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One- and Two-Level Domain Decomposition Methods for Nonlinear Problems

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Abstract

In this paper we synthesize the results in [1] – [6] concerning the convergence rate of the one- and two-level methods for some nonlinear problems: nonlinear variational inequalities, inequalities with contraction operators, variational inequalities of the second kind and quasi-variational inequalities. Also, we verify that the convergence rates obtained by numerical tests are really in concordance with the theoretical ones. We comparatively illustrate the convergence rates of the one- and two-level methods by numerical experiments for the solution of the two-obstacle problem of a nonlinear elastic membrane.

Keywords: domain decomposition methods, nonlinear variational inequalities, fixed-point problems, quasi-variational inequalities, multigrid and multilevel methods, contact problems with friction, nonlinear obstacle problems.

1 Introduction

The literature on the domain decomposition methods is very large. We can see, for instance, the papers in the proceedings of the annual conferences on domain decomposition methods starting in 1987 with [11] or those cited in the books [16], [22], [24] and [28]. Naturally, the most of the papers dealing with these methods are dedicated to the linear elliptic problems. For the variational inequalities, the convergence proofs refer in general to the inequalities coming from the minimization of quadratic functionals. Also, the most of the papers consider the convex set decomposed according to the space decomposition as a sum of convex subsets. To our knowledge, very few papers really deal with the application of these methods to nonlinear problems. We can cite in this direction the papers written by Boglaev [8], Dryja and Hackbusch [10], Lui [17], [18] and [19], Tai and Espedal [25], and Tai and Xu [26], for nonlinear equations, Hoffmann and Zhou [13], Zeng and Zhou [29], for inequalities having nonlinear source terms, and Badea [1]–[3], for the minimization of non-quadratic functionals.

The multilevel or multigrid methods can be viewed as domain decomposition methods and we can cite the results obtained by Kornhuber [14]–[16], Mandel [20], [21], Smith, Bjørstad and Gropp [24], Tarvainen [27], Badea, Tai and Wang [7], and Badea [2]. Evidently, this list is not exhaustive and it can be completed with other papers.

In this paper we synthesize the results in [1] – [6] concerning the convergence rate of the one- and two-level methods for some nonlinear problems: nonlinear variational inequalities, inequalities with contraction operators, variational inequalities of the second kind and quasi-variational inequalities. In these papers the same proof techniques are used. We give, under a certain assumption, general convergence results (error estimations, included) for some subspace correction algorithms in a reflexive Banach space. In Sobolev spaces, when the subspaces are associated with a domain decomposition, we get the multiplicative and additive Schwarz methods. In the finite element spaces, for the one- and two-level methods, the constants in the error estimations are explicitly written as functions of the overlapping and mesh parameters. These error estimations are similar with the familiar case of linear equations, ie. the convergence is global and optimal. Also, we verify that the convergence rates obtained by numerical tests are really in concordance with the theoretical ones. We comparatively illustrate the convergence rates of the one- and two-level methods by numerical experiments for the solution of the two-obstacle problem of a nonlinear elastic membrane.

The paper is organized as follows. In Section 2, we give a general background in which the problems in the next sections are stated. Section 3 is dedicated to the theoretical convergence results of the algorithms. In Section 4, the Schwarz methods, including the one and two-level methods, as particular cases in which the Sobolev and finite element spaces are used. Finally, Section 5 is dedicated to the numerical experiments.

2 General background

In this section, we give a general framework for all methods we propose. We consider a reflexive Banach space V, some closed subspaces of it, $V_1, \cdots, V_m$, and $K \subset V$ a non empty closed convex set. Let $F : V \to R$ be a Gâteaux differentiable functional, and we assume that there exist $p, q > 1$ such that for any $M > 0$ there exist $\alpha_M,\ \beta_M > 0$ for which

$$\begin{array}{l} \alpha_M ||v-u||^p \leq < F'(v) - F'(u), v-u >, \\ ||F'(v) - F'(u)||_{V'} \leq \beta_M ||v-u||^{q-1}, \end{array} \tag{1}$$

for any $u, v \in V$, $||u||, ||v|| \leq M$. We can prove that, if F satisfies the above properties, then

$$\begin{array}{l} \alpha_M ||v-u||^p \leq < F'(v) - F'(u), v-u > \leq \beta_M ||v-u||^q \\ < F'(u), v-u > + \frac{\alpha_M}{p} ||v-u||^p \leq F(v) - F(u) \leq \\ < F'(u), v-u > + \frac{\beta_M}{q} ||v-u||^q. \end{array}$$

Depending on the algorithm we use, we make one of the following assumptions on the convex set K. In the case of the multiplicative algorithms for inequalities we assume

Assumption 2.1 *There exists a constant $C_0 > 0$ such that for any $w, v \in K$ and $w_i \in V_i$ with $w + \sum_{j=1}^{i} w_j \in K$, $i = 1, \cdots, m$, there exist $v_i \in V_i$, $i = 1, \cdots, m$, satisfying*

$$w + \sum_{j=1}^{i-1} w_j + v_i \in K, \quad v - w = \sum_{i=1}^{m} v_i, \quad \sum_{i=1}^{m} ||v_i|| \le C_0 \left(||v - w|| + \sum_{i=1}^{m} ||w_i|| \right).$$

For the additive algorithms for inequalities, we assume

Assumption 2.2 *There exists a constant $C_0 > 0$ such that for any $w, v \in K$ there exist $v_i \in V_i$, $i = 1, \cdots, m$, which satisfy*

$$v - w = \sum_{i=1}^{m} v_i, \quad w + v_i \in K, \quad \sum_{i=1}^{m} ||v_i|| \le C_0 ||v - w||.$$

Finally, in the case of equations, for both additive and multiplicative algorithms, we assume,

Assumption 2.3 *There exists a constant $C_0 > 0$ such that for any $v \in V$ there exist $v_i \in V_i$, $i = 1, \cdots, m$, which satisfy*

$$v = \sum_{i=1}^{m} v_i, \quad \sum_{i=1}^{m} ||v_i|| \le C_0 ||v||.$$

3 Subspace correction algorithms for nonlinear problems

In this section we introduce additive and multiplicative algorithms for the nonlinear problems we have mentioned in the introduction, show that they a globally convergent, and deduce error estimates.

