Spectral Theory for Schrödinger operator with magnetic field and analysis of the third critical field in superconductivity

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Main goals

Using recent results by the authors on the spectral asymptotics of the Neumann Laplacian with magnetic field, we give precise estimates on the critical field, H_{C_3}, describing the appearance of superconductivity in superconductors of type II. Furthermore, we prove that the local and global definitions of this field coincide. Near H_{C_3} only a small part, near the boundary points where the curvature is maximal, of the sample carries superconductivity. We give precise estimates on the size of this zone and decay estimates in both the normal (to the boundary) and parallel variables. We discuss also the three dimensional case.

Ginzburg-Landau functional

The Ginzburg-Landau functional is given by

$$\mathcal{E}_{\kappa,H}[\psi,\vec{A}] = \int_\Omega \left\{ |\nabla_{\kappa H \vec{A}}\psi|^2 - \kappa^2|\psi|^2 + \frac{\kappa^2}{2}|\psi|^4 + \kappa^2 H^2 |\operatorname{curl} \vec{A} - 1|^2 \right\} dx \,,$$

with Ω simply connected, $(\psi,\vec{A}) \in W^{1,2}(\Omega;\mathbb{C}) \times W^{1,2}(\Omega;\mathbb{R}^2)$ and where $\nabla_{\vec{A}} = (\nabla - i\vec{A})$.

We fix the choice of gauge by imposing that

$$\operatorname{div} \vec{A} = 0 \quad \text{in } \Omega\,,\ \vec{A}\cdot\nu = 0 \quad \text{on } \partial\Omega\,.$$

The Ginzburg-Landau functional in three space dimensions is given by

$$\mathcal{E}[\psi, \vec{A}] = \int_\Omega \left\{ |\nabla_{\kappa H \vec{A}} \psi|^2 - \kappa^2 |\psi|^2 + \frac{\kappa^2}{2} |\psi|^4 \right\} dx + \kappa^2 H^2 \int_{\mathbb{R}^3} |\text{ curl } \vec{A} - \beta|^2 \, dx \ , \quad (1)$$

Here $\vec{A}$ is such that

$$\vec{A} - \vec{F} \in \dot{H}^1(\mathbb{R}^3) \ ,$$

where $\dot{H}^1(\mathbb{R}^3)$ is the closure of $C_0^\infty(\mathbb{R}^3)$ under the norm

$$f \mapsto \|f\|_{\dot{H}^1} = \|\nabla f\|_{L^2} \ .$$

and

$$\text{curl } \vec{F} = \beta \ ,$$

with β constant magnetic field of intensity one along the z axis.

We come back to dimension two.

Minimizers $(\psi, \vec{A})$ of the functional satisfy the Ginzburg-Landau equations,

$$\left.\begin{array}{r} -\nabla^2_{\kappa H\vec{A}}\psi = \kappa^2(1-|\psi|^2)\psi \\ \text{curl}\,^2\vec{A} = -\frac{i}{2\kappa H}(\overline{\psi}\nabla\psi - \psi\nabla\overline{\psi}) - |\psi|^2\vec{A} \end{array}\right\} \quad \text{in} \quad \Omega\,; \tag{2a}$$

$$\left.\begin{array}{r} (\nabla_{\kappa H\vec{A}}\psi)\cdot\nu = 0 \\ \text{curl}\,\vec{A} - 1 = 0 \end{array}\right\} \quad \text{on} \quad \partial\Omega\,. \tag{2b}$$

Here $\text{curl}\,(A_1, A_2) = \partial_{x_1}A_2 - \partial_{x_2}A_1$,

$$\text{curl}\,^2\vec{A} = (\partial_{x_2}(\,\text{curl}\,\vec{A}), -\partial_{x_1}(\,\text{curl}\,\vec{A}))\,.$$

