Long time behavior of solutions of local and nonlocal nondegenerate Hamilton-Jacobi equations with Ornstein-Uhlenbeck operator

Thi Tuyen NGUYEN

IRMAR - University of Rennes 1, France

Joint work with: Emmanuel CHASSEIGNE (LMPT, Tours, France), Olivier LEY (INSA, Rennes, France)

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Outline of presentation

I. Introduction.
II. A *priori* Lipschitz regularity of solutions.
III. Well-posedness of solutions.
IV. Application to ergodic problem and long time behavior.

Introduction

We study long time behavior of solution of Cauchy problem

$$\begin{cases} u_t - \mathcal{F}(x,[u]) + \langle b(x), Du\rangle + H(x,Du) = f(x) \text{ in } Q, \\ u(\cdot,0) = u_0(\cdot) \ \text{ in } \mathbb{R}^N, \end{cases} \tag{HJt}$$

where $Q = \mathbb{R}^N \times (0,\infty)$, $\mathcal{F}$ can be either *local* or *nonlocal*:

$$\begin{cases} \mathcal{F}(x,[u]) = \mathrm{tr}(\sigma(x)\sigma^T(x)D^2u) \ \textbf{(local case)}, \\ \mathcal{F}(x,[u]) = \mathcal{I}(x,u,Du) \ \textbf{(nonlocal case)}, \end{cases}$$

the term $\mathcal{F} - \langle b, D\rangle$ is called the **Ornstein-Uhlenbeck operator** (O-U).

The convergence that we want to obtain:

$$u(\cdot,t) - (ct + v) \to 0 \text{ locally uniformly in } \mathbb{R}^N \text{ as } t \to \infty$$

where $(c,v) \in \mathbb{R} \times C(\mathbb{R}^N)$ is a solution of the associated ergodic problem:

$$c - \mathcal{F}(x,[v]) + \langle b(x), Dv\rangle + H(x,Dv) = f(x) \quad \text{in } \mathbb{R}^N. \tag{HJErg}$$

Main steps to prove the convergence

Goal: $u(\cdot,t)-(ct+v)\to 0$ locally uniformly in $\mathbb{R}^N$ as $t\to\infty$.

- **Step 1:** Solving the ergodic problem

$$c-\mathcal{F}(x,[v])+\langle b(x),Dv\rangle+H(x,Dv)=f(x),$$

  where $(c,v)\in\mathbb{R}\times C(\mathbb{R}^N)$.
- **Step 2:** Getting compactness property of $\{u(\cdot,t)-ct, t\geq 0\}$.
- **Step 3:** Applying strong maximum principle.

All three above main steps need Lipschitz regularity of solutions.

➪ Establishing the Lipschitz regularity of solutions is one of the **crucial steps** to study long time behavior of solutions.

Our improvements from Fujita, Ishii & Loreti (2006)'s work

$$(FIL's\ eq): \qquad u_t - \Delta u + \alpha\langle x, Du\rangle + H(Du) = f(x).$$

- Using more general diffusion terms

Δu ➪ $\mathrm{tr}(\sigma(x)\sigma^T(x)D^2u)$, with σ is Lipschitz bounded,

➪ Nonlocal integro-differential operator of fractional Laplacian type.

- Using more general class of Hamiltonians, H is sublinear and depends on x, i.e., $|H(x,p)| \leq C(1+|p|)$, $\forall x, p \in \mathbb{R}^N$ instead of H is Lipschitz, i.e., $|H(p)-H(q)| \leq C|p-q|$, $\forall p, q \in \mathbb{R}^N$.

Assumptions on datas (1)

Prescribed growth of solutions and datas

$$\phi_\mu(x) = e^{\mu\sqrt{|x|^2+1}}, \quad \mu > 0, \qquad \text{(growth)}$$

$$\mathcal{E}_\mu(\bar{Q}) = \{v : \bar{Q} \to \mathbb{R} : \lim_{|x|\to+\infty} \sup_{0\le t<T} \frac{v(x,t)}{\phi_\mu(x)} = 0\}, \qquad (\mathcal{E}_\mu)$$

where $Q = \mathbb{R}^N \times (0,\infty)$, $T > 0$. We look for solutions which belong to $\mathcal{E}_\mu$ with datas f, $u_0 \in \mathcal{E}_\mu$.

