Uniform Stabilization of a family of Boussinesq systems

Ademir Pazoto
Instituto de Matemática
Universidade Federal do Rio de Janeiro (UFRJ)
ademir@im.ufrj.br

In collaboration with
Sorin Micu - University of Craiova (Romania)

Supported by
Agence Universitaire de la Francophonie

Outline

- Description of the model: a family of Boussinesq systems
- Setting of the problem: stabilization of a coupled system of two Benjamin-Bona-Mahony **(BBM)** equations
- Main results
- Main Idea of the proofs
- Remarks and open problems

The Benjamin-Bona-Mahony (BBM) equation

The BBM equation

$$u_t + u_x - u_{xxt} + uu_x = 0, \tag{1}$$

was proposed as an alternative model for the Korteweg-de Vries equation (KdV)

$$u_t + u_x + u_{xxx} + uu_x = 0, \tag{2}$$

to describe the propagation of one-dimensional, unidirectional small amplitude long waves in nonlinear dispersive media.

- $u(x, t)$ is a real-valued functions of the real variables x and t.

In the context of shallow-water waves, $u(x, t)$ represents the displacement of the water surface at location x and time t.

The Boussinesq system

J. L. **Bona**, M. **Chen**, J.-C. **Saut** - J. Nonlinear Sci. 12 (2002).

$$\begin{cases} \eta_t + w_x + (\eta w)_x + a w_{xxx} - b\eta_{xxt} = 0 \\ w_t + \eta_x + w w_x + c\eta_{xxx} - d w_{xxt} = 0, \end{cases} \tag{3}$$

The model describes the motion of small-amplitude long waves on the surface of an ideal fluid under the gravity force and in situations where the motion is sensibly two dimensional.

η is the elevation of the fluid surface from the equilibrium position;

$w = w_\theta$ is the horizontal velocity in the flow at height θh, where h is the undisturbed depth of the liquid;

a, b, c, d, are parameters required to fulfill the relations

$$a + b = \frac{1}{2}\left(\theta^2 - \frac{1}{3}\right), \qquad c + d = \frac{1}{2}(1 - \theta^2) \geq 0,$$

where $\theta \in [0, 1]$ specifies which velocity the variable w represents.

The Boussinesq system

J. L. **Bona**, M. **Chen**, J.-C. **Saut** - J. Nonlinear Sci. 12 (2002).

$$\begin{cases} \eta_t + w_x + (\eta w)_x + a w_{xxx} - b\eta_{xxt} = 0 \\ w_t + \eta_x + w w_x + c\eta_{xxx} - d w_{xxt} = 0, \end{cases} \tag{3}$$

The model describes the motion of small-amplitude long waves on the surface of an ideal fluid under the gravity force and in situations where the motion is sensibly two dimensional.

η is the elevation of the fluid surface from the equilibrium position;

$w = w_\theta$ is the horizontal velocity in the flow at height θh, where h is the undisturbed depth of the liquid;

a, b, c, d, are parameters required to fulfill the relations

$$a + b = \frac{1}{2}\left(\theta^2 - \frac{1}{3}\right), \qquad c + d = \frac{1}{2}(1 - \theta^2) \geq 0,$$

where $\theta \in [0, 1]$ specifies which velocity the variable w represents.

The Boussinesq system

J. L. **Bona**, M. **Chen**, J.-C. **Saut** - J. Nonlinear Sci. 12 (2002).

$$\begin{cases} \eta_t + w_x + (\eta w)_x + a w_{xxx} - b\eta_{xxt} = 0 \\ w_t + \eta_x + w w_x + c\eta_{xxx} - d w_{xxt} = 0, \end{cases} \tag{3}$$

The model describes the motion of small-amplitude long waves on the surface of an ideal fluid under the gravity force and in situations where the motion is sensibly two dimensional.

η is the elevation of the fluid surface from the equilibrium position;

$w = w_\theta$ is the horizontal velocity in the flow at height θh, where h is the undisturbed depth of the liquid;

a, b, c, d, are parameters required to fulfill the relations

$$a + b = \frac{1}{2}\left(\theta^2 - \frac{1}{3}\right), \qquad c + d = \frac{1}{2}(1 - \theta^2) \geq 0,$$

where $\theta \in [0, 1]$ specifies which velocity the variable w represents.

Stabilization Results: $E(t) \leq cE(0)e^{-\omega t}, \ \omega > 0, \, c > 0$

The Boussinesq system posed on a bounded interval:

- A. Pazoto and L. Rosier, Stabilization of a Boussinesq system of KdV-KdV type, System and Control Letters 57 (2008), 595-601.
- R. Capistrano Filho, A. Pazoto and L. Rosier, Control of Boussinesq system of KdV-KdV type on a bounded domain, Preprint.

The Boussinesq system posed on the whole real axis: $(-\eta_{xx}, -w_{xx})$

- M. Chen and O. Goubet, Long-time asymptotic behavior of dissipative Boussinesq systems, Discrete Contin. Dyn. Syst. Ser. 17 (2007), 509-528.

The Boussinesq system posed on a periodic domain:

- S. Micu, J. H. Ortega, L. Rosier and B.-Y. Zhang, Control and stabilization of a family of Boussinesq systems, Discrete Contin. Dyn. Syst. 24 (2009), 273-313.

Stabilization Results: $E(t) \le cE(0)e^{-\omega t}, \ \omega > 0, \, c > 0$

The Boussinesq system posed on a bounded interval:

- A. Pazoto and L. Rosier, Stabilization of a Boussinesq system of KdV-KdV type, System and Control Letters 57 (2008), 595-601.
- R. Capistrano Filho, A. Pazoto and L. Rosier, Control of Boussinesq system of KdV-KdV type on a bounded domain, Preprint.

The Boussinesq system posed on the whole real axis: $(-\eta_{xx}, -w_{xx})$

- M. Chen and O. Goubet, Long-time asymptotic behavior of dissipative Boussinesq systems, Discrete Contin. Dyn. Syst. Ser. 17 (2007), 509-528.

The Boussinesq system posed on a periodic domain:

- S. Micu, J. H. Ortega, L. Rosier and B.-Y. Zhang, Control and stabilization of a family of Boussinesq systems, Discrete Contin. Dyn. Syst. 24 (2009), 273-313.

Stabilization Results: $E(t) \leq cE(0)e^{-\omega t}, \ \omega > 0, \, c > 0$

The Boussinesq system posed on a bounded interval:

- A. Pazoto and L. Rosier, Stabilization of a Boussinesq system of KdV-KdV type, System and Control Letters 57 (2008), 595-601.
- R. Capistrano Filho, A. Pazoto and L. Rosier, Control of Boussinesq system of KdV-KdV type on a bounded domain, Preprint.

The Boussinesq system posed on the whole real axis: $(-\eta_{xx}, -w_{xx})$

- M. Chen and O. Goubet, Long-time asymptotic behavior of dissipative Boussinesq systems, Discrete Contin. Dyn. Syst. Ser. 17 (2007), 509-528.

The Boussinesq system posed on a periodic domain:

- S. Micu, J. H. Ortega, L. Rosier and B.-Y. Zhang, Control and stabilization of a family of Boussinesq systems, Discrete Contin. Dyn. Syst. 24 (2009), 273-313.

Controllability and Stabilization

• S. Micu, J. H. Ortega, L. Rosier, B.-Y. Zhang - Discrete Contin. Dyn. Syst. 24 (2009).

$$b, d \geq 0, a \leq 0, c \leq 0 \quad \text{or} \quad b, d \geq 0, a = c > 0.$$

$$\begin{cases} \eta_t + w_x + (\eta w)_x + a w_{xxx} - b\eta_{xxt} = f(x,t) \\ w_t + \eta_x + w w_x + c\eta_{xxx} - d w_{xxt} = g(x,t) \end{cases}$$

where $0 < x < 2\pi$ and $t > 0$, with boundary conditions

$$\frac{\partial^r \eta}{\partial x^r}(0,t) = \frac{\partial^r \eta}{\partial x^r}(2\pi,t), \quad \frac{\partial^r w}{\partial x^r}(0,t) = \frac{\partial^r w}{\partial x^r}(2\pi,t)$$

and initial conditions

$$\eta(x,0) = \eta^0(x), \quad w(x,0) = w^0(x).$$

• f and g are locally supported forces.