Here $\vec{F}$ denotes the vector potential generating the constant exterior magnetic field

$$\left.\begin{aligned} \text{div } \vec{F} = 0 \\ \text{curl } \vec{F} = 1 \end{aligned}\right\} \quad \text{in } \Omega \, ,$$

and the boundary condition

$$\vec{F} \cdot \nu = 0 \quad \text{on } \partial\Omega \, .$$

In the three dimensional case, minimizers of the functional $\mathcal{E}$, have to satisfy the Euler-Lagrange equations:

$$
\begin{aligned}
&-\nabla^2_{\kappa H \vec{A}}\psi = \kappa^2(1-|\psi|^2)\psi \\
&\qquad\qquad \text{in} \quad \Omega\,, \\
&\text{curl}\ ^2\vec{A} = \Big\{ -\tfrac{i}{2\kappa H}(\overline{\psi}\nabla\psi - \psi\nabla\overline{\psi}) - |\psi|^2\vec{A}\Big\} 1_\Omega(x) \\
&\qquad\qquad \text{in} \quad \mathbb{R}^3\,, \\
&(\nabla_{\kappa H\vec{A}}\psi)\cdot\nu = 0 \\
&\qquad\qquad \text{on} \quad \partial\Omega\,.
\end{aligned}
\tag{3}
$$

Note that it does not give curl $A = \beta$ on $\partial\Omega$

Terminology for the minimizers

The pair $(0, \vec{F})$ is called the Normal State.

A minimizer (ψ, A) for which ψ never vanishes will be called SuperConducting State.

In the other cases, one will speak about Mixed State.

The general question is to determine the topology of the sets of (κ, H) corresponding to minimizers belonging to each of these three situations.

Existence of the third critical field $\underline{H}_{C3}(\kappa)$

It is known that, for given values of the parameters κ, H, the functional $\mathcal{E}$ has minimizers.

However, after some analysis of the functional, one finds (see [GiPh]) that, for given κ, there exists $H(\kappa)$ such that if $H > H(\kappa)$ then $(0, \vec{F})$ is the unique minimizer of $\mathcal{E}_{\kappa,H}$ (up to change of gauge).

Following Lu and Pan [LuPa1], we define

$$\underline{H}_{C_3}(\kappa) = \inf\{H > 0 \ : \ (0, \vec{F}) \text{ minimizer of } \mathcal{E}_{\kappa,H}\} \ .$$

A central question in the mathematical treatment of Type II superconductors is to establish the asymptotic behavior of $\underline{H}_{C_3}(\kappa)$ for large κ.

We will also discuss the relevance of this definition and describe how $\underline{H}_{C_3}(\kappa)$ can be determined by the study of a linear problem.

Our first result [FoHe] is the following strengthening of a result in [HePa].

Theorem A

Suppose Ω is a bounded simply-connected domain in $\mathbb{R}^2$ with smooth boundary. Let $k_{\max}$ be the maximal curvature of $\partial\Omega$. Then

$$\underline{H}_{C_3}(\kappa) = \frac{\kappa}{\Theta_0} + \frac{C_1}{\Theta_0^{\frac{3}{2}}} k_{\max} + \mathcal{O}(\kappa^{-\frac{1}{2}}) \,, \qquad (4)$$

where C_1, Θ_0 are universal constants.

Remark

The constants Θ_0, C_1 are defined in terms of auxiliary spectral problems.

In the case of dimension 3, under certain geometric assumptions on Ω. One deduces of the result of Helffer-Morame the following asymptotics for $H_{C_3}(\kappa)$:

Theorem A'

Suppose Ω is a smooth, bounded, simply connected domain in $\mathbb{R}^3$ satisfying "generic conditions". Then one finds

$$\underline{H}_{C_3}(\kappa) - (\frac{\kappa}{\Theta_0} - \hat{\gamma}_0 \Theta_0^{-2/3} \kappa^{1/3}) = o(\kappa^{1/3}) \,, \quad (5)$$

where $H_{C_3}(\kappa)$ denotes any of the critical fields defined above and $\hat{\gamma}_0$ is a geometrically defined (and depending on the magnetic field), positive constant.

Localization at the boundary

From the work of Helffer-Morame [HeMo2]

——(improving Del Pino-Fellmer-Sternberg and Lu-Pan) (see also Helffer-Pan [HePa] for the non-linear case)——

we know that, when H is sufficiently close to $\underline{H}_{C3}(\kappa)$, (non-trivial) minimizers of the Ginzburg-Landau functional are exponentially localized to a region near the boundary.

This is called Surface Superconductivity.

The proof is based on semi-classical Agmon estimates, but the "Agmon distance" has to be replaced by the distance to the boundary.

