Koszul Cohomology and Algebraic Geometry

Marian Aprodu

Jan Nagel

Institute of Mathematics "Simion Stoilow" of the Romanian Academy, P.O. Box 1-764, 014700 Bucharest, ROMANIA & Şcoala Normală Superioară Bucureşti, 21, Calea Griviţei, 010702 Bucharest, Sector 1 ROMANIA

E-mail address: Marian.Aprodu@imar.ro

Institut de Mathmatiques de Bourgogne, CNRS UMR 5584, Universit de Bourgogne, U.F.R. Sciences et Techniques B.P. 47870 F-21078 Dijon Cedex FRANCE

E-mail address: Johannes.Nagel@u-bourgogne.fr

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Contents

Preface

The systematic use of Koszul cohomology computations in algebraic geometry can be traced back to the foundational paper [**Gre84a**] by M. Green. In this paper, Green introduced the Koszul cohomology groups $K_{p,q}(X, L)$ associated to a line bundle L on a smooth, projective variety X, and studied the basic properties of these groups. Green noted that a number of classical results concerning the generators and relations of the (saturated) ideal of a projective variety can be rephrased naturally in terms of vanishing theorems for Koszul cohomology, and extended these results using his newly developed techniques. In a remarkable series of papers, Green and Lazarsfeld further pursued this approach. Much of their work in the late 80's centers around the shape of the minimal resolution of the ideal of a projective variety; see [**Gre89**], [**La89**] for an overview of the results obtained during this period.

Green and Lazarsfeld also stated two conjectures that relate the Koszul cohomology of algebraic curves to two numerical invariants of the curve, the *Clifford index* and the *gonality*. These conjectures became an important guideline for future research. They were solved in a number of special cases, but the solution of the general problem remained elusive. C. Voisin achieved a major breakthrough by proving the Green conjecture for general curves in [**V02**] and [**V05**]. This result soon led to a proof of the conjecture of Green–Lazarsfeld for general curves [**AV03**], [**Ap04**].

Since the appearance of Green's paper there has been a growing interaction between Koszul cohomology and algebraic geometry. Green and Voisin applied Koszul cohomology to a number of Hodge–theoretic problems, with remarkable success. This work culminated in Nori's proof of his connectivity theorem [**No93**]. In recent years, Koszul cohomology has been linked to the geometry of Hilbert schemes (via the geometric description of Koszul cohomology used by Voisin in her work on the Green conjecture) and moduli spaces of curves.

Since there already exists an excellent introduction to the subject [**Ei06**], this book is devoted to more advanced results. Our main goal was to cover the recent developments in the subject (Voisin's proof of the generic Green conjecture, and subsequent refinements) and to discuss the geometric aspects of the theory, including a number of concrete applications of Koszul cohomology to problems in algebraic geometry. The relationship between Koszul cohomology and minimal resolutions will not be treated at length, although it is important for historical reasons and provides a way to compute Koszul cohomology by computer calculations.

Outline of contents. The first two chapters contain a review of a number of basic definitions and results, which are mainly included to fix the notation and to obtain a reasonably self–contained presentation. Chapter 3 is devoted to the theory of syzygy schemes. The aim of this theory is to study Koszul cohomology classes in the groups $K_{p,1}(X, L)$ by associating a geometric object to them. This chapter includes a proof of one of the fundamental results in the subject, Green's $K_{p,1}$–theorem. In Chapter 4 we recall a number of results from Brill–Noether theory that will be needed in the sequel and state the conjectures of Green and Green–Lazarsfeld.

Chapters 5–7 form the heart of the book. Chapter 5 is devoted to Voisin's description of the Koszul cohomology groups $K_{p,q}(X, L)$ in terms of the Hilbert scheme of zero–dimensional subschemes of X. This description yields a method to prove vanishing theorems for Koszul cohomology by base change, which is used in Voisin's proof of the generic Green conjecture. In Chapter 6 we present Voisin's proof for curves of even genus, and outline the main steps of the proof for curves of odd genus; this case is technically more complicated. Chapter 7 contains a number of refinements of Voisin's result; in particular, we have included a proof of the conjectures of Green and Green–Lazarsfeld for the general curve in a given gonality stratum of the moduli space of curves. In the final chapter we discuss geometric applications of Koszul cohomology to Hodge theory and the geometry of the moduli space of curves.

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Marian Aprodu and Jan Nagel