- **Local diffusion:** σ is a bounded Lipschitz continuous in $\mathbb{R}^N$.
- **Nonlocal diffusion:** Let $x \in \mathbb{R}^N$, $\psi : \mathbb{R}^N \to \mathbb{R}$ be a C^2 function,

$$\mathcal{I}(x,\psi,D\psi) = \int_{\mathbb{R}^N} \{\psi(x+z) - \psi(x) - \langle D\psi(x), z\rangle \mathbf{1}_B(z)\}\nu(dz),$$

ν : Lévy measure satisfying $\int_B |z|^2\nu(dz), \int_{B^c} \phi_\mu(z)\nu(dz) \le C_\nu$, $C_\nu > 0$,

$$\text{typical case:} \qquad \nu(dz) = \frac{e^{-\mu|z|}dz}{|z|^{N+\beta}}, \ \mu > 0, \ \beta \in (0,2).$$

Assumptions on datas (2)

- The **Ornstein-Uhlenbeck drift** term:

$$\text{There exists } \alpha > 0 \text{ such that } \langle b(x) - b(y), x - y \rangle \geq \alpha |x - y|^2, \quad \text{(O-U)}$$

where α is called *size* of the Ornstein-Uhlenbeck operator which has a crutial role in the equation. It gives a kind of supersolution of the equation for large x, that is

$$-\mathcal{F}(x, [\phi_\mu]) + \langle b(x), D\phi_\mu \rangle - C|D\phi_\mu| \geq \phi_\mu - K, \qquad \text{(super pro)}$$

where $K > 0$, and ϕ_μ is the growth function.

- The **Hamiltonian**:

$$|H(x, p)| \leq C_H(1 + |p|), \quad x, p \in \mathbb{R}^N. \qquad \text{(sublinear)}$$

Lipschitz regularity results for nondegenerate elliptic equations

Let $\lambda > 0$, we consider the approximate stationary problem

$$\lambda u^\lambda - \mathcal{F}(x,[u^\lambda]) + \langle b(x), Du^\lambda\rangle + H(x,Du^\lambda) = f(x) \qquad (\mathrm{HJ}\lambda)$$

in $\mathbb{R}^N$. This equation is called nondegenerate elliptic if

- **Local case**, $\mathcal{F}(x,[u^\lambda]) = \mathrm{tr}(A(x)D^2u^\lambda)$, A satisfies $A(x) \geq \rho Id$, $\rho > 0$.
- **Nonlocal case**, $\mathcal{F}(x,[u^\lambda]) = \mathcal{I}(x,u^\lambda,Du^\lambda)$, with measures satisfy

(M) Let $\beta \in (1,2)$, $a \in \mathbb{R}^N$.
There exist $0 < \eta < 1$ and $C_\nu > 0$ such that

$$\forall \gamma > 0 \quad \int_{\mathcal{C}_{\eta,\gamma}(a)} |z|^2 \nu(dz) \geq C_\nu \eta^{\frac{N-1}{2}} \gamma^{2-\beta},$$

where $\mathcal{C}_{\eta,\gamma}(a) := \{z \in B_\gamma; (1-\eta)|z||a| \leq |a.z|\}$

here β is called the order of nonlocal operator, see [BCCI-12].

Statement of results

Recall:

$$\phi_\mu(x) = e^{\mu\sqrt{|x|^2+1}}, \quad \mu > 0,$$

$$\mathcal{E}_\mu(\mathbb{R}^N) = \{v : \mathbb{R}^N \to \mathbb{R} : \lim_{|x|\to+\infty} \frac{v(x)}{\phi_\mu(x)} = 0\}.$$

Theorem 1 (Gradient bound for (HJλ) in the **nondegenerate** case)

Let $u^\lambda \in \mathcal{E}_\mu$ with ϕ_μ-growth be a continuous solution of (HJλ). Suppose that (HJλ) is **nondegenerate elliptic** *(in the both local and nonlocal case), O-U holds with any size $\alpha > 0$, H is sublinear, $f \in \mathcal{E}_\mu$. Then*

$$|u^\lambda(x) - u^\lambda(y)| \le C(\phi_\mu(x) + \phi_\mu(y))|x - y|, \ C \text{ **independent** of } \lambda.$$

➪ Same results for parabolic equations (HJt) with $C > 0$ **independent** of t.

