

Koszul Cohomology and Algebraic Geometry

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2000 *Mathematics Subject Classification.* Primary 14H51, 14C20, 14F99, 13D02

Contents

Preface	vii
Chapter 1. Basic definitions	1
1.1. The Koszul complex	1
1.2. Definitions in the algebraic context	2
1.3. Minimal resolutions	3
1.4. Definitions in the geometric context	5
1.5. Functorial properties	6
1.6. Notes and comments	10
Chapter 2. Basic results	11
2.1. Kernel bundles	11
2.2. Projections and linear sections	12
2.3. Duality	17
2.4. Koszul cohomology versus usual cohomology	19
2.5. Sheaf regularity.	21
2.6. Vanishing theorems	22
Chapter 3. Syzygy schemes	25
3.1. Basic definitions	25
3.2. Koszul classes of low rank	32
3.3. The $K_{p,1}$ theorem	34
3.4. Rank-2 bundles and Koszul classes	38
3.5. The curve case	41
3.6. Notes and comments	45
Chapter 4. The conjectures of Green and Green–Lazarsfeld	47
4.1. Brill-Noether theory	47
4.2. Numerical invariants of curves	49
4.3. Statement of the conjectures	51
4.4. Generalizations of the Green conjecture.	54
4.5. Notes and comments	57
Chapter 5. Koszul cohomology and the Hilbert scheme	59
5.1. Voisin’s description	59
5.2. Examples	62
5.3. Koszul vanishing via base change	66
Chapter 6. Koszul cohomology of a $K3$ surface	69
6.1. The Serre construction, and vector bundles on $K3$ surfaces	69
6.2. Brill-Noether theory of curves on $K3$ surfaces	70

6.3. Voisin's proof of Green's generic conjecture: even genus	73
6.4. The odd-genus case (outline)	81
6.5. Notes and comments	83
Chapter 7. Specific versions of the syzygy conjectures	85
7.1. The specific Green conjecture	85
7.2. Stable curves with extra-syzygies	87
7.3. Curves with small Brill-Noether loci	89
7.4. Further evidence for the Green-Lazarsfeld conjecture	93
7.5. Exceptional curves	95
7.6. Notes and comments	96
Chapter 8. Applications	99
8.1. Koszul cohomology and Hodge theory	99
8.2. Koszul divisors of small slope on the moduli space.	104
8.3. Slopes of fibered surfaces	109
8.4. Notes and comments	110
Bibliography	113
Index	119

Preface

The systematic use of Koszul cohomology computations in algebraic geometry can be traced back to the foundational paper [Gre84a] by M. Green. In this paper, Green introduced the Koszul cohomology groups $K_{p,q}(X, L)$ associated to a line bundle L on a smooth, projective variety X , and studied the basic properties of these groups. Green noted that a number of classical results concerning the generators and relations of the (saturated) ideal of a projective variety can be rephrased naturally in terms of vanishing theorems for Koszul cohomology, and extended these results using his newly developed techniques. In a remarkable series of papers, Green and Lazarsfeld further pursued this approach. Much of their work in the late 80's centers around the shape of the minimal resolution of the ideal of a projective variety; see [Gre89], [La89] for an overview of the results obtained during this period.

Green and Lazarsfeld also stated two conjectures that relate the Koszul cohomology of algebraic curves to two numerical invariants of the curve, the *Clifford index* and the *gonality*. These conjectures became an important guideline for future research. They were solved in a number of special cases, but the solution of the general problem remained elusive. C. Voisin achieved a major breakthrough by proving the Green conjecture for general curves in [V02] and [V05]. This result soon led to a proof of the conjecture of Green–Lazarsfeld for general curves [AV03], [Ap04].

Since the appearance of Green's paper there has been a growing interaction between Koszul cohomology and algebraic geometry. Green and Voisin applied Koszul cohomology to a number of Hodge-theoretic problems, with remarkable success. This work culminated in Nori's proof of his connectivity theorem [No93]. In recent years, Koszul cohomology has been linked to the geometry of Hilbert schemes (via the geometric description of Koszul cohomology used by Voisin in her work on the Green conjecture) and moduli spaces of curves.

Since there already exists an excellent introduction to the subject [Ei06], this book is devoted to more advanced results. Our main goal was to cover the recent developments in the subject (Voisin's proof of the generic Green conjecture, and subsequent refinements) and to discuss the geometric aspects of the theory, including a number of concrete applications of Koszul cohomology to problems in algebraic geometry. The relationship between Koszul cohomology and minimal resolutions will not be treated at length, although it is important for historical reasons and provides a way to compute Koszul cohomology by computer calculations.

Outline of contents. The first two chapters contain a review of a number of basic definitions and results, which are mainly included to fix the notation and

to obtain a reasonably self-contained presentation. Chapter 3 is devoted to the theory of syzygy schemes. The aim of this theory is to study Koszul cohomology classes in the groups $K_{p,1}(X, L)$ by associating a geometric object to them. This chapter includes a proof of one of the fundamental results in the subject, Green's $K_{p,1}$ -theorem. In Chapter 4 we recall a number of results from Brill–Noether theory that will be needed in the sequel and state the conjectures of Green and Green–Lazarsfeld.

Chapters 5–7 form the heart of the book. Chapter 5 is devoted to Voisin's description of the Koszul cohomology groups $K_{p,q}(X, L)$ in terms of the Hilbert scheme of zero-dimensional subschemes of X . This description yields a method to prove vanishing theorems for Koszul cohomology by base change, which is used in Voisin's proof of the generic Green conjecture. In Chapter 6 we present Voisin's proof for curves of even genus, and outline the main steps of the proof for curves of odd genus; this case is technically more complicated. Chapter 7 contains a number of refinements of Voisin's result; in particular, we have included a proof of the conjectures of Green and Green–Lazarsfeld for the general curve in a given gonality stratum of the moduli space of curves. In the final chapter we discuss geometric applications of Koszul cohomology to Hodge theory and the geometry of the moduli space of curves.

Acknowledgments. MA thanks the Fourier Institute Grenoble, the University of Bayreuth, Laboratoire Painlevé Lille, and the Max Planck Institut für Mathematik Bonn, and JN thanks the Fourier Institute Grenoble, the University of Bayreuth, and the Simion Stoilow Institute Bucharest for hospitality during the preparation of this work. MA was partially supported by an ANCS contract. The authors are members of the L.E.A. “MathMode”.

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