PENCILS ON EXCEPTIONAL CURVES ON A $K3$ SURFACE (PRELIMINARY DRAFT)

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Abstract. We compute the dimensions of Brill-Noether loci of pencils on exceptional curves on a $K3$ surface.

This paper is a continuation of [AP08]. We use the notation of [AP08], and [A05].

Theorem 0.1. *Let S be a $K3$ surface with $\mathrm{Pic}(S) \cong \mathbb{Z} \cdot H \oplus \mathbb{Z} \cdot \ell$, where H is very ample, ℓ is a smooth rational curve, $H^2 = 2r - 2 \geq 4$, and $H \cdot \ell = 1$. Then for any $0 \leq n \leq 2r-2$ and any smooth curve $C \in |2H + \ell|$ we have $\dim(W^1_{2r+n}(C)) \leq max(n+2, 2n)$.*

Recall that smooth connected curves in the linear system $|2H + \ell|$ always exist [ELMS89], and, again by [ELMS89, Theorem, p. 176] if $C \in |2H + \ell|$ is smooth then it has the following invariants:

(1) $g_C = 4r - 2$;
(2) $\mathrm{gon}(C) = 2r$ (and this is the maximal gonality for the given genus);
(3) $\mathrm{Cliff}(C) = 2r - 3$ (one less than the maximal Clifford index);
(4) $\dim\left(W^1_{2r}(C)\right) = 1$ (a generic curve of even genus has finitely many minimal pencils).

Consider the variety

$$\mathcal{W}^1_{2r+1}(|2H + \ell|_s) \xrightarrow{\pi_S} |2H + \ell|_s,$$

whose fibre over $C \in |2H + \ell|_s$ coincides with $W^1_{2r+1}(C)$ scheme-theoretically [AC81]: the subscript $\cdot_s$ stands here for open subset of $|2H + \ell|$ parametrizing smooth curves.

In order to prove Theorem 0.1 we analyze the birational geometry of all the possible components $\mathcal{W}$ of the variety $\mathcal{W}^1_{2r+1}(|2H + \ell|_s)$ that dominate the linear system $|2H + \ell|$, and prove that the relative dimension cannot exceed two.

Proof of Theorem 0.1. First, remark that

$$\rho(g_C, 1, 2r + n) = 2n,$$

and this is the expected dimension of the Brill-Noether locus in this case. We follow the lines of [AP08]. Let $\mathcal{W}$ be a component of the variety $\mathcal{W}^1_{2r+1}(|2H + \ell|_s)$ dominating the linear system $|2H + \ell|$. By [ACGH85, Lemma 3.5, p. 182] a generic pair (C, A) parametrized by $\mathcal{W}$ will verify $h^0(C, A) = 2$ (as no component of W^1_{2r+n} is entirely contained in W^2_{2r+n}). As in [AP08], we distinguish two cases:

(1) either a generic member $(C, A) \in \mathcal{W}$ is such that A is base-point-free and $h^0(C, A) = 2$, or
(2) a generic member $(C, A) \in \mathcal{W}$ is such that A has base-points and $h^0(C, A) = 2$.

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In the second case, we apply an inductive argument: adding one base-point will increase the dimension of (the corresponding component of) the Brill-Noether locus by one.

In the first case, we use the Lazarsfeld-Mukai bundles [GL87], [La86], [La89], [Mu89], similarly to [AP08].

Recall that if $(C, A) \in \mathcal{W}$ is a generic member, then we have a rank-2 bundle given by an extension

$$0 \to H^0(C, A)^* \otimes \mathcal{O}_S \to E \to K_C(-A) \to 0,$$

with the following invariants:

(1) $\det(E) = \mathcal{O}_S(C) = \mathcal{O}_S(2H + \ell)$;
(2) $c_2(E) = \deg(A) = 2r + n$;
(3) $h^0(S, E) = 2h^0(C, A) - \deg(A) - 1 + g_C$ (and it equals $2r + 1 - n$);
(4) E is globally generated (outside the base locus of $|A|$ which is empty in our case);
(5) $h^1(S, E) = h^2(S, E) = 0$.

There are two subcases:

(a1) the generic (C, A) corresponds to a simple bundle $E(C, A)$.
(a2) the generic (C, A) corresponds to a non-simple bundle $E = E(C, A)$.

In case (a1) [AP08, Corollary 3.3] shows that the relative dimension of $\mathcal{W} \to |2H + \ell|_s$ is equal to $\rho(g_C, 1, 2r + n) = 2n$ and we are done.