Periodic boundary conditions

For $b,\ d > 0$ and $\beta_1,\ \beta_2, \alpha_1,\ \alpha_2 \geq 0$, we consider the system

$$\begin{aligned} &\eta_t + w_x - b\eta_{txx} + (\eta w)_x + \beta_1 M_{\alpha_1}\eta = 0, \\ &w_t + \eta_x - dw_{txx} + ww_x + \beta_2 M_{\alpha_2} w = 0, \end{aligned} \tag{4}$$

with periodic boundary conditions

$$\begin{aligned} &\eta(0,t) = \eta(2\pi,t);\ \eta_x(0,t) = \eta_x(2\pi,t), \\ &w(0,t) = w(2\pi,t);\ w_x(0,t) = w_x(2\pi,t), \end{aligned}$$

and initial conditions

$$\eta(x,0) = \eta^0(x), \quad w(x,0) = w^0(x).$$

In (4), M_{α_j} are Fourier multiplier operators given by

$$M_{\alpha_j}\left(\sum_{k\in\mathbb{Z}} v_k e^{ikx}\right) = \sum_{k\in\mathbb{Z}} (1+k^2)^{\frac{\alpha_j}{2}} \widehat{v}_k e^{ikx}.$$

The energy associated to the model is given by

$$E(t) = \frac{1}{2} \int_0^{2\pi} (\eta^2 + b\eta_x^2 + w^2 + dw_x^2) dx \tag{5}$$

and we can (formally) deduce that

$$\begin{aligned} \frac{d}{dt} E(t) = -\beta_1 \int_0^{2\pi} (M_{\alpha_1} \eta)\, \eta\, dx - \beta_2 \int_0^{2\pi} (M_{\alpha_2} w)\, w\, dx \\ - \int_0^{2\pi} (\eta w)_x\, \eta\, dx. \end{aligned} \tag{6}$$

Since $\beta_1, \beta_2 \geq 0$ and

$$(M_{\alpha_j} v, v)_{L^2(0,2\pi)} \geq 0, \qquad j = 1, 2,$$

the terms $M_{\alpha_1}\eta$ and $M_{\alpha_2}w$ play the role of feedback damping mechanisms, at least for the linearized system.

Assumptions on the Dissipation: $\int_{\mathbb{T}} M_{\alpha_i}\varphi(x)\varphi(x)dx \geq 0$

- Applications and study of asymptotic behavior os solutions:
  - J. L. Bona and J. Wu, M2AN Math. Model. Numer. Anal. (2000).
  - J.-P. Chehab, P. Garnier and Y. Mammeri, J. Math. Chem. (2001).
  - D. Dix, Comm. PDE (1992).
  - C. J. Amick, J. L. Bona and M. Schonbek, Jr. Diff. Eq. (1989).
  - P. Biler, Bull. Polish. Acad. Sci. Math. (1984).
  - J.-C. Saut, J. Math. Pures et Appl. (1979).
- Fractional derivative (Weyl fractional derivative operator):

$$h(x) = \sum_{k\in\mathbb{Z}} a_k e^{ikx} \Rightarrow W_x^{\alpha}(h)(x) = \sum_{k\in\mathbb{Z}} (ik)^{\alpha} a_k e^{ikx}, \quad \alpha \in (0,1).$$

Assumptions on the Dissipation: $\int_{\mathbb{T}} M_{\alpha_i}\varphi(x)\varphi(x)dx \geq 0$

- Applications and study of asymptotic behavior os solutions:
  - J. L. Bona and J. Wu, M2AN Math. Model. Numer. Anal. (2000).
  - J.-P. Chehab, P. Garnier and Y. Mammeri, J. Math. Chem. (2001).
  - D. Dix, Comm. PDE (1992).
  - C. J. Amick, J. L. Bona and M. Schonbek, Jr. Diff. Eq. (1989).
  - P. Biler, Bull. Polish. Acad. Sci. Math. (1984).
  - J.-C. Saut, J. Math. Pures et Appl. (1979).
- Fractional derivative (Weyl fractional derivative operator):

$$h(x) = \sum_{k\in\mathbb{Z}} a_k e^{ikx} \Rightarrow W_x^{\alpha}(h)(x) = \sum_{k\in\mathbb{Z}} (ik)^{\alpha} a_k e^{ikx}, \quad \alpha \in (0,1).$$

Main results

The energy $E(t)$ satisfies

$$\frac{dE}{dt} = -\beta_1 \int_0^{2\pi} (M_{\alpha_1}\eta)\,\eta\,dx - \beta_2 \int_0^{2\pi} (M_{\alpha_2}w)\,w\,dx - \int_0^{2\pi} (\eta w)_x\,\eta\,dx,$$

where

$$M_{\alpha_j} v = \sum_{k\in\mathbb{Z}} (1+k^2)^{\frac{\alpha_j}{2}} \widehat{v}_k e^{ikx}.$$

Firstly, we analyze the linearized system:

- $\alpha_1 = \alpha_2 = 2$ and $\beta_1, \beta_2 > 0 \Longrightarrow$ the exponential decay of solutions in the H^s-setting, for any $s \in \mathbb{R}$.
- $\max\{\alpha_1,\, \alpha_2\} \in (0,2)$, $\beta_1, \beta_2 \geq 0$ and $\beta_1^2 + \beta_2^2 > 0 \Longrightarrow$ polynomial decay rate of solutions in the H^s-setting, by considering more regular initial data.

Exponential decay estimate and contraction mapping argument $\Longrightarrow$ global well-posedness and the exponential stability property of the nonlinear system.

Main results

The energy $E(t)$ satisfies

$$\frac{dE}{dt} = -\beta_1 \int_0^{2\pi} (M_{\alpha_1}\eta)\,\eta\,dx - \beta_2 \int_0^{2\pi} (M_{\alpha_2}w)\,w\,dx - \int_0^{2\pi} (\eta w)_x\,\eta\,dx,$$

where

$$M_{\alpha_j} v = \sum_{k\in\mathbb{Z}} (1+k^2)^{\frac{\alpha_j}{2}} \widehat{v}_k e^{ikx}.$$

Firstly, we analyze the linearized system:

- $\alpha_1 = \alpha_2 = 2$ and $\beta_1, \beta_2 > 0 \Longrightarrow$ the exponential decay of solutions in the H^s-setting, for any $s \in \mathbb{R}$.
- $\max\{\alpha_1, \alpha_2\} \in (0,2)$, $\beta_1, \beta_2 \geq 0$ and $\beta_1^2 + \beta_2^2 > 0 \Longrightarrow$ polynomial decay rate of solutions in the H^s-setting, by considering more regular initial data.

Exponential decay estimate and contraction mapping argument $\Longrightarrow$ global well-posedness and the exponential stability property of the nonlinear system.

Main results

The energy $E(t)$ satisfies

$$\frac{dE}{dt} = -\beta_1 \int_0^{2\pi} (M_{\alpha_1}\eta)\,\eta\,dx - \beta_2 \int_0^{2\pi} (M_{\alpha_2}w)\,w\,dx - \int_0^{2\pi} (\eta w)_x\,\eta\,dx,$$

where

$$M_{\alpha_j} v = \sum_{k\in\mathbb{Z}} (1+k^2)^{\frac{\alpha_j}{2}} \widehat{v}_k e^{ikx}.$$

Firstly, we analyze the linearized system:

- $\alpha_1 = \alpha_2 = 2$ and $\beta_1, \beta_2 > 0 \Longrightarrow$ the exponential decay of solutions in the H^s-setting, for any $s \in \mathbb{R}$.
- $\max\{\alpha_1,\, \alpha_2\} \in (0,2)$, $\beta_1, \beta_2 \geq 0$ and $\beta_1^2 + \beta_2^2 > 0 \Longrightarrow$ polynomial decay rate of solutions in the H^s-setting, by considering more regular initial data.

Exponential decay estimate and contraction mapping argument $\Longrightarrow$ global well-posedness and the exponential stability property of the nonlinear system.

Main results

The energy $E(t)$ satisfies

$$\frac{dE}{dt} = -\beta_1 \int_0^{2\pi} (M_{\alpha_1}\eta)\,\eta\,dx - \beta_2 \int_0^{2\pi} (M_{\alpha_2}w)\,w\,dx - \int_0^{2\pi} (\eta w)_x\,\eta\,dx,$$

where

$$M_{\alpha_j} v = \sum_{k\in\mathbb{Z}} (1+k^2)^{\frac{\alpha_j}{2}} \widehat{v}_k e^{ikx}.$$

Firstly, we analyze the linearized system:

- $\alpha_1 = \alpha_2 = 2$ and $\beta_1, \beta_2 > 0 \Longrightarrow$ the exponential decay of solutions in the H^s-setting, for any $s \in \mathbb{R}$.
- $\max\{\alpha_1, \alpha_2\} \in (0,2)$, $\beta_1, \beta_2 \geq 0$ and $\beta_1^2 + \beta_2^2 > 0 \Longrightarrow$ polynomial decay rate of solutions in the H^s-setting, by considering more regular initial data.

Exponential decay estimate and contraction mapping argument $\Longrightarrow$ global well-posedness and the exponential stability property of the nonlinear system.