3.1 Nonlinear variational inequalities

Details on the results in this subsection can be found in [1] and [3]. We assume that F is coercive, ie. $\frac{F(v)}{||v||} \to \infty$ as $||v|| \to \infty$, if K is not bounded. We consider the problem

$$u \in K : \ < F'(u), v - u > \ge 0, \text{ for any } v \in K \tag{2}$$

Under the above conditions on F, this problem has an unique solution, and it is equivalent with the minimization problem

$$u \in K : F(u) \le F(v), \text{ for any } v \in K.$$

To solve problem (2), we introduce two algorithms. The first one is a multiplicative algorithm,

Algorithm 3.1 *We start the algorithm with an arbitrary* $u^0 \in K$. *At iteration* $n+1$, *having* $u^n \in K$, $n \geq 0$, *we compute sequentially for* $i = 1, \cdots, m$, $w_i^{n+1} \in V_i$, $u^{n+\frac{i-1}{m}} + w_i^{n+1} \in K$ *satisfying*

$$< F'(u^{n+\frac{i-1}{m}} + w_i^{n+1}), v_i - w_i^{n+1} >\geq 0, \textit{ for any } v_i \in V_i,\ u^{n+\frac{i-1}{m}} + v_i \in K,$$

and then we update $u^{n+\frac{i}{m}} = u^{n+\frac{i-1}{m}} + w_i^{n+1}$.

The additive algorithm is stated as

Algorithm 3.2 *We start the algorithm with an arbitrary* $u^0 \in K$. *At iteration* $n+1$, *having* $u^n \in K$, $n \geq 0$, *we compute* $w_i^{n+1} \in V_i$, $u^n + w_i^{n+1} \in K$, *the solution of the inequality*

$$\langle F'(u^n + w_i^{n+1}), v_i - w_i^{n+1} \rangle \geq 0, \textit{ for any } v_i \in V_i,\ u^n + v_i \in K$$

for $i = 1, \cdots, m$, *and then we update* $u^{n+1} = u^n + \varrho \sum_{i=1}^{m} w_i^{n+1}$, *where* $0 < \varrho \leq \frac{1}{m}$ *for any* $n \geq 0$.

We notice that, because of the choice of ϱ in this algorithm, $u^{n+1} \in K$ for any $n \geq 0$. Concerning the convergence of the above algorithms, we have the following result

Theorem 3.1 *Under the above conditions on the space* V *and the functional* F, *we have the following error estimations:*

(i) if $p = q$ *we have*

$$\begin{aligned} F(u^n) - F(u) &\leq \left(\tfrac{C_1}{C_1+1}\right)^n \left[F(u^0) - F(u)\right], \\ ||u^n - u||^p &\leq \tfrac{2}{\alpha_M} \left(\tfrac{C_1}{C_1+1}\right)^n \left[F(u^0) - F(u)\right]. \end{aligned}$$

(ii) if $p > q$ *we have*

$$\begin{aligned} F(u^n) - F(u) &\leq \frac{F(u^0) - F(u)}{\left[1 + nC_2 (F(u^0) - F(u))^{\frac{p-q}{q-1}}\right]^{\frac{q-1}{p-q}}}, \\ ||u - u^n||^p &\leq \frac{p}{\alpha_M} \frac{F(u^0) - F(u)}{\left[1 + nC_2 (F(u^0) - F(u))^{\frac{p-q}{q-1}}\right]^{\frac{q-1}{p-q}}}. \end{aligned}$$

for the multiplicative and additive Algorithms 3.1 and 3.2, if the corresponding assumption on the convex set K *holds. The constants* C_1 *and* C_2 *depend on the functional* F *(ie. on* α_M, β_M, p *and* q*), the number of subspaces* m, *the initial approximation* u^0, *and is an increasing function on* C_0 *in assumptions.*

3.2 Inequalities with contraction operators

Details on the results in this subsection can be found in [4]. We assume that p=q, and, as in the previous case, that F coercive, ie. $\dfrac{F(v)}{||v||} \to \infty$ as $||v|| \to \infty$, if K is not bounded. We consider an operator $T : V \to V'$ such that for any $M > 0$ there exists $0 < \rho_M$ such that

$$||T(v) - T(u)||_{V'} \leq \rho_M ||v - u|| \text{ for any } v, u \in V, ||v||, ||u|| \leq M. \tag{3}$$

We consider the problem

$$u \in K : \langle F'(u), v - u\rangle - \langle T(u), v - u\rangle \geq 0, \text{ for any } v \in K \tag{4}$$

This problem is equivalent with the minimization problem

$$u \in K : F(u) - \langle T(u), u\rangle \leq F(v) - \langle T(u), v\rangle, \text{ for any } v \in K.$$

We know that, under the above assumptions on V, K, F and T, if there exists a constant $0 < \theta < 1$ such that $\frac{\rho_M}{\alpha_M} \leq \theta$, for any $M > 0$, then problem (4) has a unique solution. Now, we state three multiplicative algorithms for problem (4).

Algorithm 3.3 *We start the algorithm with an arbitrary $u^0 \in K$. At iteration $n+1$, having $u^n \in K$, $n \geq 0$, we compute $w_i^{n+1} \in V_i$, $u^{n+\frac{i-1}{m}} + w_i^{n+1} \in K$, the solution of the inequality*

$$\begin{aligned}&\langle F'(u^{n+\frac{i-1}{m}} + w_i^{n+1}), v_i - w_i^{n+1}\rangle - \langle T(u^{n+\frac{i-1}{m}} + w_i^{n+1}), v_i - w_i^{n+1}\rangle \geq 0,\\ &\textit{for any } v_i \in V_i,\ u^{n+\frac{i-1}{m}} + v_i \in K\end{aligned}$$

and then we update $u^{n+\frac{i}{m}} = u^{n+\frac{i-1}{m}} + w_i^{n+1}$, for $i = 1, \cdots, m$.

Algorithm 3.4 *We start the algorithm with an arbitrary $u^0 \in K$. At iteration $n+1$, having $u^n \in K$, $n \geq 0$, we compute $w_i^{n+1} \in V_i$, $u^{n+\frac{i-1}{m}} + w_i^{n+1} \in K$, the solution of the inequality*

$$\begin{aligned}&\langle F'(u^{n+\frac{i-1}{m}} + w_i^{n+1}), v_i - w_i^{n+1}\rangle - \langle T(u^{n+\frac{i-1}{m}}), v_i - w_i^{n+1}\rangle \geq 0,\\ &\textit{for any } v_i \in V_i,\ u^{n+\frac{i-1}{m}} + v_i \in K\end{aligned}$$

and then we update $u^{n+\frac{i}{m}} = u^{n+\frac{i-1}{m}} + w_i^{n+1}$, for $i = 1, \cdots, m$.