Ideas of proof (1). Viscosity inequalities.

Arguing by contradiction, using [Ishii-Lions-90] method's ideas. Let $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ be a C^2 concave, increasing function with $\psi(0) = 0$, ϕ_μ be the growth function and set $\Phi(x, y) = \phi_\mu(x) + \phi_\mu(y)$. Suppose that

$$\sup_{x,y\in\mathbb{R}^N} \{u^\lambda(x) - u^\lambda(y) - \psi(|x-y|)\Phi(x,y)\} > 0, \text{ achieved at } (\bar{x}, \bar{y}),\ \bar{x} \neq \bar{y}.$$

Writing the viscosity inequalities with $\varphi(\bar{x}, \bar{y}) = \psi(|\bar{x} - \bar{y}|)\Phi(\bar{x}, \bar{y})$ we have

$$\begin{aligned} & \lambda(u^\lambda(\bar{x}) - u^\lambda(\bar{y})) - (\mathcal{F}(\bar{x}, [u^\lambda]) - \mathcal{F}(\bar{y}, [u^\lambda])) \\ & + \langle b(\bar{x}), D_x\varphi\rangle - \langle b(\bar{y}), -D_y\varphi\rangle \\ \leq\ & H(\bar{y}, -D_y\varphi) - H(\bar{x}, D_x\varphi) + f(\bar{x}) - f(\bar{y}), \end{aligned} \tag{1}$$

Goal: Reach a contradiction in (1).

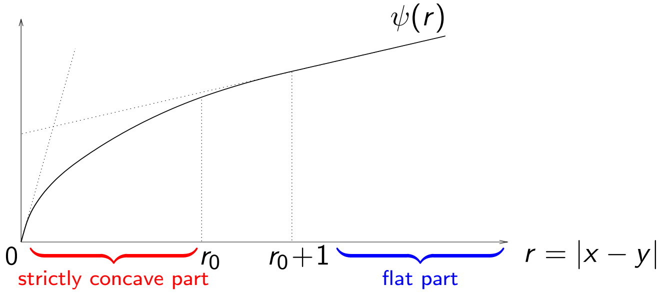
Ideas of proof (2). Estimate the different terms in (1) (local case)

• **Local terms:** $\mathcal{F}(\bar{x},[u^\lambda]) - \mathcal{F}(\bar{y},[u^\lambda]) = \mathrm{tr}(A(\bar{x})X - A(\bar{y})Y)$, using [Ishii-Lions-90]'s method.

▷ Using the assumptions on the local diffusion matrix, (O-U), H, f to estimate the different terms in (1) we get

$$
\begin{aligned}
& \Phi(\bar{x},\bar{y})\big(\overbrace{-4\rho\psi''(|\bar{x}-\bar{y}|)}^{>0 \text{ if } \psi \text{ strict.concave}} + \overbrace{\alpha\psi'(|\bar{x}-\bar{y}|)|\bar{x}-\bar{y}|}^{\text{coming from (O-U)}}\big) \\
\leq\ & \Phi(\bar{x},\bar{y})C\psi'(|\bar{x}-\bar{y}|) + C + C(\phi_\mu(\bar{x}) + \phi_\mu(\bar{y}))|\bar{x}-\bar{y}| \\
& +(\psi(|\bar{x}-\bar{y}|))\left(\mathrm{tr}(A(\bar{x})D^2\phi_\mu(\bar{x})) - \langle b(\bar{x}), D\phi_\mu(\bar{x})\rangle + C|D\phi_\mu(\bar{x})|\right) \\
& +(\psi(|\bar{x}-\bar{y}|))\left(\mathrm{tr}(A(\bar{y})D^2\phi_\mu(\bar{y})) - \langle b(\bar{y}), D\phi_\mu(\bar{y})\rangle + C|D\phi_\mu(\bar{y})|\right).
\end{aligned}
$$

Ideas of proof (3). Reach a contradiction (local case).