For any $k \in \mathbb{Z}$, we denote by $\widehat{v}_k$ the $k-$Fourier coefficient of v,

$$\widehat{v}_k = \frac{1}{2\pi} \int_0^{2\pi} v(x) e^{-ikx} dx,$$

and, for any $s \in \mathbb{R}$, we define the space

$$H_p^s(0, 2\pi) = \left\{ v = \sum_{k \in \mathbb{Z}} \widehat{v}_k e^{ikx} \in H^s(0, 2\pi) \;\middle|\; \sum_{k \in \mathbb{Z}} |\widehat{v}_k|^2 (1 + k^2)^s < \infty \right\},$$

which is a Hilbert space with the inner product defined by

$$(v, w)_s = \sum_{k \in \mathbb{Z}} \widehat{v}_k \overline{\widehat{w}_k} (1 + k^2)^s. \tag{7}$$

Then,

$$M_{\alpha_j} : H_p^{\alpha_j}(0, 2\pi) \to L^2(0, 2\pi).$$

$$M_{\alpha_j} v = \sum_{k \in \mathbb{Z}} (1 + k^2)^{\frac{\alpha_j}{2}} \widehat{v}_k e^{ikx}, \qquad j = 1, 2.$$

The Linearized System

Since

$$\begin{aligned}(I - b\partial_x^2)\eta_t + w_x + \beta_1 M_1 \eta = 0,\\ (I - d\partial_x^2)w_t + \eta_x + \beta_2 M_2 \eta = 0,\end{aligned}$$

the linear system can be written as

$$\begin{aligned}U_t + AU = 0,\\ U(0) = U_0,\end{aligned}$$

where A is given by

$$A = \begin{pmatrix} \beta_1 \left(I - b\partial_x^2\right)^{-1} M_{\alpha_1} & \left(I - b\partial_x^2\right)^{-1} \partial_x \\ \\ \left(I - d\partial_x^2\right)^{-1} \partial_x & \beta_2 \left(I - b\partial_x^2\right)^{-1} M_{\alpha_2} \end{pmatrix}. \tag{8}$$

For $\alpha > 0$, the operator $(I - \alpha\partial_x^2)^{-1}$ is defined in the following way:

$$(I - \alpha\partial_x^2)^{-1}\varphi = v \Leftrightarrow \begin{cases} v - \alpha v_{xx} = \varphi & \text{in } (0, 2\pi), \\ \\ v(0) = v(2\pi), & v_x(0) = v_x(2\pi). \end{cases}$$

Spectral Analysis

If we assume that

$$(\eta^0, w^0) = \sum_{k\in\mathbb{Z}} (\widehat{\eta}_k^0, \widehat{w}_k^0) e^{ikx},$$

the solution can be written as

$$(\eta, \omega)(x,t) = \sum_{k\in\mathbb{Z}} (\widehat{\eta}_k(t), \widehat{\omega}_k(t)) e^{ikx},$$

where the pair $(\widehat{\eta}_k(t), \widehat{w}_k(t))$ fulfills

$$\begin{aligned}
&(1+bk^2)(\widehat{\eta}_k)_t + ik\widehat{w}_k + \beta_1(1+k^2)^{\frac{\alpha_1}{2}}\widehat{\eta}_k = 0,\\
&(1+dk^2)(\widehat{w}_k)_t + ik\widehat{\eta}_k + \beta_2(1+k^2)^{\frac{\alpha_2}{2}}\widehat{w}_k = 0,\\
&\widehat{\eta}_k(0) = \widehat{\eta}_k^0, \qquad \widehat{w}_k(0) = \widehat{w}_k^0,
\end{aligned} \tag{9}$$

where $t \in (0, T)$.

We set

$$A(k) = \begin{pmatrix} \frac{\beta_1(1+k^2)^{\frac{\alpha_1}{2}}}{1+bk^2} & \frac{ik}{1+bk^2} \\ \\ \frac{ik}{1+dk^2} & \frac{\beta_2(1+k^2)^{\frac{\alpha_2}{2}}}{1+dk^2} \end{pmatrix}.$$

Then system (9) is equivalent to

$$\begin{pmatrix} \widehat{\eta}_k \\ \\ \widehat{w}_k \end{pmatrix}_t (t) + A(k) \begin{pmatrix} \widehat{\eta}_k \\ \\ \widehat{w}_k \end{pmatrix} (t) = \begin{pmatrix} 0 \\ \\ 0 \end{pmatrix},$$

$$\begin{pmatrix} \widehat{\eta}_k \\ \\ \widehat{w}_k \end{pmatrix} (0) = \begin{pmatrix} \widehat{\eta}_k^0 \\ \\ \widehat{w}_k^0 \end{pmatrix}.$$

Lemma

The eigenvalues of the matrix A are given by

$$\lambda_k^{\pm} = \frac{1}{2}\left(\frac{\beta_1(1+k^2)^{\frac{\alpha_1}{2}}}{1+bk^2} + \frac{\beta_2(1+k^2)^{\frac{\alpha_2}{2}}}{1+dk^2} \pm \frac{2|k|\sqrt{e_k^2-1}}{\sqrt{(1+bk^2)(1+dk^2)}}\right),$$

where

$$e_k = \frac{1}{2k}\left(\beta_1(1+k^2)^{\frac{\alpha_1}{2}}\sqrt{\frac{1+dk^2}{1+bk^2}} - \beta_2(1+k^2)^{\frac{\alpha_2}{2}}\sqrt{\frac{1+bk^2}{1+dk^2}}\right),$$

and $k \in \mathbb{Z}^*$.

Observe that

- $\lambda_k^{\pm} = \lambda_{-k}^{\pm}$.
- If $e_k < 1$, the eigenvalues $\lambda_k^{\pm} \in \mathbb{C}$.
- If $e_k \geq 1$, the eigenvalues $\lambda_k^{\pm} \in \mathbb{R}$.

Lemma

The solution $(\widehat{\eta}_k(t), \widehat{w}_k(t))$ *of* (9) *is given by*

$$\widehat{\eta}_k(t) = \tfrac{1}{1-\zeta_k^2}\left[\left(\widehat{\eta}_k^0 + i\alpha_k\zeta_k\widehat{w}_k^0\right)e^{-\lambda_k^+ t} - \left(\zeta_k^2\widehat{\eta}_k^0 + i\alpha_k\zeta_k\widehat{w}_k^0\right)e^{-\lambda_k^- t}\right],$$

$$\widehat{w}_k(t) = \tfrac{1}{1-\zeta_k^2}\left[\left(i\theta_k\zeta_k\widehat{\eta}_k^0 - \zeta_k^2\widehat{w}_k^0\right)e^{-\lambda_k^+ t} - \left(i\theta_k\zeta_k\widehat{\eta}_k^0 - \widehat{w}_k^0\right)e^{-\lambda_k^- t}\right],$$

if $|e_k| \neq 1$ *and* $k \neq 0$,

$$\widehat{\eta}_k(t) = \left[\left(1 - \tfrac{k\zeta_k}{\sqrt{(1+bk^2)(1+dk^2)}}t\right)\widehat{\eta}_k^0 - \tfrac{ikt}{1+bk^2}\widehat{w}_k^0\right]e^{-\lambda_k^+ t},$$

$$\widehat{w}_k(t) = \left[-\tfrac{ikt}{1+dk^2}\widehat{\eta}_k^0 + \left(1 + \tfrac{k\zeta_k}{\sqrt{(1+bk^2)(1+dk^2)}}t\right)\widehat{w}_k^0\right]e^{-\lambda_k^+ t},$$

if $|e_k| = 1$ *and* $k \neq 0$, *and finally,*

$$\widehat{\eta}_0(t) = \widehat{\eta}_0^0 e^{-\beta_1 t}, \qquad \widehat{w}_0(t) = \widehat{w}_0^0 e^{-\beta_2 t}.$$

Here, $\alpha_k = \sqrt{\tfrac{1+dk^2}{1+bk^2}}$, $\theta_k = \sqrt{\tfrac{1+bk^2}{1+dk^2}}$ *and* $\zeta_k = e_k - \sqrt{e_k^2 - 1}$.

The case $s = 0$

For any $t \geq 0$ and $k \in \mathbb{Z}$, we have that

$$b|\widehat{\eta}_k(t)|^2 + d|\widehat{w}_k(t)|^2 \leq M\left(b|\widehat{\eta}_k^0|^2 + d|\widehat{w}_k^0|^2\right) e^{-2t\min\left\{|\Re(\lambda_k^+)|,\, |\Re(\lambda_k^-)|\right\}},$$

where

$$\min\{|\Re(\lambda_k^+)|,\, |\Re(\lambda_k^-)|\} \geq D > 0,$$

and D is a positive number, depending on the parameters β_1, β_2, α_1, α_2, b and d.

Moreover,

- If $\beta_1\beta_2 = 0$, then $\Re(\lambda_k^{\pm}) \to 0$, as $|k| \to \infty$, and we cannot expect uniform exponential decay of the solutions.
- The fact that the decay of the solutions is not exponential is equivalent to the non uniform decay rate: given any non increasing positive function Θ, there is an initial data $\left(\eta^0, w^0\right)$ such that the $H_p^s \times H_p^s-$norm of the corresponding solution decays slower that Θ.