Algorithm 3.5 *We start the algorithm with an arbitrary $u^0 \in K$. At iteration $n+1$, having $u^n \in K$, $n \geq 0$, we compute $w_i^{n+1} \in V_i$, $u^{n+\frac{i-1}{m}} + w_i^{n+1} \in K$, the solution of the inequality*

$$\begin{aligned}&\langle F'(u^{n+\frac{i-1}{m}} + w_i^{n+1}), v_i - w_i^{n+1}\rangle - \langle T(u^n), v_i - w_i^{n+1}\rangle \geq 0,\\ &\textit{for any } v_i \in V_i,\ u^{n+\frac{i-1}{m}} + v_i \in K\end{aligned}$$

and then we update $u^{n+\frac{i}{m}} = u^{n+\frac{i-1}{m}} + w_i^{n+1}$, for $i = 1, \cdots, m$.

Also, two additive algorithms can be given for the solution of problem (4),

Algorithm 3.6 *We start the algorithm with an arbitrary $u^0 \in K$. At iteration $n+1$, having $u^n \in K$, $n \geq 0$, we compute $w_i^{n+1} \in V_i$, $u^n + w_i^{n+1} \in K$, the solution of the inequality*

$$\begin{array}{l}\langle F'(u^n + w_i^{n+1}), v_i - w_i^{n+1}\rangle - \langle T(u^n + w_i^{n+1}), v_i - w_i^{n+1}\rangle \geq 0, \\ \textit{for any } v_i \in V_i,\ u^n + v_i \in K\end{array}$$

for $i = 1, \cdots, m$, and then we update $u^{n+1} = u^n + \varrho \sum_{i=1}^{m} w_i^{n+1}$, where $0 < \varrho \leq \frac{1}{m}$ for any $n \geq 0$ ($u^{n+1} \in K$).

Algorithm 3.7 *We start the algorithm with an arbitrary $u^0 \in K$. At iteration $n+1$, having $u^n \in K$, $n \geq 0$, we compute $w_i^{n+1} \in V_i$, $u^n + w_i^{n+1} \in K$, the solution of the inequality*

$$\begin{array}{l}\langle F'(u^n + w_i^{n+1}), v_i - w_i^{n+1}\rangle - \langle T(u^n), v_i - w_i^{n+1}\rangle \geq 0, \\ \textit{for any } v_i \in V_i,\ u^n + v_i \in K\end{array}$$

for $i = 1, \cdots, m$, and then we update $u^{n+1} = u^n + \varrho \sum_{i=1}^{m} w_i^{n+1}$, where $0 < \varrho \leq \frac{1}{m}$ for any $n \geq 0$ ($u^{n+1} \in K$).

The following theorem give an error estimation for the above algorithms.

Theorem 3.2 *Under the above assumptions on V, F and T, let u be the solution of our problem, and u^n, $n \geq 0$, be its approximations obtained from one of the above Algorithms 3.3–3.7. If the assumption corresponding to the algorithm holds, then there exist a function $\theta_{\max} : \mathbf{R}_+ \to \mathbf{R}_+$ and a constant $C_1 > 0$ such that if $\frac{\rho_M}{\alpha_M} < \theta_{\max}(M)$, for any $M > 0$, then, for any $u^0 \in K$, we have the error estimates*

$$\begin{array}{l}F(u^n) - \langle T(u), u^n\rangle - F(u) + \langle T(u), u\rangle \leq \\ \left(\frac{C_1}{C_1+1}\right)^n \left[F(u^0) - \langle T(u), u^0\rangle - F(u) + \langle T(u), u\rangle\right], \\ ||u^n - u||^2 \leq \frac{2}{\alpha_{M_0}} \left(\frac{C_1}{C_1+1}\right)^n \left[F(u^0) - \langle T(u), u^0\rangle - F(u) + \langle T(u), u\rangle\right]\end{array}$$

hold for any $n \geq 1$, where $M_0 = \max(\ ||u||,\ \sup\{||v|| : F(v) - \langle T(u), v\rangle \leq F(u^0) - \langle T(u), u^0\rangle\}\ )$. Constant C_1 depends on the functional F (ie. on α_M, β_M), the operator T (ie. on ρ_M), the number of subspaces m, the initial approximation u^0, and is an increasing function on C_0 in assumptions.

3.3 Variational inequalities of the second kind

Details on the results in this subsection can be found in [5] and [6]. Let $\varphi : K \to \mathbf{R}$ be a convex, lower semicontinuous functional s.t. $F + \varphi$ is coercive, ie. $\frac{F(v) + \varphi(v)}{\|v\|} \to \infty$, as $\|v\| \to \infty,\ v \in K$, if K is not bounded. In addition to the hypotheses of Assumption 2.1, we suppose that

$$\sum_{i=1}^{m} [\varphi(w + \sum_{j=1}^{i-1} w_j + v_i) - \varphi(w + \sum_{j=1}^{i-1} w_j + w_i)] \leq \varphi(v) - \varphi(w + \sum_{i=1}^{m} w_i) \tag{5}$$

for $v, w \in K$, and $v_i, w_i \in V_i$, $i = 1, \dots, m$, in Assumption 2.1. We consider the problem

$$u \in K : \langle F'(u), v - u\rangle + \varphi(v) - \varphi(u) \geq 0, \text{ for any } v \in K. \tag{6}$$

which is equivalent with

$$u \in K : F(u) + \varphi(u) \leq F(v) + \varphi(v), \text{ for any } v \in K.$$

It is well known that problem (6) has a unique solution. We state the following algorithm for the solution of the above problem.

Algorithm 3.8 *We start the algorithm with an arbitrary $u^0 \in K$. At iteration $n+1$, having $u^n \in K$, $n \geq 0$, we compute sequentially for $i = 1, \cdots, m$, the local corrections $w_i^{n+1} \in V_i$ as the solution of the variational inequality*

$$\begin{aligned}&\langle F'(u^{n+\frac{i-1}{m}} + w_i^{n+1}), v_i - w_i^{n+1}\rangle + \varphi(u^{n+\frac{i-1}{m}} + v_i) - \\ &\varphi(u^{n+\frac{i-1}{m}} + w_i^{n+1}) \geq 0, \textit{ for any } v_i \in V_i,\ u^{n+\frac{i-1}{m}} + v_i \in K,\end{aligned}$$

and then we update $u^{n+\frac{i}{m}} = u^{n+\frac{i-1}{m}} + w_i^{n+1}$.