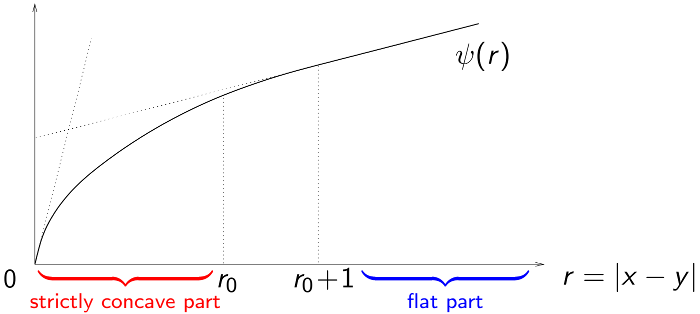
- $r := |\bar{x} - \bar{y}| \leq r_0$. We take profit of the ellipticity of the equation which is related with the strictly concave property of ψ to control bad terms which come form the Hamiltonian the other terms.
- $r := |\bar{x} - \bar{y}| \geq r_0$. Using the effect of the Ornstein-Uhlenbeck operator to control everything.

▷ All of the parameters in two cases are chosen **independently** of λ.

Ideas of proof (4). Reach a contradiction (nonlocal case).

- The ideas follow the ones in the local case in order that we can apply [Ishii-Lions-90] method's ideas.
- **Nonlocal terms:**
  $\mathcal{F}(\bar{x}, [u^\lambda]) - \mathcal{F}(\bar{y}, [u^\lambda]) = \mathcal{I}(\bar{x}, u^\lambda, D_x\varphi) - \mathcal{I}(\bar{y}, u^\lambda, -D_y\varphi)$, inspired by [BCCI-12].
- The estimates for the nonlocal terms are **more technical and complicated** than those of the local ones because of **nonlocal property of the operator and unbounded settings of solutions**.
- We need to construct concave test functions which are different with the one in the local case.
- We can not obtain the result directly. We first establish τ-Hölder continuity for solutions, $\forall \tau \in (0,1)$, then we improve that Hölder regularity to Lipschitz regularity.

Ideas of proof (5). Concave test functions.



The concave test functions for $r \leq r_0$:

- **Local case:** $\psi_{local}(r) = 1 - e^{-C_1 r}$.
- **Nonlocal case:** $\begin{cases} \psi_{nonlocal1}(r) = 1 - e^{-C_1 r^{\tau}}, \ \tau \in (0,1), \\ \psi_{nonlocal2}(r) = r - \varrho r^{1+\theta}, \ \theta \in (0,1), \varrho > 0. \end{cases}$

Well-posedness of solutions (1)

$*$ **Existence:**

➪ Using truncation arguments (inspired by [Barles-Souganidis-01]) to build a continuous solution for problems. Let $m, n \geq 1$,

$$f_m(x) = \min\{f(x) + \frac{1}{m}\phi_\mu(x), m\}, \ \phi_\mu \text{ is the growth function},$$

$$H_m(x,p) = \begin{cases} H(x,p) & |x| \leq m \\ H(m\frac{x}{|x|}, p) & |x| \geq m, \end{cases} \quad H_{mn}(x,p) = \begin{cases} H_m(x,p) & |p| \leq n \\ H_m(x, n\frac{p}{|p|}) & |p| \geq n. \end{cases}$$

▷ $f_m \in BUC(\mathbb{R}^N)$ and $H_{mn} \in BUC(\mathbb{R}^N \times \mathbb{R}^N)$ with modulus of continuity depending on $m, n \Rightarrow$ existence of BUC solution of

$$\lambda u - \mathcal{F}(x, [u]) + \langle b(x), Du\rangle + H_{mn}(x, Du) = f_m \ \text{ in } \mathbb{R}^N.$$

Well-posedness of solutions (2)

➪ Using the local gradient bounds and applying Ascoli Theorem to get the convergence of the bounded solution to the truncation equation to the initial one when $n, m \to \infty$.

∗ **Uniqueness:**

- It is a direct consequence of the comparison principle.
- In our case, the comparison principle only holds if either sub or supersolution is locally Lipschitz continuous.