The case $s = 0$

For any $t \geq 0$ and $k \in \mathbb{Z}$, we have that

$$b|\widehat{\eta}_k(t)|^2 + d|\widehat{w}_k(t)|^2 \leq M\left(b|\widehat{\eta}_k^0|^2 + d|\widehat{w}_k^0|^2\right) e^{-2t\min\left\{|\Re(\lambda_k^+)|,\, |\Re(\lambda_k^-)|\right\}},$$

where

$$\min\{|\Re(\lambda_k^+)|,\, |\Re(\lambda_k^-)|\} \geq D > 0,$$

and D is a positive number, depending on the parameters β_1, β_2, α_1, α_2, b and d.

Moreover,

- If $\beta_1\beta_2 = 0$, then $\Re(\lambda_k^\pm) \to 0$, as $|k| \to \infty$, and we cannot expect uniform exponential decay of the solutions.
- The fact that the decay of the solutions is not exponential is equivalent to the non uniform decay rate: given any non increasing positive function Θ, there is an initial data $\left(\eta^0, w^0\right)$ such that the $H_p^s \times H_p^s-$norm of the corresponding solution decays slower that Θ.

The case $s = 0$

For any $t \geq 0$ and $k \in \mathbb{Z}$, we have that

$$b|\widehat{\eta}_k(t)|^2 + d|\widehat{w}_k(t)|^2 \leq M \left(b|\widehat{\eta}_k^0|^2 + d|\widehat{w}_k^0|^2\right) e^{-2t \min\left\{|\Re(\lambda_k^+)|, \, |\Re(\lambda_k^-)|\right\}},$$

where

$$\min\{|\Re(\lambda_k^+)|, \, |\Re(\lambda_k^-)|\} \geq D > 0,$$

and D is a positive number, depending on the parameters β_1, β_2, α_1, α_2, b and d.

Moreover,

- If $\beta_1\beta_2 = 0$, then $\Re(\lambda_k^\pm) \to 0$, as $|k| \to \infty$, and we cannot expect uniform exponential decay of the solutions.
- The fact that the decay of the solutions is not exponential is equivalent to the non uniform decay rate: given any non increasing positive function Θ, there is an initial data $\left(\eta^0, w^0\right)$ such that the $H_p^s \times H_p^s-$norm of the corresponding solution decays slower that Θ.

Let us introduce the space

$$V^s = H_p^s(0,2\pi) \times H_p^s(0,2\pi).$$

Then, the following holds:

Theorem (Micu, P., Preprint, 2016)

The family of linear operators $\{S(t)\}_{t\geq 0}$ *defined by*

$$S(t)(\eta^0, w^0) = \sum_{k\in\mathbb{Z}} (\widehat{\eta}_k(t), \widehat{w}_k(t))e^{ikx}, \qquad (\eta^0, w^0) \in V^s, \tag{10}$$

is an analytic semigroup in V^s *and verifies the following estimate*

$$\|S(t)(\eta^0, w^0)\|_{V^s} \leq C\|(\eta^0, w^0)\|_{V^s}, \tag{11}$$

where C *is a positive constant. Moreover, its infinitesimal generator is the compact operator* $(D(A), A)$, *where* $D(A) = V^s$ *and* A *is given by*

$$A = \begin{pmatrix} \beta_1 \left(I - b\partial_x^2\right)^{-1} M_{\alpha_1} & \left(I - b\partial_x^2\right)^{-1} \partial_x \\ \left(I - d\partial_x^2\right)^{-1} \partial_x & \beta_2 \left(I - b\partial_x^2\right)^{-1} M_{\alpha_2} \end{pmatrix}. \tag{12}$$

Definition

The solutions *decay exponentially in V^s* if there exist two positive constants M and μ, such that

$$\|S(t)(\eta^0, w^0)\|_{V^s} \leq Me^{-\mu t}\|(\eta^0, w^0)\|_{V^s}, \tag{13}$$

$\forall t \geq 0$ and $(\eta^0, w^0) \in V^s$.

We have the following result:

Theorem (Micu, P., Preprint, 2016)

The solutions decay exponentially in V^s if and only if $\alpha_1 = \alpha_2 = 2$ and $\beta_1,\ \beta_2 > 0$. Moreover, μ from (13) is given by

$$\mu = \inf_{k\in\mathbb{Z}} \left\{ \left|\Re(\lambda_k^+)\right|,\ \left|\Re(\lambda_k^-)\right| \right\}, \tag{14}$$

where $\lambda_k^\pm$ are the eigenvalues of the operator A.

Definition

The solutions *decay exponentially in* V^s if there exist two positive constants M and μ, such that

$$\|S(t)(\eta^0, w^0)\|_{V^s} \leq Me^{-\mu t}\|(\eta^0, w^0)\|_{V^s}, \tag{13}$$

$\forall t \geq 0$ and $(\eta^0, w^0) \in V^s$.

We have the following result:

Theorem (Micu, P., Preprint, 2016)

The solutions decay exponentially in V^s *if and only if* $\alpha_1 = \alpha_2 = 2$ *and* $\beta_1,\ \beta_2 > 0$. *Moreover,* μ *from* (13) *is given by*

$$\mu = \inf_{k\in\mathbb{Z}} \left\{ \left|\Re(\lambda_k^+)\right|, \left|\Re(\lambda_k^-)\right| \right\}, \tag{14}$$

where $\lambda_k^{\pm}$ *are the eigenvalues of the operator* A.

Theorem (Micu, P., Preprint, 2016)

Suppose that $\beta_1, \beta_2 \geq 0$, $\beta_1^2 + \beta_2^2 > 0$ *and* $\min\{\alpha_1, \alpha_2\} \in [0, 2)$. *Then, there exists* δ *and* $M > 0$, *such that*

$$\|S(t)(\eta^0, w^0)\|_{V^s} \leq \frac{M}{(1+t)^{\frac{1}{\delta}(q-\frac{1}{2})}} ||(\eta^0, w^0)||_{V^{s+q}}, \, \forall t > 0,$$

where $s \in \mathbb{R}$ *and* $q > \frac{1}{2}$. *Moreover,* $\delta > 0$ *is defined by*

$$\delta = \begin{cases} 2 - \max\{\alpha_1, \alpha_2\} & \text{if } \alpha_1 + \alpha_2 \leq 2, \quad \max\{\alpha_1, \alpha_2\} \leq 1, \\ \\ \max\{\alpha_1, \alpha_2\} & \text{if } \alpha_1 + \alpha_2 \leq 2, \quad \max\{\alpha_1, \alpha_2\} > 1, \\ \\ 2 - \min\{\alpha_1, \alpha_2\} & \text{if } \alpha_1 + \alpha_2 > 2. \end{cases}$$

Remark: If $\alpha_1 = \alpha_2 = 2$ and $\beta_1 = 0$ or $\beta_2 = 0$, then $\delta = 2$.

The nonlinear problem

Theorem (Micu, P., Preprint, 2016)

Let $s \geq 0$ and suppose that $\beta_1, \beta_2 > 0$ and $\alpha_1 = \alpha_2 = 2$. There exist $r > 0$, $C > 0$ and $\mu > 0$, such that, for any $(\eta^0, w^0) \in V^s$, satisfying

$$||(\eta^0, w^0)||_{V^s} \leq r,$$

the system admits a unique solution $(\eta, w) \in C([0, \infty); V^s)$ which verifies

$$\|(\eta(t), w(t))\|_{V^s} \leq Ce^{-\mu t}\|(\eta^0, w^0)\|_{V^s}, \quad t \geq 0.$$

Moreover, μ may be taken as in the linearized problem.

The energy $E(t)$ satisfies

$$\frac{dE}{dt} = -\beta_1 \int_0^{2\pi} (M_{\alpha_1}\eta)\, \eta\, dx - \beta_2 \int_0^{2\pi} (M_{\alpha_2} w)\, w\, dx - \int_0^{2\pi} (\eta w)_x\, \eta\, dx.$$

We define the space

$$Y_{s,\mu} = \{(\eta, w) \in C_b(\mathbb{R}^+; V^s) : e^{\mu t}(\eta, w) \in C_b(\mathbb{R}^+; V^s)\},$$

with the norm

$$||(\eta, w)||_{Y_{s,\mu}} := \sup_{0\leq t<\infty} ||e^{\mu t}(\eta, w)(t)||_{V^s},$$

and the function $\Gamma : Y_{s,\mu} \to Y_{s,\mu}$ by

$$\Gamma(\eta, w)(t) = S(t)(\eta^0, w^0) - \int_0^t S(t-\tau)N(\eta, w)(\tau)\,\mathrm{d}\tau,$$

where $N(\eta, w) = (-(I - b\partial_x^2)^{-1}(\eta w)_x, -(I - d\partial_x^2)^{-1}ww_x)$ and $\{S(t)\}_{t\geq 0}$ is the semigroup associated to the linearized system.

Combining the estimates obtained for the linearized system we have

$$||\Gamma(\eta, w)(t)||_{V^s} \leq Me^{-\mu t}||(\eta^0, w^0)||_{V^s} + MCe^{-\mu t} \sup_{0\leq\tau\leq t} ||e^{\mu\tau}(\eta, w)||_{V^s},$$

for any $t \geq 0$ and some positive constants M and C.