Concerning the convergence of this algorithm, we have

Theorem 3.3 *Under the above assumptions on V, F and φ, let u be the solution of problem (6), and u^n, $n \geq 0$, be its approximations obtained from the multiplicative Algorithm 3.8. If Assumption 2.1 together with (5) hold, then there exists $M > 0$ such that $\max(\|u\|, \|u^0\|, \max_{n\geq 0, 1\leq i\leq m} \|u^{n+\frac{i}{m}}\|) \leq M$ and we have the following error estimations:*

(i) if $p = q$ we have

$$\begin{aligned}&F(u^n) + \varphi(u^n) - F(u) - \varphi(u) \leq \\ &\left(\tfrac{C_1}{C_1+1}\right)^n [F(u^0) + \varphi(u^0) - F(u) - \varphi(u)],\end{aligned}$$

$$\|u^n - u\|^p \leq \tfrac{p}{\alpha_M} \left(\tfrac{C_1}{C_1+1}\right)^n [F(u^0) + \varphi(u^0) - F(u) - \varphi(u)].$$

(ii) if $p > q$ we have

$$F(u^n) + \varphi(u^n) - F(u) - \varphi(u) \leq \frac{F(u^0)+\varphi(u^0)-F(u)-\varphi(u)}{\left[1+nC_2(F(u^0)+\varphi(u^0)-F(u)-\varphi(u))^{\frac{p-q}{q-1}}\right]^{\frac{q-1}{p-q}}},$$

$$\|u - u^n\|^p \leq \frac{p}{\alpha_M} \frac{F(u^0)+\varphi(u^0)-F(u)-\varphi(u)}{\left[1+nC_2(F(u^0)+\varphi(u^0)-F(u)-\varphi(u))^{\frac{p-q}{q-1}}\right]^{\frac{q-1}{p-q}}}.$$

Constants C_1 and C_2 depend on the functionals F and φ, the number of subspaces m, the initial approximation u^0, and is an increasing function on C_0 in Assumption 2.1.

The above theorem has been used in [6] to prove that the multiplicative Schwarz method converge for the contact problem with Tresca friction.

3.4 Quasi-variational inequalities

Details on the results in this subsection can be found in [5] and [6]. In this subsection, we consider $p = q = 2$. Let $\varphi : K \times K \to \mathbf{R}$ be a functional such that, for any $u \in K$, $\varphi(u, \cdot) : K \to \mathbf{R}$ is convex, lower semicontinuous, and $F(\cdot) + \varphi(u, \cdot)$ is coercive, ie. $\dfrac{F(v) + \varphi(u, v)}{\|v\|} \to \infty$ as $\|v\| \to \infty, v \in K$, if K is not bounded. We assume that for any $M > 0$ there exists $c_M > 0$ such that

$$|\varphi(v_1, w_2) + \varphi(v_2, w_1) - \varphi(v_1, w_1) - \varphi(v_2, w_2)| \leq c_M \|v_1 - v_2\| \|w_1 - w_2\| \quad (7)$$

for any $v_1,\ v_2,\ w_1\ w_2 \in K$, $\|v_1\|,\ \|v_2\|,\ \|w_1\|\ \|w_2\| \leq M$. In addition to the hypotheses of Assumption 2.1, we suppose

$$\begin{aligned} &\sum_{i=1}^{m} [\varphi(u, w + \sum_{j=1}^{i-1} w_j + v_i) - \varphi(u, w + \sum_{j=1}^{i-1} w_j + w_i)] \leq \\ &\varphi(u, v) - \varphi(u, w + \sum_{i=1}^{m} w_i) \end{aligned} \quad (8)$$

for any $u \in K$ and for $v, w \in K$ and $v_i, w_i \in V_i$, $i = 1, \ldots, m$, in Assumption 2.1. We consider the problem

$$u \in K : \langle F'(u), v - u \rangle + \varphi(u, v) - \varphi(u, u) \geq 0, \text{ for any } v \in K. \quad (9)$$

which is equivalent with the minimization problem

$$u \in K : F(u) + \varphi(u, u) \leq F(v) + \varphi(u, v), \text{ for any } v \in K.$$

With a similar proof to that of Theorem 2.1 in [23], we can show that these problems have a unique solution if there exists a constant $\kappa < 1$ such that $\frac{c_M}{\alpha_M} \leq \kappa$ for any $M > 0$. Similarly with the case of inequalities with contraction operators, we can state three multiplicative algorithms to solve problem (9),

Algorithm 3.9 *We start the algorithm with an arbitrary $u^0 \in K$. At iteration $n + 1$, having $u^n \in K$, $n \geq 0$, we compute sequentially for $i = 1, \cdots, m$, the local corrections $w_i^{n+1} \in V_i$ satisfying*

$$\begin{aligned}&\langle F'(u^{n+\frac{i-1}{m}} + w_i^{n+1}), v_i - w_i^{n+1}\rangle + \varphi(u^{n+\frac{i-1}{m}} + w_i^{n+1}, u^{n+\frac{i-1}{m}} + v_i)- \\ &\varphi(u^{n+\frac{i-1}{m}} + w_i^{n+1}, u^{n+\frac{i-1}{m}} + w_i^{n+1}) \geq 0, \textit{ for any } v_i \in V_i,\ u^{n+\frac{i-1}{m}} + v_i \in K,\end{aligned}$$

and then we update $u^{n+\frac{i}{m}} = u^{n+\frac{i-1}{m}} + w_i^{n+1}$.

Algorithm 3.10 *We start the algorithm with an arbitrary $u^0 \in K$. At iteration $n + 1$, having $u^n \in K$, $n \geq 0$, we compute sequentially for $i = 1, \cdots, m$, the local corrections $w_i^{n+1} \in V_i$ satisfying*

$$\begin{aligned}&\langle F'(u^{n+\frac{i-1}{m}} + w_i^{n+1}), v_i - w_i^{n+1}\rangle + \varphi(u^{n+\frac{i-1}{m}}, u^{n+\frac{i-1}{m}} + v_i)- \\ &\varphi(u^{n+\frac{i-1}{m}}, u^{n+\frac{i-1}{m}} + w_i^{n+1}) \geq 0, \textit{ for any } v_i \in V_i,\ u^{n+\frac{i-1}{m}} + v_i \in K\end{aligned}$$

and then we update $u^{n+\frac{i}{m}} = u^{n+\frac{i-1}{m}} + w_i^{n+1}$.

