (23) and we get that

$$\int_{t_1}^{t_2}|\dot\gamma(s)|^\alpha ds \le c + c\mathcal{K}^{B,F}_{t_1,t_2} + c(\mathcal{K}^{B,F}_{t_1,t_2})^{\frac{\alpha}{\alpha-1}} .$$

Since $\|\dot\gamma\|_{\mathbb{L}^1(t_1,t_2)} \le (t_2-t_1)^{\frac{\alpha-1}{\alpha}} \|\dot\gamma\|_{\mathbb{L}^\alpha(t_1,t_2)}$, with (22) we have

$$|\dot\gamma(t)| \le c\left[1+\mathcal{K}^{B,F}_{t_1,t_2} + (\mathcal{K}^{B,F}_{t_1,t_2})^{\frac{\alpha}{\alpha-1}}\right]\left\{(t_2-t_1)^{-\frac{1}{\alpha}} + (t_2-t_1)^{\frac{\alpha-1}{\alpha}}\mathcal{K}^{B,F}_{t_1,t_2}\right\} + c\mathcal{K}^{B,F}_{t_1,t_2}$$

and we obtain (20). □

4. An asymptotic property of fBm

The paper [3] essentially uses the fact that the Brownian motion has periods of arbitrary length and arbitrary small amplitude oscillation as time goes to $-\infty$. In this section, we will prove that a similar property holds for the fBm defined on the all real line $(-\infty,+\infty)$. The result, which is interesting in itself, is the following.

Theorem 17. *For all $\varepsilon>0$, $T>0$, for almost-all ω, there exists a sequence of random time $(t_n(\omega))_{n\ge1}$, such that $t_n(\omega)\to-\infty$ and*

$$\forall\ n\ ,\ \textstyle\sum_{k\ge1}\left\{\|F_k\|_{C^2_b(\mathbb{R})} \displaystyle\sup_{t_n-T\le s\le r\le t_n} |B_k(r)-B_k(s)|\right\}\ \le\varepsilon\ .$$

Before proving this theorem, we will recall and prove some basic facts about the fBm defined on the real line $\mathbb{R}$. We first recall the moving average representation of the fBm $(B(t))_{t\in\mathbb{R}}$. For $s,t\in\mathbb{R}$, we define

$$f_t(s) = \frac{1}{c_H}\left((t-s)_+^{H-\frac12} - (-s)_+^{H-\frac12}\right)$$

with

$$c_H = \left(\int_0^\infty \left((1+s)^{H-\frac12} - s^{\frac12}\right)^2 ds + \frac{1}{2H}\right)^{\frac12}\ .$$

Notice that $\int_{\mathbb{R}} f_t^2(s)ds<\infty$ and more precisely, if $H\neq1/2$, $s\mapsto f_t(s)$ behaves like $(-s)^{H-3/2}$ when $s\to-\infty$ which is square integrable at $-\infty$. With these notations, the fBm can be written as

$$B(t) = \int_{\mathbb{R}} f_t(s)dW_s \tag{24}$$

where the process $(W_t)_{t\in\mathbb{R}}$ is a two sided classical Brownian motion which is obtained by gluing two independent copies of one sided Brownian motions together at time $t=0$.

Since we are interested in the oscillations of the fBm, we express its increments for $t<t'<0$ as

$$\begin{aligned}
B(t)-B(t') &= \int_{\mathbb{R}} c_H\left\{(t-r)_+^{H-\frac12} - (t'-r)_+^{H-\frac12}\right\}dW_r\\
&= \int_{-\infty}^t c_H\left\{(t-r)^{H-\frac12} - (t'-r)^{H-\frac12}\right\}dW_r + \int_t^{t'} c_H(t'-r)^{H-\frac12}dW_r\\
&= \int_{\mathbb{R}} g_{t,t'}(r)dW_r
\end{aligned}$$

where

$$\frac{g_{t,t'}(r)}{c_H} = \left\{(t-r)^{H-\frac{1}{2}} - (t'-r)^{H-\frac{1}{2}}\right\} \mathbb{1}_{]-\infty,t]}(r) + (t'-r)^{H-\frac{1}{2}} \mathbb{1}_{[t,t']}(r) \ .$$

Let $\mathcal{F}_s$ the sigma-algebra generated by the family of random variables $\{B(r); -\infty < r \leq s\}$. We remark that for $s \leq 0$,

$$\mathcal{F}_s \subseteq \sigma\left\{W_r; -\infty < r \leq s\right\} = \sigma\{\widetilde{W}_r; r \geq -s\} := \mathcal{F}^{\widetilde{W}}_{-s,\infty}$$

where $\widetilde{W}$ is the Brownian motion in $\mathbb{R}^+$ defined by $\widetilde{W}_r = W_{-r}$ for any $r \geq 0$.