- If we take $(\eta, w) \in B_R(0) \subset Y_{s,\mu}$, the following estimate holds

  $$||\Gamma(\eta, w)||_{Y_{s,\mu}} \leq M||(\eta^0, w^0)||_{V^s} + MC||(\eta, w)||^2_{Y_{s,\mu}} \leq MR + MCR^2.$$

- A similar calculations shows that,

  $$||\Gamma(\eta_1, w_1) - \Gamma(\eta_2, w_2)||_{Y_{s,\mu}} \leq 2RMC||(\eta_1, w_1) - (\eta_2, w_2)||_{Y_{s,\mu}},$$

  for any $(\eta_1, w_1), (\eta_2, w_2) \in B_R(0)$.

A suitable choice of R guarantees that Γ is a contraction.

Combining the estimates obtained for the linearized system we have

$$||\Gamma(\eta, w)(t)||_{V^s} \leq Me^{-\mu t}||(\eta^0, w^0)||_{V^s} + MCe^{-\mu t} \sup_{0 \leq \tau \leq t} ||e^{\mu\tau}(\eta, w)||_{V^s},$$

for any $t \geq 0$ and some positive constants M and C.

- If we take $(\eta, w) \in B_R(0) \subset Y_{s,\mu}$, the following estimate holds

  $$||\Gamma(\eta, w)||_{Y_{s,\mu}} \leq M||(\eta^0, w^0)||_{V^s} + MC||(\eta, w)||^2_{Y_{s,\mu}} \leq MR + MCR^2.$$

- A similar calculations shows that,

  $$||\Gamma(\eta_1, w_1) - \Gamma(\eta_2, w_2)||_{Y_{s,\mu}} \leq 2RMC||(\eta_1, w_1) - (\eta_2, w_2)||_{Y_{s,\mu}},$$

  for any $(\eta_1, w_1), (\eta_2, w_2) \in B_R(0)$.

A suitable choice of R guarantees that Γ is a contraction.

Dirichlet boundary conditions

We consider the BBM-BBM system

$$\begin{aligned} &\eta_t + w_x - b\eta_{txx} + \varepsilon a(x)\eta = 0, \qquad && x \in (0, 2\pi),\ t > 0, \\ &w_t + \eta_x - d w_{txx} = 0, && x \in (0, 2\pi),\ t > 0, \end{aligned}$$

with boundary conditions

$$\eta(t, 0) = \eta(t, 2\pi) = w(t, 0) = w(t, 2\pi) = 0, \quad t > 0,$$

and initial conditions

$$\eta(0, x) = \eta^0(x), \quad w(0, x) = w^0(x), \qquad x \in (0, 2\pi).$$

We assume that

- $b, d > 0$ and $\varepsilon > 0$ are parameters.
- $a = a(x)$ is a nonnegative real-valued function satisfying

$$a(x) \geq a_0 > 0, \qquad \text{in } \Omega \subset (0, 2\pi).$$

Dirichlet boundary conditions

We consider the BBM-BBM system

$$\begin{aligned} &\eta_t + w_x - b\eta_{txx} + \varepsilon a(x)\eta = 0, \qquad && x \in (0, 2\pi),\ t > 0, \\ &w_t + \eta_x - dw_{txx} = 0, && x \in (0, 2\pi),\ t > 0, \end{aligned}$$

with boundary conditions

$$\eta(t, 0) = \eta(t, 2\pi) = w(t, 0) = w(t, 2\pi) = 0, \quad t > 0,$$

and initial conditions

$$\eta(0, x) = \eta^0(x), \quad w(0, x) = w^0(x), \qquad x \in (0, 2\pi).$$

We assume that

- $b, d > 0$ and $\varepsilon > 0$ are parameters.
- $a = a(x)$ is a nonnegative real-valued function satisfying

$$a(x) \geq a_0 > 0, \qquad \text{in } \Omega \subset (0, 2\pi).$$

The energy associated to the model is given by

$$E(t) = \frac{1}{2}\int_0^{2\pi} (\eta^2 + b\eta_x^2 + w^2 + dw_x^2)dx \qquad (15)$$

and we can (formally) deduce that

$$\frac{d}{dt}E(t) = -\varepsilon \int_0^{2\pi} a(x)\eta^2(t,x)dx. \qquad (16)$$

Does $E(t)$ converges to zero as $t \to \infty$?

If yes, at which rates it decays?

The energy associated to the model is given by

$$E(t) = \frac{1}{2} \int_0^{2\pi} (\eta^2 + b\eta_x^2 + w^2 + dw_x^2)dx \tag{15}$$

and we can (formally) deduce that

$$\frac{d}{dt}E(t) = -\varepsilon \int_0^{2\pi} a(x)\eta^2(t,x)dx. \tag{16}$$

Does $E(t)$ converges to zero as $t \to \infty$?

If yes, at which rates it decays?

Lack of Compactness

There exist $T > 0$ and $C > 0$ such that

$$E(0) \leq C \int_0^T \Big[\int_0^{2\pi} \varepsilon a(x)\eta^2(x,t)dx \Big] dt, \qquad (17)$$

for every finite energy solution. Indeed, from (17) and the energy dissipation law, we have that

$$E(T) \leq \frac{C}{C+1} E(0). \qquad (18)$$

Since $E(t) \leq E(kT) \leq \gamma^k E(0)$, for $0 < \gamma < 1$ and $k > 0$,

$$E(t) \leq \frac{1}{\gamma} E(0) e^{\frac{\ln \gamma}{T} t}, \text{ where } \gamma = \frac{C}{C+1}. \qquad (19)$$

(17) does not hold for the BBM-BBM model.

Lack of Compactness

There exist $T > 0$ and $C > 0$ such that

$$E(0) \leq C \int_0^T \Big[\int_0^{2\pi} \varepsilon a(x) \eta^2(x,t) dx \Big] dt, \qquad (17)$$

for every finite energy solution. Indeed, from (17) and the energy dissipation law, we have that

$$E(T) \leq \frac{C}{C+1} E(0). \qquad (18)$$

Since $E(t) \leq E(kT) \leq \gamma^k E(0)$, for $0 < \gamma < 1$ and $k > 0$,

$$E(t) \leq \frac{1}{\gamma} E(0) e^{\frac{\ln \gamma}{T} t}, \text{ where } \gamma = \frac{C}{C+1}. \qquad (19)$$

(17) does not hold for the BBM-BBM model.

Lack of Compactness

There exist $T > 0$ and $C > 0$ such that

$$E(0) \leq C \int_0^T \Big[\int_0^{2\pi} \varepsilon a(x) \eta^2(x,t) dx \Big] dt, \tag{17}$$

for every finite energy solution. Indeed, from (17) and the energy dissipation law, we have that

$$E(T) \leq \frac{C}{C+1} E(0). \tag{18}$$

Since $E(t) \leq E(kT) \leq \gamma^k E(0)$, for $0 < \gamma < 1$ and $k > 0$,

$$E(t) \leq \frac{1}{\gamma} E(0) e^{\frac{\ln \gamma}{T} t}, \text{ where } \gamma = \frac{C}{C+1}. \tag{19}$$

(17) does not hold for the BBM-BBM model.

Main results

We assume that $a = a(x)$ is nonnegative and

$$\begin{aligned} & a(x) \geq a_0 > 0, \text{ in } \Omega \subset (0, 2\pi), \\ & a \in W^{2,\infty}(0, 2\pi), \text{ with } a(0) = a'(0) = 0. \end{aligned} \tag{20}$$

Theorem (Micu, P., Journal d'Analyse Mathématique)

There exits ε_0*, such that, for any* $\varepsilon \in (0, \varepsilon_0)$ *and* (η^0, w^0) *in* $(H_0^1(0, 2\pi))^2$*, the solution* (η, w) *of the system verifies*

$$\lim_{t \to \infty} \|(\eta(t), w(t))\|_{(H_0^1(0,2\pi))^2} = 0.$$

Moreover, the decay of the energy is not exponential, i. e., there exists no positive constants M *and* ω*, such that*

$$\|(\eta(t), w(t))\|_{(H_0^1(0,2\pi))^2} \leq M e^{-\omega t}, \qquad t \geq 0.$$

Spectral analysis and eigenvectors expansion of solutions

Since

$$\begin{array}{ll}(I-b\partial_x^2)\eta_t + w_x + \varepsilon a(x)\eta = 0, & x\in(0,2\pi),\ t>0,\\ (I-d\partial_x^2)w_t + \eta_x = 0, & x\in(0,2\pi),\ t>0,\end{array}$$

the system can be written as

$$\begin{array}{l}U_t + \mathcal{A}_\varepsilon U = 0,\\ U(0) = U_0,\end{array}$$

where $\mathcal{A}_\varepsilon : (H_0^1(0,2\pi))^2 \to (H_0^1(0,2\pi))^2$ is given by

$$\mathcal{A}_\varepsilon = \begin{pmatrix} \varepsilon\left(I-b\partial_x^2\right)^{-1} a(\cdot)\, I & \left(I-b\partial_x^2\right)^{-1}\partial_x \\ \left(I-d\partial_x^2\right)^{-1}\partial_x & 0 \end{pmatrix}. \qquad (21)$$

We have that

$\mathcal{A}_\varepsilon \in \mathcal{L}((H_0^1(0,2\pi))^2)$ and $\mathcal{A}_\varepsilon$ is a compact operator.