Algorithm 3.11 *We start the algorithm with an arbitrary $u^0 \in K$. At iteration $n + 1$, having $u^n \in K$, $n \geq 0$, we compute sequentially for $i = 1, \cdots, m$, the local corrections $w_i^{n+1} \in V_i$ satisfying*

$$\begin{aligned}&\langle F'(u^{n+\frac{i-1}{m}} + w_i^{n+1}), v_i - w_i^{n+1}\rangle + \varphi(u^n, u^{n+\frac{i-1}{m}} + v_i)- \\ &\varphi(u^n, u^{n+\frac{i-1}{m}} + w_i^{n+1}) \geq 0, \textit{ for any } v_i \in V_i,\ u^{n+\frac{i-1}{m}} + v_i \in K\end{aligned}$$

and then we update $u^{n+\frac{i}{m}} = u^{n+\frac{i-1}{m}} + w_i^{n+1}$.

The following theorem proves that if C_M is small enough in comparison with α_M and β_M, then the multiplicative Algorithms 3.9–3.11 are convergent.

Theorem 3.4 *Under the above assumptions on V, F and φ, let u be the solution of problem (9), and u^n, $n \geq 0$, be its approximations obtained from one of the multiplicative Algorithms 3.9–3.11. If Assumption 2.1 together with (8) hold, and if $\frac{\alpha_M}{2} \geq mC_M + \sqrt{2m(25C_0 + 8)\beta_M C_M}$, for any $M > 0$, then there exists an $M > 0$ such that $\max(\|u\|, \|u^0\|, \max_{n\geq 0, 1\leq i\leq m} \|u^{n+\frac{i}{m}}\|) \leq M$ and we have the following error estimations*

$$\begin{aligned}&F(u^n) + \varphi(u, u^n) - F(u) - \varphi(u, u) \leq \\ &\left(\frac{C_1}{C_1+1}\right)^n \left[F(u^0) + \varphi(u, u^0) - F(u) - \varphi(u, u)\right],\end{aligned}$$

$$\|u^n - u\|^2 \leq \frac{2}{\alpha_M}\left(\frac{C_1}{C_1+1}\right)^n \left[F(u^0) + \varphi(u^0) - F(u) - \varphi(u)\right].$$

Constant C_1 depends on the functionals F and φ, the number of subspaces m, the initial approximation u^0, and is an increasing function on C_0 in Assumption 2.1.

This result concerning the convergence of Algorithms 3.9–3.11 has been applied in [6] to prove that the multiplicative Schwarz methods converge for the contact problem with Coulomb friction.

4 Schwarz methods as subspace correction methods

Details on the results in this section can be found in [2]. Let $\Omega \subset \mathbf{R}^d$ be an open bounded domain in $\mathbf{R}^d$ with Lipschitz continuous boundary $\partial\Omega$, and we consider an overlapping decomposition $\Omega = \cup_{i=1}^{m}\Omega_i$, where Ω_i are open subdomains with Lipschitz continuous boundary, too. We define the Sobolev space $V = W_0^{1,s}(\Omega)$, $1 < s < \infty$, let $K \subset V$ be a convex closed set, and $V_i = W_0^{1,s}(\Omega_i)$, $i = 1, \cdots, m$. With these spaces, associated to the domain decomposition, the multiplicative and additive algorithms introduced in the previous section represent Schwarz methods. The above spaces correspond to Dirichlet boundary conditions. Similar results can be obtained if we consider the problems with mixed boundary conditions. Also, we have considered problems having solution in $W^{1,s}(\Omega)$, but all the obtained results hold for problems in $[W^{1,s}(\Omega)]^N$, $N \geq 2$. We assume that the convex set K has the following

Property 4.1 *If $v, w \in K$, and if $\theta \in C^1(\bar{\Omega})$ with $0 \leq \theta \leq 1$, then $\theta v + (1-\theta)w \in K$.*

We can prove (see [1] for the multiplicative case, and [3] for the additive one),

Proposition 4.1 *Assumptions 2.1–2.3 hold for any convex set K having Property 4.1.*

Consequently, the error estimations in Theorems 3.1–3.4 hold for convex sets having the above property, provided that the functional F and the operator T or functionals φ have appropriate proprieties. The convergence rate of these methods depends essentially on the constant C_0 and is an increasing function of this constant. On the other hand, the constant C_0 depends on the domain decomposition. For the one- and two-level methods, we can write this constant C_0 as a function of the overlapping and mesh parameters.

4.1 One-level method

We consider a simplicial regular mesh partition $\mathcal{T}_h$ of mesh size h over $\Omega \subset \mathbf{R}^d$. We assume that $\mathcal{T}_h$ supplies a mesh partition for each subdomain Ω_i, $i = 1, \ldots, m$, and the overlapping parameter of the domain decomposition is δ. We use the piecewise linear finite element spaces

$$V_h = \{v \in C^0(\bar{\Omega}) : v|_\tau \in P_1(\tau),\ \tau \in \mathcal{T}_h,\ v = 0 \text{ on } \partial\Omega\} \text{ and}$$
$$V_h^i = \{v \in V_h : v = 0 \text{ in } \Omega\backslash\Omega_i\}, \quad i = 1, \ldots, m.$$

The spaces V_h and V_h^i, $i = 1, \ldots, m$, are considered as subspaces of $W^{1,s}$. We assume that convex set $K_h \subset V_h$ satisfies

Property 4.2 *If $v, w \in K_h$, and if $\theta \in C^0(\bar{\Omega})$, $\theta|_\tau \in C^1(\tau)$ for any $\tau \in \mathcal{T}_h$, and $0 \leq \theta \leq 1$, then $L_h(\theta v + (1-\theta)w) \in K_h$, where L_h is the P_1-Lagrangian interpolation.*

The following proposition estimates the constant C_0 as a function of the number of subdomains and the overlapping parameter.

Proposition 4.2 *Assumptions 2.1–2.3 hold for the piecewise linear finite element spaces, $V = V_h$ and $V_i = V_h^i$, $i = 1, \ldots, m$, and for any convex set $K = K_h \subset V_h$ having Property 4.2. The constant C_0 can be written as*

$$C_0 = C(m+1)(1 + \frac{m-1}{\delta}),$$

in Assumption 2.1, and

$$C_0 = Cm\,(1 + 1/\delta)\,,$$

in Assumptions 2.2 and 2.3, where C is independent of the mesh parameter and the domain decomposition.