From semi-classical Agmon estimates to weak localization

Note that the Agmon estimates give first, for some $\alpha > 0$,

$$|| \exp \alpha \, \kappa \, d(x, \partial\Omega) \psi ||_2^2 \leq C \, ||\psi||_2^2 \, ,$$

which imply

$$||\psi||_2^2 \leq M \int_{d(x,\partial\Omega)\leq \frac{M}{\kappa}} |\psi(x)|^2 dx \, ,$$

We will need the following weak form of this localization :

$$||\psi||_{L^2(\Omega)} \leq C \, \kappa^{-\frac{1}{4}} \, ||\psi||_{L^4(\Omega)} \, , \tag{6}$$

which is true for κ large enough.

The proof is similar in Dimension 3.

Localization at the points of maximal curvature

The statement in dimension 2 is that, when H is rather close to the third critical field, the minimizers are also localized in the tangential variable to a small zone around the points of maximal curvature.

This leads in particular to the better

$$||\psi||_{L^2(\Omega)} \leq C\kappa^{-\frac{3}{8}}||\psi||_{L^4(\Omega)} \,, \tag{7}$$

One can hope similar results in dimension 3 :

Localization near the curve where the external magnetic field vector is tangent to the boundary (see Pan and Helffer-Morame)

Inside this curve, localization due to some “magnetic curvature”.

Discussion of critical fields

Actually, we should define more than one critical field, instead of just $\underline{H}_{C_3}$.

We should also a priori define an upper third critical field, by

$$\begin{aligned}&\overline{H}_{C_3}(\kappa)\\&\quad = \inf\{H > 0 \ : \ \forall H' > H\,, (0, \vec{F})\\&\qquad\qquad\qquad \text{unique minimizer of } \mathcal{E}_{\kappa,H'}\}\ ,\end{aligned}$$

Of course we have

$$\underline{H}_{C_3}(\kappa) \leq \overline{H}_{C_3}(\kappa)\ .$$

Note that one can prove that the asymptotics given before is valid for both fields.

The Schrödinger operator with magnetic field

Let, for $B \in \mathbb{R}_+$, the magnetic Neumann Laplacian $\mathcal{H}(B)$ be the self-adjoint operator (with Neumann boundary conditions) associated to the quadratic form

$$W^{1,2}(\Omega) \ni u \mapsto Q_B(u) := \int_\Omega |(-i\nabla - B\vec{F})u|^2 \, dx \,,$$

We define $\lambda_1(B)$ as the lowest eigenvalue of $\mathcal{H}(B)$.

The local upper critical fields can now be defined :

$$\overline{H}_{C_3}^{\mathrm{loc}}(\kappa) = \inf\{H > 0 \;:\; \forall H' > H, \lambda_1(\kappa H') \geq \kappa^2\}\;,$$

and

$$\underline{H}_{C_3}^{\mathrm{loc}}(\kappa) = \inf\{H > 0 \;:\; \lambda_1(\kappa H) \geq \kappa^2\}\;.$$

The coincidence between $\overline{H}_{C_3}^{\mathrm{loc}}(\kappa)$ and $\underline{H}_{C_3}^{\mathrm{loc}}(\kappa)$ is immediately related to lack of strict monotonicity of $B \mapsto \lambda_1(B)$.

These local critical fields appear when analyzing the (local) stability of the normal solution $(0, \vec{F})$.

Next goal is to compare the various fields ([FoHe]).

Comparison Theorem C

Let Ω be a bounded simply-connected domain in $\mathbb{R}^2$ with smooth boundary and let $\kappa > 0$, then the following general relations hold

$$\overline{H}_{C_3}(\kappa) \geq \overline{H}_{C_3}^{\mathrm{loc}}(\kappa) \,,$$

and

$$\underline{H}_{C_3}(\kappa) \geq \underline{H}_{C_3}^{\mathrm{loc}}(\kappa) \,.$$

EASY and GENERAL.

Next theorem is new and more delicate !