We need the following expression: for any $-\infty < s \leq t \leq t' \leq 0$

$$\mathbb{E}(B(t) - B(t')|\mathcal{F}_s) = \mathbb{E}\left[\int_{-\infty}^{s} c_H \left\{(t-r)^{H-\frac{1}{2}} - (t'-r)^{H-\frac{1}{2}}\right\} dW_r \middle| \mathcal{F}_s\right] \ . \tag{25}$$

Indeed, with $-s \geq -t$ we write

$$\begin{aligned} B(t) - B(t') &= \int_{-t}^{+\infty} g_{t,t'}(-r) dW_{-r} \\ &= \int_{-t}^{+\infty} g_{t,t'}(r) d\widetilde{W}_r \\ &= \int_{-t}^{-s} g_{t,t'}(r) d\widetilde{W}_r + \int_{-s}^{+\infty} g_{t,t'}(r) d\widetilde{W}_r \\ &= I_1(s) - I_2(s) \ , \end{aligned}$$

with obvious notations. The random variable $I_1(s)$ is independent of $\mathcal{F}^{\widetilde{W}}_{-s,\infty} \supseteq \mathcal{F}_s$ hence

$$\mathbb{E}(I_1(s)|\mathcal{F}_s) = \mathbb{E}\left[\mathbb{E}(I_1(s)|\,\mathcal{F}^{\widetilde{W}}_{-s,\infty})|\mathcal{F}_s\right] = 0 \ .$$

Consequently,

$$\mathbb{E}(B(t) - B(t')|\mathcal{F}_s) = \mathbb{E}\left[\int_{-s}^{+\infty} g_{t,t'}(r) d\widetilde{W}_r \middle| \mathcal{F}_s\right] = \mathbb{E}\left[\int_{-\infty}^{s} g_{t,t'}(r) dW_r \middle| \mathcal{F}_s\right]$$

and we have proved (25).

The proof of Theorem 17 is based on the following reversed conditional Borel-Cantelli lemma.

Lemma 18. *Let $(\mathcal{F}_n)_{n\geq 1}$ be a decreasing sequence of $\sigma-$fields and $(A_n)_{n\geq 1}$ a sequence of events such that $A_n \in \mathcal{F}_n$. Then the events*

$$\left\{\sum_{k\geq 1} \mathbb{1}_{A_k} < \infty\right\} \qquad \textit{and} \qquad \left\{\sum_{k\geq 1} \mathbb{E}\left(1_{A_k}|\mathcal{F}_{k+1}\right) < \infty\right\}$$

are almost-surely equal.

Proof. Let $M_n = \mathbb{1}_{A_n} - \mathbb{E}(\mathbb{1}_{A_n}|\mathcal{F}_{n+1})$. We have $\mathbb{E}(M_n|\mathcal{F}_{n+1}) = 0$ so $(M_n)_{n\geq 1}$ is a reversed martingale difference sequence. So $\sum_{k\geq 1} \mathbb{E}(M_k^2|\mathcal{F}_{k+1}) < \infty$ implies that $\sum_{k\geq 1} M_k$ is convergent almost-surely (see Stout [15, Theorem 2.8.7]). We have

$$\begin{aligned}
\mathbb{E}(M_k^2|\mathcal{F}_{k+1}) &= \mathbb{E}(\mathbb{1}_{A_k}^2|\mathcal{F}_{k+1}) - (\mathbb{E}(\mathbb{1}_{A_k}|\mathcal{F}_{k+1}))^2 \\
&= \mathbb{E}(\mathbb{1}_{A_k}|\mathcal{F}_{k+1})\Big[1 - \mathbb{E}(\mathbb{1}_{A_k}|\mathcal{F}_{k+1})\Big] \\
&\leq \mathbb{E}(\mathbb{1}_{A_k}|\mathcal{F}_{k+1}) \ .
\end{aligned}$$

Hence $\sum_{k\geq 1} M_k$ is almost surely convergent and since $\sum_{k\geq 1} M_k = \sum_{k\geq 1} \mathbb{1}_{A_k} - \sum_{k\geq 1} \mathbb{E}(\mathbb{1}_{A_k}|\mathcal{F}_{k+1})$, we deduce that

$$\left\{ \sum_{k\geq 1} \mathbb{1}_{A_k} < \infty \right\} \supset \left\{ \sum_{k\geq 1} \mathbb{E}\left(1_{A_k}|\mathcal{F}_{k+1}\right) < \infty \right\} \ .$$

It is clear that if $\sum_{k\geq 1} \mathbb{1}_{A_k} < \infty$ then $\sum_{k\geq 1} \mathbb{E}(1_{A_k}|\mathcal{F}_{k+1})$ is integrable so almost-surely finite. So we have the equality. □

Now we prove Theorem 17.