4.2 Two-level method

In the case of the two-level method, we consider two regular simplicial mesh partitions $\mathcal{T}_h$ and $\mathcal{T}_H$ on $\Omega \subset \mathbf{R}^d$, $\mathcal{T}_h$ being a refinement of $\mathcal{T}_H$. As in the previous case, we assume that $\mathcal{T}_h$ supplies a mesh partition for each Ω_i, $1 \le i \le M$, and the overlapping parameter of the domain decomposition is δ. Also, we assume that $\text{diam}(\Omega_i) \le CH$, $i = 1, \ldots, m$, C being independent of the mesh partitions. The domain Ω may be different from $\Omega_0 = \cup_{\tau \in \mathcal{T}_H} \tau$, but we assume that if a node of $\mathcal{T}_H$ lies on $\partial\Omega_0$, then it lies on $\partial\Omega$, too, and $\text{dist}(x, \Omega_0) \le CH$ for any node x of $\mathcal{T}_h$, C being independent of both meshes. The spaces V_h, V_h^i, $i = 1, \ldots, m$ are defined as in case of the one-level method, but we introduce a new piecewise linear finite element space corresponding to the H-level,

$$V_H^0 = \left\{ v \in C^0(\bar{\Omega}_0) : v|_\tau \in P_1(\tau),\ \tau \in \mathcal{T}_H,\ v = 0 \text{ on } \partial\Omega_0 \right\}.$$

The convex set $K_h \subset V_h$ is assumed to satisfy Property 4.2. The two-level Schwarz methods are obtained from the multiplicative and additive algorithms with $V = V_h$, $K = K_h$, and the subspaces $V_0 = V_H^0$, $V_1 = V_h^1$, $V_2 = V_h^2$, …, $V_m = V_h^m$. As in the case of the one-level method, spaces V_h, V_H^0, V_h^1, $V_h^2, \ldots, V_h^m$, are considered as subspaces of $W^{1,s}$ for $1 \le s \le \infty$. The following proposition shows that the constant C_0 in Assumptions 2.1–2.3 is independent of the mesh and domain decomposition parameters if H/δ and H/h are constant.

Proposition 4.3 *Assumptions 2.1–2.3 are verified for the piecewise linear finite element spaces $V = V_h$ and $V_0 = V_H^0$, $V_i = V_h^i$, $i = 1, \ldots, m$, and any convex set $K = K_h$ satisfying Property 4.2. The constant C_0 can be taken of the form*

$$C_0 = Cm\left(1 + (m-1)\frac{H}{\delta}\right) C_{d,s}(H, h),$$

in Assumption 2.1, and

$$C_0 = C(m+1)\,(1 + H/\delta)\, C_{d,s}(H, h),$$

in Assumptions 2.2 and 2.3, where C is independent of the mesh and domain decomposition parameters, and

$$C_{d,s}(H,h) = \begin{cases} 1 & \text{if } d = s = 1 \text{ or } 1 \le d < s \le \infty \\ \left(\ln \frac{H}{h} + 1\right)^{\frac{d-1}{d}} & \text{if } 1 < d = s < \infty \\ \left(\frac{H}{h}\right)^{\frac{d-s}{s}} & \text{if } 1 \le s < d < \infty, \end{cases} \tag{10}$$

Remark 4.1 In the cases of the inequalities of the second kind and the quasi-variational inequalities we have introduced conditions on (5) and (8), respectively. These conditions refer to the functionals φ and the function decompositions in Assumptions 2.1 – 2.3. It is proved in [5], in the general setting, and in [6], in the case of the contact problems with friction, that if we consider some linear approximations of φ, then conditions (5) and (8) hold. Consequently, it follows from Theorems 3.1–3.4, the one- and two-level methods are globally convergent. The convergence rate essentially depends on the mesh and overlapping parameters through the constant C_0 given in Propositions 4.2 and 4.3.

5 Numerical example

The numerical experiments refer to the two-obstacle problem of a nonlinear elastic membrane without body forces. In a domain $\Omega \subset \mathbf{R}^2$, we consider the problem

$$u \in K \equiv [a,b] : \int_\Omega |\nabla u|^{s-2} \nabla u \nabla (v-u) \ge 0, \text{ for any } v \in [a,b].$$

where $a \le b$, $a, b \in W_0^{1,s}(\Omega)$, $1 < s < \infty$. This problem is equivalent with the

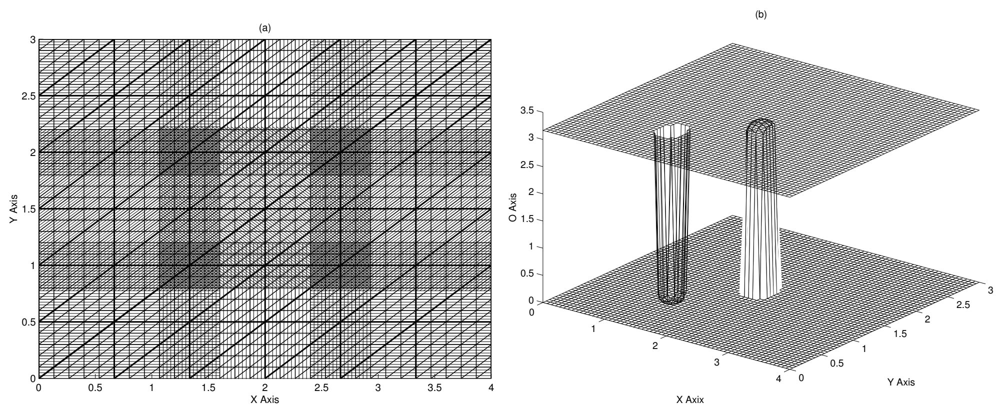

Figure 1: (a) Meshes $\mathcal{T}_H$, $\mathcal{T}_h$, and the domain decomposition, (b) Obstacles a and b.

minimization problem

$$u \in K : F(u) = \min_{v \in K} \frac{1}{s} \int_\Omega |\nabla v|^s.$$

The above problem is of the type (2) and has been solved by the multiplicative Algorithm 3.1. We know that if $1 < s \leq 2$, then (see [12]) for any $v, u \in W_0^{1,s}(\Omega)$,

$$
\begin{array}{l}
< F'(v) - F'(u), v - u > \geq \alpha ||v - u||_{1,s}^2 / (||v||_{1,s} + ||u||_{1,s})^{2-s} \\
\beta ||v - u||_{1,s}^{s-1} \geq ||F'(v) - F'(u)||_{V'},
\end{array}
$$

where $\alpha,\ \beta > 0$ are constants. Consequently, the functions introduced in (1) can be written as $\alpha_M = \alpha/(2M)^{2-s}$, $\beta_M = \beta$ with $p = 2$ and $q = s$.
If $s \geq 2$, then (see [9]) for any $v, u \in W_0^{1,s}(\Omega)$, we have

$$
\begin{array}{l}
< F'(v) - F'(u), v - u > \geq \alpha ||v - u||_{1,s}^s \\
\beta(||v||_{1,s} + ||u||_{1,s})^{s-2} ||v - u||_{1,s} \geq ||F'(v) - F'(u)||_{V'},
\end{array}
$$

where $\alpha,\ \beta > 0$ are constants. Therefore, $\alpha_M = \alpha,\ \beta_M = \beta(2M)^{s-2}$, $p = s$ and $q = 2$ in (1).