In case (a2) by the the Donagi-Morrison description [DM89] (see also [CP95]) we know that a non-simple E is given by an extension

$$0 \to M \to E \to N \otimes I_\xi \to 0 \tag{0.1}$$

with

(i) $h^0(S, M) \geq 2$, $h^0(S, N) \geq 2$;
(ii) N is base-point-free;
(iii) if $h^0(S, M \otimes N^\vee) = 0$ then $\mathrm{supp}(\xi) = \emptyset$ and the sequence is split.

Write $M = aH + b\ell$ and $N = a'H + b'\ell$. From the equations

$$\det(E) = M \otimes N = \mathcal{O}_S(2H + \ell) \tag{0.2}$$

and

$$2r + n = c_2(E) = M \cdot N + \lg(\xi), \tag{0.3}$$

using the conditions (i) and (ii) above we deduce that

$$M = \mathcal{O}_S(H + \ell), \; N = \mathcal{O}_S(H) \text{ and } \lg(\xi) = n + 1. \tag{0.4}$$

Next, as in [AP08], we use (0.4) to obtain a concrete description of the parameter space $\mathcal{P}$ of non-simple Lazarsfeld-Mukai bundles.

Lemma 0.2. *For any non simple bundle Lazarsfeld-Mukai bundle E as above, the extension (0.1) is uniquely determined.*

Proof. The proof of Lemma 3.6 [AP08] also goes through, as we have a short exact sequence

$$0 \to \mathcal{O}_S \to E(-H - \ell) \to \mathcal{O}_S(-\ell) \otimes I_\xi \to 0$$

and $h^0(S, \mathcal{O}_S(-\ell) \otimes I_\xi) = 0$. □

Lemma 0.3. *For any zero-dimensional subscheme ξ of length two, we have*

$$\dim \left(\mathrm{Ext}^1(\mathcal{O}_S(H) \otimes I_\xi, \mathcal{O}_S(H + \ell))\right) = n + 1.$$

Proof. We compute

$$\dim\left(\mathrm{Ext}^1(\mathcal{O}_S(H)\otimes I_\xi, \mathcal{O}_S(H+\ell))\right) = h^1(S, M^\vee \otimes N \otimes I_\xi) = h^1(S, I_\xi(-\ell))$$

using the exact sequence

$$0 \to I_\xi(-\ell) \to \mathcal{O}_S(-\ell) \to \mathcal{O}_\xi \to 0.$$

Since $h^0(\mathcal{O}_\xi) = n+1$, $h^0(S, \mathcal{O}_S(-\ell)) = 0$ and $h^1(S, \mathcal{O}_S(-\ell)) = 0$ (from the exact sequence $0 \to \mathcal{O}_S(-\ell) \to \mathcal{O}_S \to \mathcal{O}_\ell \to 0$, we obtain the exact sequence $0 \to H^0(\mathcal{O}_S) \to H^0(\mathcal{O}_\ell) \to H^1(\mathcal{O}_S(-\ell)) \to H^1(\mathcal{O}_S) = 0$), it follows that $h^1(S, I_\xi(-\ell)) = n+1$. □

Lemmas 0.2 and 0.3 eventually show the following statement

Proposition 0.4. *The parameter space $\mathcal{P}$ of non-simple Lazarsfeld-Mukai bundles is birational to a $\mathbb{P}^n$-bundle over $S^{[n+1]}$, hence it is of dimension $3n+2$.*

As in [AP08], consider the fibre bundle $\mathcal{G} \xrightarrow{p} \mathcal{P}$ whose fibre over a point $[E] \in \mathcal{P}$ is the Grassmannian:

$$p^{-1}([E]) = G(2, H^0(S, E)).$$

The dimension of $\mathcal{G}$ equals

$$\dim(\mathcal{G}) = \dim(\mathcal{P}) + \dim(G(2, H^0(S,E))) = 3n+2+2(2r-1-n) = 4r+n.$$

The last step in the proof of Theorem 0.1 is represented by the following

Proposition 0.5. *If $\mathcal{W}$ is a dominating component whose generic element (C, A) corresponds to a non-simple Lazarsfeld-Mukai bundle $E = E(C, A)$, then the generic fibres of the morphism $\mathcal{W} \to |2H+\ell|_s$ are $(n+2)$-dimensional.*

For the proof of Proposition 0.5, observe that Proposition 3.9 of [AP08] goes through, hence $\mathcal{W}$ is birational to $\mathcal{G}$, and $\dim(\mathcal{W}) = 4r+n$ for the subcase $a2$). Since $\dim|2H+\ell| = g_C = 4r-2$, we have

$$\dim(\mathcal{W}) - \dim|2H+\ell| = n+2.$$

□

Note that if $n = 0$, there are no components corresponding to non-simple vector bundles, and if $n = 1$, then these components are three-dimensional. By excess linear series, in the case $n = 1$ *any* smooth curve in $C \in |2H+\ell|$ will have $\dim(W^1_{2r+1}(C)) = 3$.

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