Proof. Let $\varepsilon > 0$ and $T > 0$ be fixed. Let $(t_n)_{n\geq 1}$ be a decreasing sequence of negative real numbers such that

$$\begin{cases}
\lim_{n\to\infty} t_n = -\infty \ ; \\
t_{n+1} < t_n - T \text{ and} \\
\sum_{n\geq 1} (t_n - t_{n+1})^{2H-2} < \infty.
\end{cases}$$

First we prove the result for a single fBm. More precisely if $(B(t))_{t\in\mathbb{R}}$ is a fBm then

$$\lim_{n\to\infty} \inf \sup_{t_n - T\leq t,s\leq t_n} |B_t - B_s| \leq \varepsilon$$

Using Hypothesis 1(i), this will imply the Theorem 17.

As before we denote $\mathcal{F}_{t_n} = \sigma\{B(r); -\infty < r \leq t_n\}$. For $t \geq t_{n+1}$ we set $B^{n+1}(t) = \mathbb{E}(B(t)|\mathcal{F}_{t_{n+1}})$ and $\overline{B}^{n+1}(t) = B(t) - B^{n+1}(t)$. By the gaussian properties of the fBm it follows that $\overline{B}^{n+1}(t)$ is independent of $\mathcal{F}_{t_{n+1}}$. We set

$$\begin{aligned}
A_n(\varepsilon) &= \left\{ \sup_{t_n - T\leq t,s\leq t_n} |B(t) - B(s)| \leq \varepsilon \right\}, \\
\widetilde{A}_n(\varepsilon) &= \left\{ \sup_{t_n - T\leq t,s\leq t_n} \left|B^{n+1}(t) - B^{n+1}(s)\right| \leq \varepsilon \right\}, \\
\overline{A}_n(\varepsilon) &= \left\{ \sup_{t_n - T\leq t,s\leq t_n} \left|\overline{B}^{n+1}(t) - \overline{B}^{n+1}(s)\right| \leq \varepsilon \right\}.
\end{aligned}$$

Then obviously one has $\overline{A}_n(\varepsilon/2) \subset A_n(\varepsilon) \cup (\widetilde{A}_n(\varepsilon/2))^c$. This implies

$$\mathbb{1}_{A_n(\varepsilon)} + \mathbb{1}_{(\widetilde{A}_n(\varepsilon/2))^c} \geq \mathbb{1}_{\overline{A}_n(\varepsilon/2)} \ .$$

We take the conditional expectation with respect to $\mathcal{F}_{t_{n+1}}$ and we deduce that

$$\mathbb{E}\left(1\!\!1_{A_n(\varepsilon)}|\mathcal{F}_{t_{n+1}}\right) \geq \mathbb{P}\left(\overline{A}_n(\varepsilon/2)\right) - 1\!\!1_{(\widetilde{A}_n(\varepsilon/2))^c}$$

because $\overline{A}_n(\varepsilon/2)$ is independent of $\mathcal{F}_{t_{n+1}}$, while $\widetilde{A}_n(\varepsilon/2)$ belongs to $\mathcal{F}_{t_{n+1}}$.

Arguing as above we also obtain

$$\mathbb{P}(\overline{A}_n(\varepsilon/2)) + \mathbb{P}((\widetilde{A}_n(\varepsilon/4))^c) \geq \mathbb{P}(A_n(\varepsilon/4)) \ .$$

We add these inequalities and we get

$$\mathbb{E}\left(1\!\!1_{A_n(\varepsilon)}|\mathcal{F}_{t_{n+1}}\right) \geq \mathbb{P}(A_n(\varepsilon/4)) - \mathbb{P}((\widetilde{A}_n(\varepsilon/4))^c) - 1\!\!1_{(\widetilde{A}_n(\varepsilon/2))^c}. \tag{26}$$

We will show hereafter that one has

$$\sum_{n\geq 1} \mathbb{P}((\widetilde{A}_n(\varepsilon))^c) < \infty \ , \tag{27}$$

while

$$\mathbb{P}(A_n(\varepsilon)) \geq \exp\left(\frac{-cT}{\varepsilon^H}\right). \tag{28}$$