Theorem D

Let Ω be a bounded simply-connected domain in $\mathbb{R}^2$ with smooth boundary. Then $\exists\ \kappa_0 > 0$ such that, for $\kappa > \kappa_0$, we have

$$\overline{H}_{C_3}(\kappa) = \overline{H}^{\mathrm{loc}}_{C_3}(\kappa)\ ,$$

and

$$\underline{H}_{C_3}(\kappa) = \underline{H}^{\mathrm{loc}}_{C_3}(\kappa)\ ,$$

So if the monotonicity of $\lambda_1(B)$ for B large is established, or if we prove by other means that $\underline{H}_{C_3}^{\mathrm{loc}}(\kappa) = \overline{H}_{C_3}^{\mathrm{loc}}(\kappa)$ this gives immediately the coincidence of the four fields !!

In the 2 dimensional case, this monotonicity has been shown in great generality under generic assumptions by Fournais-Helffer, who get in addition a complete asymptotic expansion.

Unfortunately no such complete expansion is available in the case of dimension 3. The best result is the result which is deduced of the result of Helffer-Morame on the groundstate of the Neumann realization of the Schrödinger operator with magnetic field.

Around the proof of Theorem D

We treat the case of Dimension 2, but the proof can be modified for getting the case of Dimension 3.

The crucial point leads in the following argument.

If, for some H, there is a non trivial minimizer (ψ, A), i.e. satisfying

$$\mathcal{E}(\psi, \vec{A}) \leq 0 \; .$$

then

$$0 < \Delta := \kappa^2 ||\psi||_2^2 - Q_{\kappa H \vec{A}}[\psi] = \kappa^2 ||\psi||_4^4 \; ,$$

where $Q_{\kappa H \vec{A}}[\psi]$ is the energy of ψ.

The last equality is a consequence of the first G-L equation.

Combining with (6), this gives

$$||\psi||_2 \leq C\kappa^{-\frac{3}{4}}\Delta^{\frac{1}{4}} \,.$$

By comparison of the quadratic forms Q respectively associated with $\vec{A}$ et $\vec{F}$, we get, with $\vec{a} = \vec{A} - \vec{F}$:

$$\Delta \leq \left[\kappa^2 - (1-\rho)\lambda_1(\kappa H)\right] \|\psi\|_2^2 + \rho^{-1}(\kappa H)^2 \int_\Omega |\vec{a}\psi|^2 \, dx \,, \tag{8}$$

for all $\rho > 0$.

Note that by the regularity of the system Curl-Div, combined with the Sobolev's injection theorem, we get

$$\|\vec{a}\|_4 \leq C_1 \|\vec{a}\|_{W^{1,2}} \leq C_2 \| \text{ curl } \vec{a}\|_2 \,.$$

Now Δ is also controlling $\|$ curl $\vec{a}\|_2^2$, so we get :

$$(\kappa H)^2 \|\vec{a}\|_4^2 \leq C\, \Delta \;.$$

Combining all these inequalities leads to :

$$\begin{aligned} 0 < \Delta \leq & \\ & \leq \left[\kappa^2 - (1-\rho)\lambda_1(\kappa H)\right] \|\psi\|_2^2 + \rho^{-1}(\kappa H)^2 \|\vec{a}\|_4^2 \|\psi\|_4^2 \\ & \leq \left[\kappa^2 - \lambda_1(\kappa H)\right] \|\psi\|_2^2 \\ & \qquad + C\rho\lambda_1(\kappa H)\Delta^{\frac{1}{2}}\kappa^{-\frac{3}{2}} + C\rho^{-1}\Delta^{\frac{3}{2}}\kappa^{-1} \;. \end{aligned}$$

Chosing $\rho = \sqrt{\Delta}\kappa^{-\frac{3}{4}}$, and using the rough upper bound $\lambda_1(\kappa H) < C\kappa^2$, we find

$$0 < \Delta \leq \left[\kappa^2 - \lambda_1(\kappa H)\right] \|\psi\|_2^2 + C\Delta\kappa^{-\frac{1}{4}} \;.$$

This shows finally, for κ large enough independently of H sufficiently close to "any" third critical field (they have the same asymptoics)

$$0 < \Delta \leq \tilde{C}\big[\kappa^2 - \lambda_1(\kappa H)\big]\,\|\psi\|_2^2\ ,$$

so in particular

$$\kappa^2 - \lambda_1(\kappa H) > 0\ .$$

Coming back to the definitions this leads to the statement.