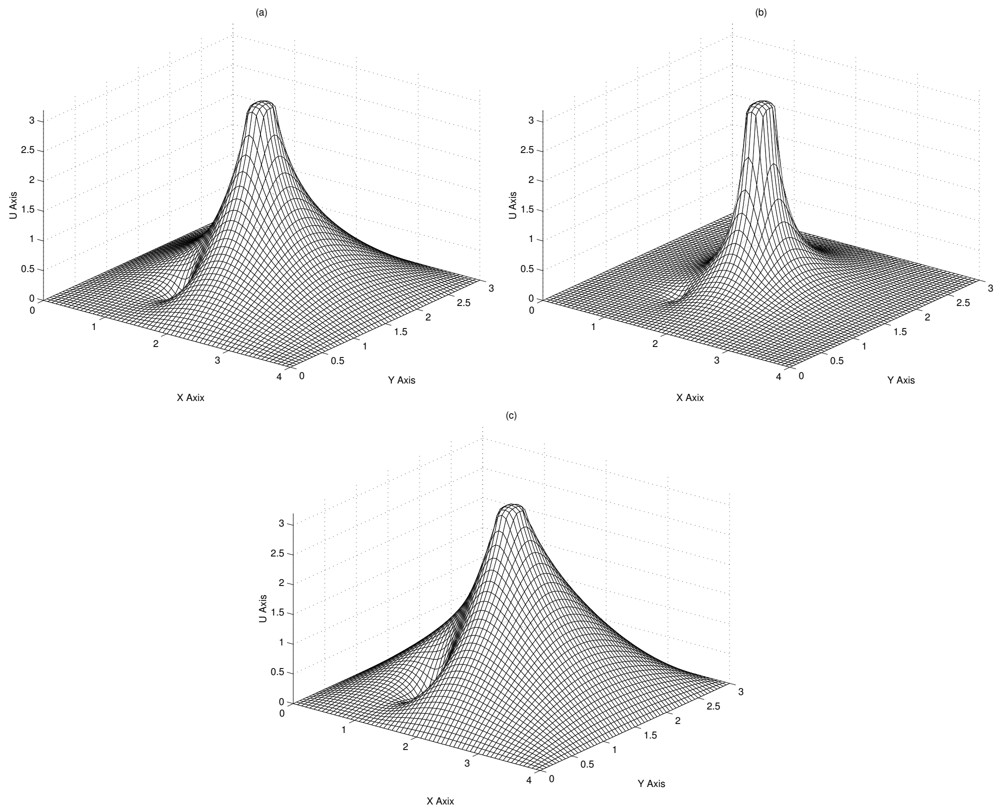

Figure 2: Solutions for: (a) s=2, (b) s=1.5, (c) s=3.

In our numerical experiments, we have taken $\Omega = (0, 4) \times (0, 3)$, $\mathcal{T}_h$, $\mathcal{T}_H$ contain right-angled triangles, and the same number of equal segments are considered on each side of the rectangular domain Ω. In Figure 1.a, this number is 30 for $\mathcal{T}_h$ and 6 for $\mathcal{T}_H$. In the same figure we have shown the domain decomposition, the number of the

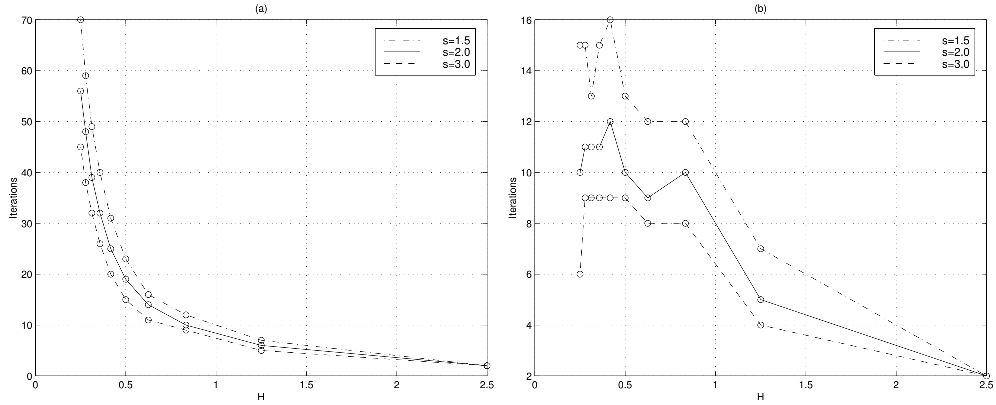

Figure 3: Iterations for H/h and H/δ constant: (a) one level, (b) two levels.

subdomains Ω_i is 9, and the width of the overlaps is of 2 triangles in $\mathcal{T}_h$. The obstacles a and b are also shown in Figure 1.b. Each of them is composed by a plane, a cylinder and a semisphere. The computed solutions for $s = 2.0$, $s = 1.5$ and $s = 3.0$ are plotted in Figure 2 for a mesh $\mathcal{T}_h$ having 60 nodes on each side of the rectangular domain Ω. In all the numerical tests the calculus has been stopped at a relative error of 1.E-03 at the nodes of $\mathcal{T}_h$ between two consecutive computed solutions.

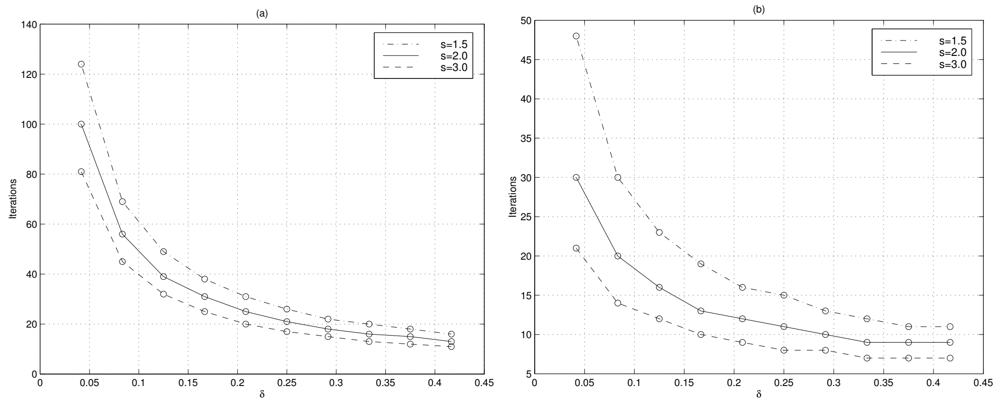

Figure 4: Iterations for H and h constant, and δ variable: (a) one level, (b) two levels.