Assume for a moment that these inequalities hold true. Then from (26) we deduce that $\sum_{n\geq 1} \mathbb{E}(1\!\!1_{A_n(\varepsilon)}|\mathcal{F}_{t_{n+1}}) = \infty$ a.s. and by Lemma 18 we obtain $\sum_{n\geq 1} 1\!\!1_{A_n(\varepsilon)} = \infty$ a.s., which implies

$$\lim_{n\to\infty}\inf \sup_{t_n - T\leq t,s\leq t_n} |B_t - B_s| \leq \varepsilon.$$

Proof of (27)

Let $t_n - T \leq s \leq t \leq t_n$. By (25) we have

$$B^{n+1}(t) - B^{n+1}(s) = \mathbb{E}\left[\int_{-\infty}^{t_{n+1}} c_H \left\{(s-r)^{H-\frac{1}{2}} - (t-r)^{H-\frac{1}{2}}\right\} dW_r \middle| \mathcal{F}_{t_{n+1}}\right]$$

and for $p \geq 1$ we obtain

$$\mathbb{E}\left(|B^{n+1}(t) - B^{n+1}(s)|^{2p}\right) \leq c\left(\int_{-\infty}^{t_{n+1}} \left|(s-r)^{H-\frac{1}{2}} - (t-r)^{H-\frac{1}{2}}\right|^2 dr\right)^p \ .$$

In the above integral we make successively the changes of variables $v = r - s$ and $u = v/(t-s)$. This yield

$$\begin{aligned}\left(\mathbb{E}\left(|B^{n+1}(t) - B^{n+1}(s)|^{2p}\right)\right)^{\frac{1}{p}} &\leq c(t-s)^{2H} \int_{-\infty}^{\frac{t_{n+1}-s}{t-s}} \left|(-u)^{H-\frac{1}{2}} - (1-u)^{H-\frac{1}{2}}\right|^2 du \\ &\leq c(t-s)^{2H} \int_{-\infty}^{\frac{t_{n+1}-s}{t-s}} (-u)^{2H-3} du\end{aligned}$$

where we have used the fact that for $-u$ sufficiently big (and positive), $|(-u)^{H-\frac{1}{2}} - (1-u)^{H-\frac{1}{2}}| \le c(-u)^{H-\frac{3}{2}}$. The above inequality is then true for sufficiently large n. Finally we obtain that

$$\mathbb{E}\left(|B^{n+1}(t) - B^{n+1}(s)|^{2p}\right) \le c\left((t-s)(t_n - t_{n+1})^{H-1}\right)^{2p} . \tag{29}$$

Now we use the Garsia-Rodemich-Rumsey inequality (see [6]): let f be a continuous function, ρ and g two continuous stricly increasing functions on $[0,\infty)$ with $\rho(0) = g(0) = 0$ and $\lim_{x\to\infty} \rho(x) = \infty$. Then it holds

$$\begin{aligned} |f(t) - f(s)| &\le 8 \int_0^{t-s} \rho^{-1}\left(\frac{4C_{s,t}}{u^2}\right) dg(u) \\ \text{with } \ C_{s,t} &= \int_s^t \int_s^t \rho\left(\frac{|f(t') - f(s')|}{g(|t'-s'|)}\right) ds'dt' . \end{aligned}$$

Let $0 < \epsilon_0 < 1$. We apply the above inequality with $\rho(u) = u^{2/(1-\epsilon_0)}$ and $g(u) = u$. Thus there exists a constant c and a random variable δ_n such that

$$|B^{n+1}(t) - B^{n+1}(s)| \le \delta_n \times |t-s|^{\epsilon_0} \qquad \text{with}$$

$$\delta_n = c\left(\int_{t_n - T}^{t_n} \int_{t_n - T}^{t_n} \rho\left(\frac{|B^{n+1}(t') - B^{n+1}(s')|}{|t'-s'|}\right)^{\frac{2}{1-\epsilon_0}} ds'dt'\right)^{\frac{1-\epsilon_0}{2}} .$$

By (29) and the Jensen inequality, it is clear that

$$\mathbb{E}(|\delta_n|^{2p}) \le cT^{2p(1-\epsilon_0)}(t_n - t_{n+1})^{2p(H-1)} ,$$

and we obtain that

$$\sup_{t_n - T \le t, s \le t_n} \left|B^{n+1}(t) - B^{n+1}(s)\right| \le c\, T^{\epsilon_0}\ \delta_n .$$

Now we write that

$$\mathbb{P}((\widetilde{A}_n(\varepsilon))^c) \le c\, T^{2\epsilon_0}\mathbb{E}(\delta_n^2)/\epsilon^2 \le c\, T^2\ (t_n - t_{n+1})^{2H-2}/\varepsilon^2$$

and since $\sum_{n\ge 1}(t_n - t_{n+1})^{2H-2} < \infty$, we obtain (27).