The operator $\mathcal{A}_\varepsilon$ has a family of eigenvalues $(\lambda_n)_{n\geq 1}$, such that:

1. $|\Re(\lambda_n)| \leq \dfrac{c}{|n|^2}, \ \forall\, n \geq n_0,$ and $\Re(\lambda_n) < 0, \ \ \forall\, n.$
2. The corresponding eigenfunctions $(\Phi_n)_{n\geq 1}$ form a Riesz basis in $(H_0^1(0,2\pi))^2$.

Then,

$$(\eta(t), w(t)) = \sum_{n\geq 1} a_n e^{\lambda_n t}\Phi_n$$

and

$$c_1 \sum_{n\geq n_0} |a_n|^2 e^{2\Re(\lambda_n)t} \leq \|(\eta(t), w(t))\|^2_{(H_0^1(0,2\pi))^2} \leq c_2 \sum_{n\geq 1} |a_n|^2 e^{2\Re(\lambda_n)t}.$$

The operator $\mathcal{A}_\varepsilon$ has a family of eigenvalues $(\lambda_n)_{n\geq 1}$, such that:

1. $|\Re(\lambda_n)| \leq \dfrac{c}{|n|^2}, \ \forall\, n \geq n_0,$ and $\Re(\lambda_n) < 0, \quad \forall\, n.$
2. The corresponding eigenfunctions $(\Phi_n)_{n\geq 1}$ form a Riesz basis in $(H_0^1(0,2\pi))^2$.

Then,

$$(\eta(t), w(t)) = \sum_{n\geq 1} a_n e^{\lambda_n t}\Phi_n$$

and

$$c_1 \sum_{n\geq n_0} |a_n|^2 e^{2\Re(\lambda_n)t} \leq \|(\eta(t), w(t))\|^2_{(H_0^1(0,2\pi))^2} \leq c_2 \sum_{n\geq 1} |a_n|^2 e^{2\Re(\lambda_n)t}.$$

The operator $\mathcal{A}_\varepsilon$ has a family of eigenvalues $(\lambda_n)_{n\geq 1}$, such that:

1. $|\Re(\lambda_n)| \leq \dfrac{c}{|n|^2}, \ \forall\, n \geq n_0,$ and $\Re(\lambda_n) < 0, \quad \forall\, n.$
2. The corresponding eigenfunctions $(\Phi_n)_{n\geq 1}$ form a Riesz basis in $(H_0^1(0, 2\pi))^2$.

Then,

$$(\eta(t), w(t)) = \sum_{n\geq 1} a_n e^{\lambda_n t} \Phi_n$$

and

$$c_1 \sum_{n\geq n_0} |a_n|^2 e^{2\Re(\lambda_n)t} \leq \|(\eta(t), w(t))\|^2_{(H_0^1(0,2\pi))^2} \leq c_2 \sum_{n\geq 1} |a_n|^2 e^{2\Re(\lambda_n)t}.$$

Theorem (Micu, P., Journal d'Analyse Mathématique)

Let (η, w) *be a finite energy solution of the system with* $a \equiv 0$*. If there exist* $T > 0$ *and an open set* $\Omega \subset (0, 2\pi)$*, such that*

$$\eta(x,t) = 0 \quad , \forall \ (t,x) \in (0,T) \times \Omega, \tag{22}$$

then

$$\eta = w \equiv 0 \quad \textit{in} \quad \mathbb{R} \times (0, 2\pi).$$

Idea of the proof:

- $\mathcal{A}_\varepsilon$ is a compact operator in $(H_0^1(0,2\pi))^2 \Rightarrow$ analyticity in time of solutions $\Rightarrow$ property (22) holds for $t \in \mathbb{R}$.
- Fourier decomposition of solutions.
- Unique continuation principle for each eigenfunction.

Theorem (Micu, P., Journal d'Analyse Mathématique)

Let (η, w) *be a finite energy solution of the system with* $a \equiv 0$*. If there exist* $T > 0$ *and an open set* $\Omega \subset (0, 2\pi)$*, such that*

$$\eta(x,t) = 0 \quad , \forall \ (t,x) \in (0,T) \times \Omega, \tag{22}$$

then

$$\eta = w \equiv 0 \quad \textit{in} \quad \mathbb{R} \times (0, 2\pi).$$

Idea of the proof:

- $\mathcal{A}_\varepsilon$ is a compact operator in $(H_0^1(0, 2\pi))^2 \Rightarrow$ analyticity in time of solutions $\Rightarrow$ property (22) holds for $t \in \mathbb{R}$.
- Fourier decomposition of solutions.
- Unique continuation principle for each eigenfunction.

Theorem (Micu, P., Journal d'Analyse Mathématique)

Let (η, w) *be a finite energy solution of the system with* $a \equiv 0$*. If there exist* $T > 0$ *and an open set* $\Omega \subset (0, 2\pi)$*, such that*

$$\eta(x,t) = 0 \quad , \forall\ (t,x) \in (0,T) \times \Omega, \tag{22}$$

then

$$\eta = w \equiv 0 \quad \textit{in} \quad \mathbb{R} \times (0, 2\pi).$$

Idea of the proof:

- $\mathcal{A}_\varepsilon$ is a compact operator in $(H_0^1(0,2\pi))^2 \Rightarrow$ analyticity in time of solutions $\Rightarrow$ property (22) holds for $t \in \mathbb{R}$.
- Fourier decomposition of solutions.
- Unique continuation principle for each eigenfunction.

Theorem (Micu, P., Journal d'Analyse Mathématique)

Let (η, w) *be a finite energy solution of the system with* $a \equiv 0$. *If there exist* $T > 0$ *and an open set* $\Omega \subset (0, 2\pi)$, *such that*

$$\eta(x,t) = 0 \quad , \forall\ (t,x) \in (0,T) \times \Omega, \tag{22}$$

then

$$\eta = w \equiv 0 \quad \textit{in} \quad \mathbb{R} \times (0, 2\pi).$$

Idea of the proof:

- $\mathcal{A}_\varepsilon$ is a compact operator in $(H_0^1(0, 2\pi))^2 \Rightarrow$ analyticity in time of solutions $\Rightarrow$ property (22) holds for $t \in \mathbb{R}$.
- Fourier decomposition of solutions.
- Unique continuation principle for each eigenfunction.

Main steps of the proof

1. The spectrum of the differential operator corresponding $\mathcal{A}_\varepsilon$ is located in the left open half-plane of the complex plane. We also obtain the asymptotic behavior (of the spectrum) .

2. There exists a Riesz basis $(\Phi_m)_{m\geq 1} \subset (H_0^1(0,2\pi))^2$ consisting of generalized eigenvectors of the differential operator $\mathcal{A}_\varepsilon$.

We obtain the asymptotic behavior of the high eigenfunctions and prove that they are quadratically close to a Riesz basis $(\Psi_m)_{m\geq 1}$ formed by eigenvectors of a well chosen dissipative differential operator with constant coefficients:

$$\sum_{m\geq N+1} ||\Phi_m - \Psi_m||^2_{(H_0^1(0,2\pi))^2} \sim \frac{1}{m^2}.$$

This is done by using less common two dimensional versions of the Shooting Method and Rouché's Theorem.

Main steps of the proof

1. The spectrum of the differential operator corresponding $\mathcal{A}_\varepsilon$ is located in the left open half-plane of the complex plane. We also obtain the asymptotic behavior (of the spectrum) .

2. There exists a Riesz basis $(\Phi_m)_{m\geq 1} \subset (H_0^1(0,2\pi))^2$ consisting of generalized eigenvectors of the differential operator $\mathcal{A}_\varepsilon$.

We obtain the asymptotic behavior of the high eigenfunctions and prove that they are quadratically close to a Riesz basis $(\Psi_m)_{m\geq 1}$ formed by eigenvectors of a well chosen dissipative differential operator with constant coefficients:

$$\sum_{m\geq N+1} ||\Phi_m - \Psi_m||^2_{(H_0^1(0,2\pi))^2} \sim \frac{1}{m^2}.$$

This is done by using less common two dimensional versions of the Shooting Method and Rouché's Theorem.

Main steps of the proof

1. The spectrum of the differential operator corresponding $\mathcal{A}_\varepsilon$ is located in the left open half-plane of the complex plane. We also obtain the asymptotic behavior (of the spectrum) .

2. There exists a Riesz basis $(\Phi_m)_{m\geq 1} \subset (H_0^1(0,2\pi))^2$ consisting of generalized eigenvectors of the differential operator $\mathcal{A}_\varepsilon$.

