MULTIPARAMETER SCHEMES FOR WAVE EQUATIONS

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Abstract. Multiparameter extensions (MP) of (linear and nonlinear) descent methods have been proposed for the solution of finite dimensional time independent problems; these new methods are based on a different treatment of several blocks of components of the solution, basically via the substitution of a scalar relaxation by a (suitable)matricial relaxation. Similarly, the Nonlinear Galerkin Method (NLG), that stems from the dynamical system theory, propose to apply distinct temporal integration schemes to different sets of data scales when solving dissipative PDEs. In this paper, the algebraic similarity of Richardson iteration and Forward-Euler time integration is extended to new grounds through the expansion of the realm of MP methods to the field of the numerical integration of non dissipative PDEs. The separation of the structures is realized by the utilization of hierarchical preconditioners in finite differences, which are conjugated to a MP temporal integration steeming from NLG theory.

A first attempt consists in combining two techniques

- preparation of the initial data by the two-grid preconditioner used by Ignat and Zuazua in the context of the control of the wave equationL
- multilevel stabilization by chehab and Costa

for non dissipative equations. We rst start with the damped wave or Dalemberts equation on the unit interval

$$\left\{\begin{array}{lr} \dfrac{\partial^2 u}{\partial t^2}+k\dfrac{\partial u}{\partial t}-\dfrac{\partial^2 u}{\partial x^2}=0, & x\in(0,1), t>0\\ u(0,t)=u(1,t)=0, & t>0\\ u(x,0)=f(x), u_t(x,0)=g(x), & x\in(0,1). \end{array}\right. \tag{0.1}$$

Here $k\geq 0$ is a damping factor.

1. Stabilization

We will use a block decomposition of the discretiazton matri of the negative Laplacian finite difference matrix A written in the incremental unknown basis. We set

$$S^{-1}AS=\begin{pmatrix} A_{11} & A_{12}\\ A_{21} & A_{22}\end{pmatrix}.$$

The spatial discretization of the wave equation leads to the differential system

$$\begin{cases} \dfrac{\partial^2 U}{\partial t^2} + k\dfrac{\partial U}{\partial t} + AU = 0 \\ U(0) = f, U_t(0) = g. \end{cases} \tag{1.2}$$

We now introduce the separation of scales with the transfer matrix S: $S\widehat{U} = U$ with $\widehat{U} = (Y, Z)^t$. We have

$$\begin{cases} \dfrac{\partial^2 \widehat{U}}{\partial t^2} + k\dfrac{\partial U}{\partial t} + S^{-1}AS\widehat{U} = 0 \\ \widehat{U}(0) = S^{-1}f, \widehat{U}_t(0) = S^{-1}g. \end{cases} \tag{1.3}$$

More precisely, this reads in terms of Y and Z

$$\begin{cases} \dfrac{\partial^2 \widehat{Y}}{\partial t^2} + k\dfrac{\partial Y}{\partial t} + A_{11}Y + A_{12}Z = 0 \\ \dfrac{\partial^2 \widehat{Z}}{\partial t^2} + k\dfrac{\partial Z}{\partial t} + A_{21}Y + A_{22}Z = 0 \\ + \text{ initial conditions in } Y \text{ and } Z. \end{cases} \tag{1.4}$$

At this point we will consider two different time marching schemes

- The original coupled Crank-Nicolson scheme

$$\frac{Y^{n+1} - 2Y^n + Y^{n-1}}{(\Delta t)^2} + k\frac{Y^{n+1} - Y^{n-1}}{2\Delta t} + \frac{1}{2}\Big(A_{11}(Y^{n+1} + Y^{n-1}) + A_{12}(Z^{n+1} + Z^{n-1})\Big) = 0$$

$$\frac{Z^{n+1} - 2Z^n + Z^{n-1}}{(\Delta t)^2} + k\frac{Z^{n+1} - Z^{n-1}}{2\Delta t} + \frac{1}{2}\Big(A_{21}(Y^{n+1} + Y^{n-1}) + A_{22}(Z^{n+1} + Z^{n-1})\Big) = 0$$

- Explicit in Z in the Y equation

$$\frac{Y^{n+1} - 2Y^n + Y^{n-1}}{(\Delta t)^2} + k\frac{Y^{n+1} - Y^{n-1}}{2\Delta t} + \frac{1}{2}\Big(A_{11}(Y^{n+1} + Y^{n-1}) + A_{12}(Z^{n+1} + Z^{n-1})\Big) = 0$$

$$\frac{Z^{n+1} - 2Z^n + Z^{n-1}}{(\Delta t)^2} + k\frac{Z^{n+1} - Z^{n-1}}{2\Delta t} + \frac{1}{2}\Big(A_{21}(Y^{n+1} + Y^{n-1}) + A_{22}(Z^{n+1} + Z^{n-1})\Big) = 0$$

The advantage in replacing $(Z^{n+1} + Z^{n-1})$ by $2Z^n$ is that the time marching scheme reduces to a block lower triangular system to be solved at each step. THis can have consequences on the stability of the method (lack of implicity) and necessitates to stabilize the eqution in Z.

- Explicit in Z in the Y equation + stabilization

$$\frac{Y^{n+1}-2Y^{n}+Y^{n-1}}{(\Delta t)^2}+k\frac{Y^{n+1}-Y^{n-1}}{2\Delta t}+\frac{1}{2}\Big(A_{11}(Y^{n+1}+Y^{n-1})+A_{12}(Z^{n+1}+Z^{n-1})\Big)=0$$

$$\frac{Z^{n+1}-2Z^{n}+Z^{n-1}}{(\Delta t)^2}+k\frac{Z^{n+1}-Z^{n-1}}{2\Delta t}+\beta\Delta t\frac{Z^{n+1}-Z^{n-1}}{2\Delta t}+ \\ +\frac{1}{2}\Big(A_{21}(Y^{n+1}+Y^{n-1})+A_{22}(Z^{n+1}+Z^{n-1})\Big)=0$$

For each scheme we have done some numerical simulation in order to obtain some information about the stabilization parameter which has to be tuned. Also there are some range of discretization steps which guarantee the stability of the above schemes.

It remains to obtain a rigorous proof of the stabilization of the above schemes.

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