Proof of (28)

This inequality is a consequence of Talagrand's small ball estimate (see [16] or [12, Theorem 3.8]). Indeed, one needs al least $T\varepsilon^{-H}$ balls of radius ε under the Dudley metric $d(s,t) = \left(\mathbb{E}|B(t) - B(s)|^2\right)^{1/2}$ to cover the time interval $[t_n - T, t_n]$. It follows that that there exists a constant c such that

$$\log \mathbb{P}\left(\sup_{t_n - T \le t, s \le t_n} |B(t) - B(s)| \le \varepsilon\right) \ge -c\, \frac{T}{\varepsilon^H}$$

and we deduce (28).

□

References

[1] Dafermos, C. M. (1977). Generalized characteristics and the structure of solutions of hyperbolic conservation laws. *Indiana Univ. Math. J.* **26**, 6, 1097–1119.

[2] Decreusefond, L. (2005). Stochastic integration with respect to Volterra processes. *Ann. Inst. H. Poincaré Probab. Statist.* **41**, 2, 123–149.

[3] E, W., Khanin, K., Mazel, A., and Sinai, Y. (2000). Invariant measures for Burgers equation with stochastic forcing. *Ann. of Math. (2)* **151**, 3, 877–960.

[4] Feng, J. and Nualart, D. (2008). Stochastic scalar conservation laws. *J. Funct. Anal.* **255**, 2, 313–373.

[5] Fleming, W. H. and Soner, H. M. (1993). *Controlled Markov processes and viscosity solutions.* Applications of Mathematics (New York), Vol. **25**. Springer-Verlag, New York.

[6] Garsia, A. M., Rodemich, E., and Rumsey, Jr., H. (1970/1971). A real variable lemma and the continuity of paths of some Gaussian processes. *Indiana Univ. Math. J. 20*, 565–578.

[7] Holden, H. and Risebro, N. H. (1997). Conservation laws with a random source. *Appl. Math. Optim.* **36**, 2, 229–241.

[8] Hörmander, L. (1997). *Lectures on nonlinear hyperbolic differential equations.* Mathématiques & Applications (Berlin) [Mathematics & Applications], Vol. **26**. Springer-Verlag, Berlin.

[9] Hult, H. (2003). Approximating some Volterra type stochastic integrals with applications to parameter estimation. *Stochastic Process. Appl.* **105**, 1, 1–32.

[10] Kim, J. U. (2003). On a stochastic scalar conservation law. *Indiana Univ. Math. J.* **52**, 1, 227–256.

[11] Lax, P. D. (1957). Hyperbolic systems of conservation laws. II. *Comm. Pure Appl. Math. 10*, 537–566.

[12] Li, W. V. and Shao, Q.-M. (2001). Gaussian processes: inequalities, small ball probabilities and applications. In *Stochastic processes: theory and methods.* Handbook of Statist., Vol. **19**. North-Holland, Amsterdam, 533–597.

[13] Oleĭnik, O. A. (1957). Discontinuous solutions of non-linear differential equations. *Uspehi Mat. Nauk (N.S.)* **12**, 3(75), 3–73.

[14] Serre, D. (1999). *Systems of conservation laws. 1.* Cambridge University Press, Cambridge. Hyperbolicity, entropies, shock waves, Translated from the 1996 French original by I. N. Sneddon.

[15] Stout, W. F. (1974). *Almost sure convergence.* Academic Press [A subsidiary of Harcourt Brace Jovanovich, Publishers], New York-London. Probability and Mathematical Statistics, Vol. 24.

[16] Talagrand, M. (1996). Transportation cost for Gaussian and other product measures. *Geom. Funct. Anal.* **6**, 3, 587–600.

[17] Vallet, G. and Wittbold, P. On a stochastic first order hyperbolic equation in a bounded domain. *Infin. Dimens. Anal. Quantum Probab. Relat. Top.*.

[18] Young, L. C. (1936). An inequality of the Hölder type connected with Stieltjes integration. *Acta Math. 67*, 251–282.

[19] Zähle, M. (1998). Integration with respect to fractal functions and stochastic calculus. I. *Probab. Theory Related Fields* **111**, 3, 333–374.

Contents