We obtain the asymptotic behavior of the high eigenfunctions and prove that they are quadratically close to a Riesz basis $(\Psi_m)_{m\geq 1}$ formed by eigenvectors of a well chosen dissipative differential operator with constant coefficients:

$$\sum_{m\geq N+1} ||\Phi_m - \Psi_m||^2_{(H_0^1(0,2\pi))^2} \sim \frac{1}{m^2}.$$

This is done by using less common two dimensional versions of the Shooting Method and Rouché's Theorem.

Main steps of the proof

1. The spectrum of the differential operator corresponding $\mathcal{A}_\varepsilon$ is located in the left open half-plane of the complex plane. We also obtain the asymptotic behavior (of the spectrum) .

2. There exists a Riesz basis $(\Phi_m)_{m\geq 1} \subset (H_0^1(0,2\pi))^2$ consisting of generalized eigenvectors of the differential operator $\mathcal{A}_\varepsilon$.

We obtain the asymptotic behavior of the high eigenfunctions and prove that they are quadratically close to a Riesz basis $(\Psi_m)_{m\geq 1}$ formed by eigenvectors of a <u>well chosen</u> dissipative differential operator with constant coefficients:

$$\sum_{m\geq N+1} ||\Phi_m - \Psi_m||^2_{(H_0^1(0,2\pi))^2} \sim \frac{1}{m^2}.$$

This is done by using less common two dimensional versions of the Shooting Method and Rouché's Theorem.

To control the low frequencies we use a result originally proved for a unbounded operator:

- B. Z. Guo, Riesz basis approach to the stabilization of a flexible beam with a tip mass, SIAM J. Control Optim. 39 (2001), 1736–1747.
- B. Z. Guo and R. Yu, The Riesz basis property of discrete operators and application to a Euler-Bernoulli beam equation with boundary linear feedback control, IMA J. Math. Control Inform. 18 (2001), 241–251.

It was extended to the bounded case:

- X. Zhang and E. Zuazua, Unique continuation for the linearized Benjamin-Bona-Mahony equation with space-dependent potential, Math. Ann. 325 (2003), 543-582.

We consider the spectral problem

$$\mathcal{A}_\varepsilon \begin{pmatrix} \eta \\ w \end{pmatrix} = \mu \begin{pmatrix} \eta \\ w \end{pmatrix},$$

where

$$\mathcal{A}_\varepsilon : (H_0^1(0,2\pi))^2 \to (H_0^1(0,2\pi))^2,$$

which is equivalent to the BVP

$$\begin{cases} \eta - b\eta_{xx} + \mu w_x + \varepsilon a(x)\mu\eta = 0 & \text{for } x \in (0,2\pi) \\ w - dw_{xx} + \mu\eta_x = 0 & \text{for } x \in (0,2\pi) \\ \eta(0) = \eta(2\pi) = 0 \\ w(0) = w(2\pi) = 0. \end{cases} \tag{23}$$

From (23) we obtain a family of eigenvalues and eigenfunctions.......

A two dimensional "shooting method"

For each $(\mu, \gamma) \in \mathbb{C}^2$, consider the IVP

$$\begin{cases} \eta - b\eta_{xx} + \mu w_x + \varepsilon a(x)\mu\eta = 0 & \text{for } x \in (0, 2\pi) \\ w - d w_{xx} + \mu \eta_x = 0 & \text{for } x \in (0, 2\pi) \\ \eta(0) = 0, \ \eta_x(0) = 1 \\ w(0) = 0, \ w_x(0) = \gamma. \end{cases} \tag{24}$$

and the map $F : \mathbb{C}^2 \to \mathbb{C}^2$, given by $F(\mu, \gamma) = \begin{pmatrix} \eta(\mu, \gamma, 2\pi) \\ w(\mu, \gamma, 2\pi) \end{pmatrix}$.

Then, μ is an eigenvalue of (23) with corresponding eigenfunction $\begin{pmatrix} \eta \\ w \end{pmatrix}$, if and only if,

$$F(\mu, \gamma) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The spectrum of the differential operator with variable potential is given by the zeros of the map F.

A two dimensional "shooting method"

For each $(\mu, \gamma) \in \mathbb{C}^2$, consider the IVP

$$\begin{cases} \eta - b\eta_{xx} + \mu w_x + \varepsilon a(x)\mu\eta = 0 & \text{for } x \in (0, 2\pi) \\ w - dw_{xx} + \mu\eta_x = 0 & \text{for } x \in (0, 2\pi) \\ \eta(0) = 0,\ \eta_x(0) = 1 \\ w(0) = 0,\ w_x(0) = \gamma. \end{cases} \tag{24}$$

and the map $F : \mathbb{C}^2 \to \mathbb{C}^2$, given by $F(\mu, \gamma) = \begin{pmatrix} \eta(\mu, \gamma, 2\pi) \\ w(\mu, \gamma, 2\pi) \end{pmatrix}$.

Then, μ is an eigenvalue of (23) with corresponding eigenfunction $\begin{pmatrix} \eta \\ w \end{pmatrix}$, if and only if,

$$F(\mu, \gamma) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The spectrum of the differential operator with variable potential is given by the zeros of the map F.

A two dimensional "shooting method"

For each $(\mu, \gamma) \in \mathbb{C}^2$, consider the IVP

$$\begin{cases} \eta - b\eta_{xx} + \mu w_x + \varepsilon a(x)\mu\eta = 0 & \text{for } x \in (0, 2\pi) \\ w - dw_{xx} + \mu\eta_x = 0 & \text{for } x \in (0, 2\pi) \\ \eta(0) = 0, \ \eta_x(0) = 1 \\ w(0) = 0, \ w_x(0) = \gamma. \end{cases} \tag{24}$$

and the map $F : \mathbb{C}^2 \to \mathbb{C}^2$, given by $F(\mu, \gamma) = \begin{pmatrix} \eta(\mu, \gamma, 2\pi) \\ w(\mu, \gamma, 2\pi) \end{pmatrix}$.

Then, μ is an eigenvalue of (23) with corresponding eigenfunction $\begin{pmatrix} \eta \\ w \end{pmatrix}$, if and only if,

$$F(\mu, \gamma) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The spectrum of the differential operator with variable potential is given by the zeros of the map F.

Next, we define the map $G = G(\sigma, \beta)$, associated to a spectral problem with constant coefficients, for which the eigenfunctions form a Riesz basis:

$$\begin{cases} -b\psi_{xx} + \sigma u_x + \varepsilon a_0 \sigma \psi = 0 & \text{for } x \in (0, 2\pi) \\ -du_{xx} + \sigma \psi_x = 0 & \text{for } x \in (0, 2\pi) \\ \psi(0) = 0, \ \psi_x(0) = 1 & \\ u(0) = 0, \ u_x(0) = \beta. & \end{cases} \tag{25}$$

The map $G : \mathbb{C}^2 \to \mathbb{C}^2$ is given by $G(\sigma, \beta) = \begin{pmatrix} \psi(\sigma, \beta, 2\pi) \\ u(\sigma, \beta, 2\pi) \end{pmatrix}$.

The corresponding BVP $(\psi(0) = u(0) = \psi(2\pi) = u(2\pi) = 0)$

- has a double indexed family of complex eigenvalues $(\sigma_n^j)_{n \in \mathbb{Z}^*, \, j \in \{1,2\}}$ and
- the family of corresponding eigenfunctions $(\Psi_n^j)_{n \in \mathbb{Z}^*, \, j \in \{1,2\}}$ forms a Riesz basis in $(H_0^1)^2$.

Next, we define the map $G = G(\sigma, \beta)$, associated to a spectral problem with constant coefficients, for which the eigenfunctions form a Riesz basis:

$$\begin{cases} -b\psi_{xx} + \sigma u_x + \varepsilon a_0 \sigma \psi = 0 & \text{for } x \in (0, 2\pi) \\ -du_{xx} + \sigma \psi_x = 0 & \text{for } x \in (0, 2\pi) \\ \psi(0) = 0, \ \psi_x(0) = 1 \\ u(0) = 0, \ u_x(0) = \beta. \end{cases} \tag{25}$$

The map $G : \mathbb{C}^2 \to \mathbb{C}^2$ is given by $G(\sigma, \beta) = \begin{pmatrix} \psi(\sigma, \beta, 2\pi) \\ u(\sigma, \beta, 2\pi) \end{pmatrix}$.

The corresponding BVP $(\psi(0) = u(0) = \psi(2\pi) = u(2\pi) = 0)$

- has a double indexed family of complex eigenvalues $(\sigma_n^j)_{n \in \mathbb{Z}^*,\, j \in \{1,2\}}$ and
- the family of corresponding eigenfunctions $(\Psi_n^j)_{n \in \mathbb{Z}^*,\, j \in \{1,2\}}$ forms a Riesz basis in $(H_0^1)^2$.