Application to ergodic problem

Problem: Finding $(c, v) \in \mathbb{R} \times C(\mathbb{R}^N)$ which is the solution (HJErg)

$$c - \mathcal{F}(x, [v]) + \langle b(x), Dv(x)\rangle + H(x, Dv(x)) \quad = \quad f(x).$$

Let $u^\lambda \in C(\mathbb{R}^N)$, $\lambda \in (0,1)$ be a solution of (HJλ). Then

$$w^\lambda := u^\lambda(x) - u^\lambda(0) \qquad \text{is a solution of}$$

$$\lambda w^\lambda - \mathcal{F}(x, [w^\lambda]) + \langle b(x), Dw^\lambda(x)\rangle + H(x, Dw^\lambda(x)) \quad = \quad f(x) - \lambda u^\lambda(0).$$

Main tools: using local uniform Lipschitz regularity of u^λ, Ascoli Theorem we get convergences, up to some subsequence,

$$\begin{aligned} w^\lambda &\to v, \quad \text{locally uniformly in } \mathbb{R}^N \\ \lambda u^\lambda(0) &\to c, \quad \text{as } \lambda \to 0. \end{aligned}$$

Thanks to Stability results ([Crandall et al.- 92,Barles-Imbert-08]) we get a solution for (HJErg).

Application to long time behavior

Let $\mu > 0$, fix $\theta \in (0, \mu)$. Recall that

$$\phi_\theta(x) = e^{\theta\sqrt{|x|^2+1}},$$

$$\mathcal{E}_\theta(\bar{Q}) = \{v : \bar{Q} \to \mathbb{R} : \lim_{|x|\to+\infty} \sup_{0\le t<T} \frac{v(x,t)}{\phi_\theta(x)} = 0\},$$

where $Q = \mathbb{R}^N \times [0, \infty)$.

Theorem 2 (Long time behavior of solution)

Let $u \in \mathcal{E}_\theta(Q)$ be the unique solution of (HJt) and $(c, v) \in \mathbb{R} \times \mathcal{E}_\gamma(\mathbb{R}^N)$, $\gamma > \theta$, be a solution of (HJErg). Suppose that the equations are **nondegenerate** *(in the both local and nonlocal case). Then there is a constant $a \in \mathbb{R}$ such that*

$$\lim_{t\to\infty} \max_{B(0,R)} |u(x,t) - (ct + v(x) + a)| = 0 \quad \textit{for all } R > 0.$$

Main ideas of proof (1) (inspired by [FIL-06]'s work).

• By the Lipschitz regularity of solutions of the (HJt) we have the compactness property of the set $\{u(\cdot,t)-ct, t\geq 0\}$.

• The functions

$$\begin{aligned} v^+(x) &= \limsup_{t\to\infty}(u(x,t)-ct), \\ v^-(x) &= \liminf_{t\to\infty}(u(x,t)-ct) \end{aligned}$$

are respectively a sub and supersolution of (HJErg).

By strong maximum principle ([Bardi-Da Lio-99, Ciomaga-12]), there are $a, b \in \mathbb{R}$ such that

$$v^+(x) = v(x)+a, \quad v^-(x) = v(x)+b, \quad \Rightarrow b \leq a.$$

$\Rightarrow$ We only need to prove $a \leq b$.

Main ideas of proof (2)

We prove $a \le b$: Let $T, \epsilon > 0$, $\gamma > \theta > 0$, ϕ_γ be the growth function, $K > e^\gamma$ be fixed so that

$$-\mathcal{F}(x, [\phi_\gamma]) + \langle b(x), D\phi_\gamma \rangle - C|D\phi_\gamma)| \ge \phi_\gamma(x) - K, \quad \forall x \in \mathbb{R}^N.$$

Hence, the functions

$$\eta^T(x,t) = u(x,t+T) - ct - v(x) - b, \quad g_\epsilon(x,t) = \epsilon(\phi_\gamma(x) - K)e^{-t} + \epsilon(1+K)$$

are respectively a sub and supersolution of

$$\eta_t - \mathcal{F}(x, [\eta]) + \langle b(x), D\eta \rangle - L_H|D\eta| = 0 \text{ in } \mathbb{R}^N \times (0, \infty).$$

By comparison principle we get $\eta^T \le g_\epsilon$. Taking $x = 0$ we have

$$\begin{aligned} u(0, t+T) &\le v(0) + b + \epsilon(\phi_\gamma(0) - K)e^{-t} + \epsilon(1+K) \\ &\le v(0) + b + \epsilon(1+K). \end{aligned}$$

Sending $t \to \infty$ along a sequence yields $v^+(0) \le v(0) + b + \epsilon(1+K)$ which implies $a \le b$, as $\epsilon > 0$ is arbitrary.

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Thank you for your attention!