We have seen in the previous section that the convergence rate depends on $1/\delta$ in the case of the one-level method, and on H/h and H/δ for the two-level method. We have tried to verify it by numerical tests for the nonlinear membrane problem taking various values of H, h and δ.

In the tests in Figure 3, $H/h = 6$ and $H/\delta = 2$ stay unchanged while the coarse mesh parameter H varies and it corresponds to 2, 4, . . . , 18, 20 segments on each side of the rectangular domain Ω. We see that the number of the iterations is bounded for the two-level method, and is an increasing function of $1/\delta$ for the one-level method.

This is in concordance with our predictions.

In the tests in Figures 4, 5 and 6, two of the parameters H, h or δ are constant and the third one is variable.

For the tests in Figure 4, we have taken $H = 5.0/12$, $h = 5.0/120$ and $\delta = 1h, 2h, \cdots, 10h$. We see that, in both cases, the number of iterations is a decreasing function of δ, and it is concordance with the expressions of C_0 in the Propositions 4.2 and 4.3.

The tests in Figure 5 have been made for $H = 5.0/6$, $\delta = 5.0/12$, and h corresponds to partitions $\mathcal{T}_h$ with $12, 24, 36, \cdots, 120$ segments on each side of the rectangular domain Ω. For the one-level method, the number of iterations is constant for $h \leq 5/24$, and it is in concordance with C_0 in Proposition 4.2. In the case of the two-level method, the number of iterations is a decreasing function of h for $s = 1.5$ and $s = 2$, and it is also in concordance with C_0 in Proposition 4.3. For $s = 3 > d = 2$, $C_{d,s}(H, h)$ in (10) is equal to 1, and the number of iterations should be bounded. In Figure 5.b, the number of iterations for $s = 3$ becomes constant for values of h less than $5.0/60$.

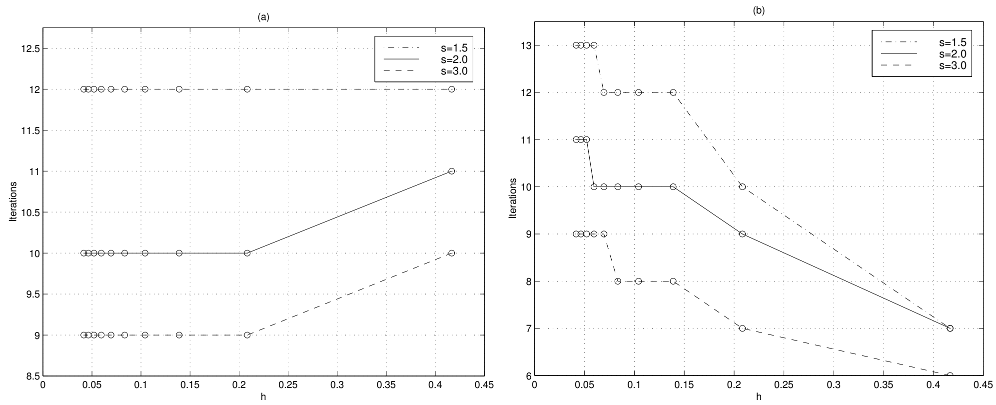

Figure 5: Iterations for H and δ constant, and h variable: (a) one level, (b) two levels.

In the tests in Figure 6 we have taken $h = 5.0/120$, $\delta = 5.0/20$ and $H =$ 5.0/20, 5.0/12, 5.0/10, 5.0/8 and 5.0/6. The number of iterations is also in concordance with the predictions in the previous section.

Finally, we see from our numerical tests that the number of iterations for the two-level method is always much less than that one for the one-level method. For instance, for $H = 5.0/10$, $h = 5.0/60$, $\delta = 5.0/20$, when the number of unknowns is 3481, the number of iterations is:

- for one-level: 23 for $s = 1.5$, 19 for $s = 2.0$, 15 for $s = 3.0$

- for two-levels: 13 for $s = 1.5$, 10 for $s = 2.0$, 9 for $s = 3.0$.

But, the two-level method is more complicated than the one-level method because $K_h \subset V_h$ and we look for corrections in V_H, too. To see the efficiency of the two-level method in comparison with the one-level method, we have compared the computing

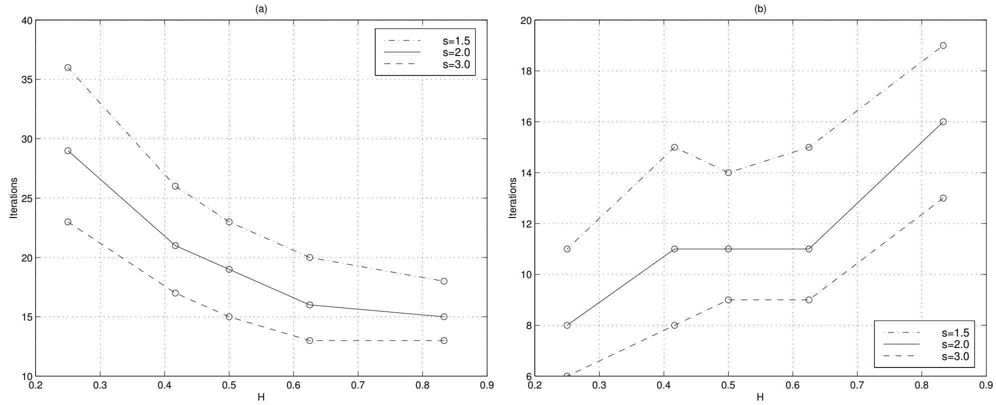

Figure 6: Iterations for h and δ constant, and H variable: (a) one level, (b) two levels.

time on a PC with one processor Pentium III of 600MHz for the above example:

- for one-level: 18min45sec for $s = 1.5$, 6min16sec for $s = 2.0$, 17min8sec for $s = 3.0$

- for two-levels: 13min54sec for $s = 1.5$, 4min43sec for $s = 2.0$, 14min27sec for $s = 3.0$

We see that the computing time of the two-level method is also less than that in the case of the one-level method. We notice that the computing time for $s = 2.0$ is much less than that for $s = 1.5$ or $s = 3.0$. It is natural, because, for $s = 2.0$, we minimize quadratic functionals.

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