Theorem (N. G. Lloyd, J. London Math. Soc. 2 (1979))

Let $\mathcal{D}$ be a bounded domain in $\mathbb{C}^N$ and h, G holomorphic maps of $\overline{\mathcal{D}}$ into $\mathbb{C}^N$ such that $\|h(z)\| < \|G(z)\|$ for $z \in \partial\mathcal{D}$. Then G has finitely many zeros in $\mathcal{D}$, and G and $h+G$ have the same number of zeros in $\mathcal{D}$, counting multiplicity.

Given a zero (σ_n^j, β_n^j) of the map G, we define the domain

$$D_n^j(\delta) = \left\{ (\mu,\gamma) \in \mathbb{C}^2 \,:\, \sqrt{|\mu - \sigma_n^j|^2 + |\gamma - \beta_n^j|^2} \le \frac{\delta}{|n|} \right\}.$$

If $h = F - G$ we obtain

- $\|G(\mu,\gamma)\| \ge \dfrac{C}{n^2}$,
- $\|F(\mu,\gamma) - G(\mu,\gamma)\| \le \dfrac{C}{n^2}, \quad \forall\, (\mu,\gamma) \in \partial D_n^j(\delta)$.

Theorem (N. G. Lloyd, J. London Math. Soc. 2 (1979))

Let $\mathcal{D}$ be a bounded domain in $\mathbb{C}^N$ and h, G holomorphic maps of $\overline{\mathcal{D}}$ into $\mathbb{C}^N$ such that $\|h(z)\| < \|G(z)\|$ for $z \in \partial\mathcal{D}$. Then G has finitely many zeros in $\mathcal{D}$, and G and $h + G$ have the same number of zeros in $\mathcal{D}$, counting multiplicity.

Given a zero (σ_n^j, β_n^j) of the map G, we define the domain

$$D_n^j(\delta) = \left\{(\mu, \gamma) \in \mathbb{C}^2 \,:\, \sqrt{|\mu - \sigma_n^j|^2 + |\gamma - \beta_n^j|^2} \leq \frac{\delta}{|n|}\right\}.$$

If $h = F - G$ we obtain

- $\|G(\mu, \gamma)\| \geq \dfrac{C}{n^2}$,
- $\|F(\mu, \gamma) - G(\mu, \gamma)\| \leq \dfrac{C}{n^2}, \quad \forall\, (\mu, \gamma) \in \partial D_n^j(\delta)$.

Finally, we obtain an ansatz $\begin{pmatrix} \varphi(\mu,\gamma,x) \\ z(\mu,\gamma,x) \end{pmatrix}$ for the solutions of the IVP

$$\begin{cases} \eta - b\eta_{xx} + \mu w_x + \varepsilon a(x)\mu\eta = 0 & \text{for } x \in (0,2\pi) \\ w - dw_{xx} + \mu\eta_x = 0 & \text{for } x \in (0,2\pi) \\ \eta(0) = 0,\ \eta_x(0) = 1 \\ w(0) = 0,\ w_x(0) = \gamma. \end{cases}$$

More precisely, $\begin{pmatrix} \eta(\mu,\gamma,x) \\ w(\mu,\gamma,x) \end{pmatrix} = \begin{pmatrix} \varphi(\mu,\gamma,x) \\ z(\mu,\gamma,x) \end{pmatrix} + \mathcal{O}\left(\frac{1}{\mu^2}\right)$, where

$$\begin{cases} \varphi(\mu,\gamma,x) = & \frac{\sqrt{bd}}{\mu}\sinh(\alpha(x)) + \frac{\gamma d}{\mu}\cosh(\alpha(x)) - \frac{\gamma d}{\mu + da(x)} \\ z(\mu,\gamma,x) = & \frac{b}{\mu}\left(\cosh(\alpha(x)) - 1\right) + \frac{\gamma\sqrt{bd}}{\mu}\sinh(\alpha(x)) + \frac{\gamma d}{\mu}\int_0^x a(s)ds, \end{cases}$$

and $\alpha(x) = \frac{\mu x}{\sqrt{bd}} + \frac{1}{2}\sqrt{\frac{d}{b}}\int_0^x a(s)ds$.

$$\|F(\mu,\gamma) - G(\mu,\gamma)\|$$

$$\leq \left\| F(\mu,\gamma) - \left(\begin{array}{c} \varphi(\mu\gamma, 2\pi) \\ z(\mu,\gamma,2\pi) \end{array} \right) \right\| + \left\| \left(\begin{array}{c} \varphi(\mu\gamma, 2\pi) \\ z(\mu,\gamma,2\pi) \end{array} \right) - G(\mu,\gamma) \right\|$$

$\leq \frac{C_1}{|\mu|^2}$ (from the ansatz property)

$+\frac{C_2}{|\mu|^2}$ (by choosing conveniently the constant potential $a_0 = \frac{1}{2\pi}\int_0^{2\pi} a(s)ds$)

$$\leq \frac{C}{|\mu|^2}.$$

$$\|F(\mu,\gamma) - G(\mu,\gamma)\|$$

$$\leq \left\| F(\mu,\gamma) - \begin{pmatrix} \varphi(\mu\gamma, 2\pi) \\ z(\mu,\gamma,2\pi) \end{pmatrix} \right\| + \left\| \begin{pmatrix} \varphi(\mu\gamma, 2\pi) \\ z(\mu,\gamma,2\pi) \end{pmatrix} - G(\mu,\gamma) \right\|$$

$$\leq \frac{C_1}{|\mu|^2}$$ (from the ansatz property)

$$+\frac{C_2}{|\mu|^2}$$ (by choosing conveniently the constant potential $a_0 = \frac{1}{2\pi}\int_0^{2\pi} a(s)ds$)

$$\leq \frac{C}{|\mu|^2}.$$

$$\|F(\mu,\gamma) - G(\mu,\gamma)\|$$

$$\leq \left\| F(\mu,\gamma) - \begin{pmatrix} \varphi(\mu\gamma, 2\pi) \\ z(\mu,\gamma,2\pi) \end{pmatrix} \right\| + \left\| \begin{pmatrix} \varphi(\mu\gamma, 2\pi) \\ z(\mu,\gamma,2\pi) \end{pmatrix} - G(\mu,\gamma) \right\|$$

$$\leq \frac{C_1}{|\mu|^2} \text{ (from the ansatz property)}$$

$$+\frac{C_2}{|\mu|^2} \text{ (by choosing conveniently the constant potential } a_0 = \frac{1}{2\pi}\int_0^{2\pi} a(s)ds\text{)}$$

$$\leq \frac{C}{|\mu|^2}.$$

Similar strategies have been successfully used by

- S. Cox and E. Zuazua, The rate at which energy decays in a damped string, Comm. Partial Differential Equations 19 (1994), 213–243.
- A. Benaddi and B. Rao, Energy decay rate of wave equations with indefinite damping, J. Differential Equations 161 (2000), 337–357.
- X. Zhang and E. Zuazua, Unique continuation for the linearized Benjamin-Bona-Mahony equation with space-dependent potential, Math. Ann. 325 (2003), 543–582.

Remarks and open problems

- **Dirichlet boundary conditions:**
- Less regularity for the potential a.
- Stabilization results for the nonlinear problem.
- Dissipative mechanisms, like $-[a(x)\varphi_x]_x$, ensures the uniform decay?
- The mixed KdV-BBM system is exponentially stabilizable?
- **Periodic boundary conditions:**
- The decay of solutions of a nonlinear problem with a linearized part that does not decay uniformly.
- Unique Continuation Property for the BBM-BBM system.
- Dissipative mechanisms, like $a(x)\varphi$ or $-[a(x)\varphi_x]_x$, ensures the uniform decay?

- Boundary stabilization results.
- The 2-d Boussinesq system.....

Remarks and open problems

- **Dirichlet boundary conditions:**
- Less regularity for the potential a.
- Stabilization results for the nonlinear problem.
- Dissipative mechanisms, like $-[a(x)\varphi_x]_x$, ensures the uniform decay?
- The mixed KdV-BBM system is exponentially stabilizable?
- **Periodic boundary conditions:**
- The decay of solutions of a nonlinear problem with a linearized part that does not decay uniformly.
- Unique Continuation Property for the BBM-BBM system.
- Dissipative mechanisms, like $a(x)\varphi$ or $-[a(x)\varphi_x]_x$, ensures the uniform decay?

- Boundary stabilization results.
- The 2-d Boussinesq system.....

Remarks and open problems

- **Dirichlet boundary conditions:**
- Less regularity for the potential a.
- Stabilization results for the nonlinear problem.
- Dissipative mechanisms, like $-[a(x)\varphi_x]_x$, ensures the uniform decay?
- The mixed KdV-BBM system is exponentially stabilizable?
- **Periodic boundary conditions:**
- The decay of solutions of a nonlinear problem with a linearized part that does not decay uniformly.
- Unique Continuation Property for the BBM-BBM system.
- Dissipative mechanisms, like $a(x)\varphi$ or $-[a(x)\varphi_x]_x$, ensures the uniform decay?

- Boundary stabilization results.
- The 2-d Boussinesq system.....