Intensity of the onset

We know that if (ψ, A) is a minimizer, then

$$||\psi||_\infty \leq 1 \; .$$

This is far to explain the mechanism of onset of superconductivity.

With the technique given before, we obtain for $H < H_{C_3}(\kappa)$ with H sufficiently close to $H_{C_3}(\kappa)$

$$||\psi||_\infty \leq C_\epsilon (H_{C_3}(\kappa) - H)^{\frac{1}{2}} \kappa^{\frac{3}{2}+\epsilon} \; , \; \forall \epsilon > 0 \; .$$

This estimate is probably not optimal.

New results on Diamagnetism

Theorem E
If Ω has regular boundary then $B \mapsto \lambda_1(B)$ is monotonically increasing for B large.

The case of the disk was known and has to be done independently.

We now assume that Ω is NOT a disk.

For a suitable choice of ground state eigenfunction $\psi_1(B)$ of $\tilde{\mathcal{H}}(B)$ we can therefore calculate for $\beta > 0$,

$$
\begin{aligned}
\lambda'_{1,+}(B) &= \langle \psi_1(B); \big(\mathbf{A}' \cdot p_{B\mathbf{A}'} + p_{B\mathbf{A}'} \cdot \mathbf{A}'\big)\psi_1(B)\rangle \\
&= \langle \psi_1(B); \{\frac{\tilde{\mathcal{H}}(B+\beta) - \tilde{\mathcal{H}}(B)}{\beta} - \beta(\mathbf{A}')^2\}\psi_1(B)\rangle \\
&\geq \frac{\lambda_1(B+\beta) - \lambda_1(B)}{\beta} - \beta \int_\Omega (\mathbf{A}')^2 |\psi_1(B)|^2 \, dx.
\end{aligned} \tag{9}
$$

Here $\mathbf{A}'$ is any magnetic potential such that curl $\mathbf{A}' = 1$.
The trick is that for a suitable gauge, we can estimate for some $\epsilon_0 > 0$

$$
\begin{aligned}
\int_\Omega (\mathbf{A}')^2 |\psi_1(B)|^2 \, dx \leq C \int_\Omega \text{dist } (x, \partial\Omega)^2 |\psi_1(B)|^2 \, dx \\
+ \|\mathbf{A}'\|_\infty^2 \int_{\Omega \setminus \Omega'(\epsilon_0)} |\psi_1(B)|^2 \, dx.
\end{aligned} \tag{10}
$$

Using some version of the Agmon estimates, we therefore find

$$\int_\Omega (\mathbf{A}')^2|\psi_1(B)|^2\,dx \leq CB^{-1}. \tag{11}$$

Choose $\beta = \eta B$, where $\eta > 0$ is arbitrary. Using a weak asymptotics for $\lambda_1(B)$, we therefore find

$$\liminf_{B\to\infty} \lambda'_{1,+}(B) \geq \Theta_0 - \eta C. \tag{12}$$

Since η was arbitrary this implies

$$\liminf_{B\to\infty} \lambda'_{1,+}(B) \geq \Theta_0. \tag{13}$$

Applying the same argument to the left side derivative, $\lambda'_{1,-}(B)$, we get (the inequality gets turned since $\beta < 0$)

$$\limsup_{B\to\infty} \lambda'_{1,-}(B) \leq \Theta_0. \tag{14}$$

Since, by perturbation theory, $\lambda'_{1,+}(B) \leq \lambda'_{1,-}(B)$

for all B, we get

$$\lim_{B\to\infty} \lambda'_{1,-}(B) = \Theta_0 = \lim_{B\to\infty} \lambda'_{1,+}(B)\ , \qquad (15)$$

hence the monotonicity of $\lambda_1(B)$.

Questions, other results and Perspectives

This is far to be the end of the story. Here are some additional questions or remarks :

1. The case of corners was analyzed by Hadallah, Bonnaillie, and a numerical analysis of the tunneling in polygons was performed by Dauge-Bonnaillie.

2. Analyze the case when Ω is not simply connected !

3. Analyze the situation between $H_{C3}(\kappa)$ and $H_{C2}(\kappa)$ (Pan, Fournais-Helffer, Almog-Helffer, Sandier-Serfaty).

4. Analyze other conditions than Neumann (see the analysis of Lu-Pan and Kachmar for the De Gennes (Robin) conditions.

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