# Boundary value problems with $\phi$-Laplacians

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# Contents

# Chapter 1

# Rezumat (Abstract)

**Rezumat.** In aceasta lucrare prezentam unele rezultate de existenta, unicitate si multiplicitate pentru probleme la limita neliniare cu $\phi$-Laplacieni. Aceste probleme au aplicatii in geometria diferentiala si teoria relativitatii.

Primul capitol contine un rezumat a ceea ce se prezinta in lucrare iar in al doilea capitol se gaseste o introducere a rezultatelor principale.

In capitolul 3, utilizand teorema de punct fix Schauder, demonstram rezultate de existenta a solutiilor radiale pentru probleme Dirichlet in bila unitate si in domenii circulare, asociate operatorilor curburii medii in spatii euclidieni si Minkowski.

In capitolul 4 studiem existenta solutiilor radiale pentru probleme Neumann pe bile sau domenii circulare, asociate operatorilor curburii medii in spatii euclidieni si Minkowski. Instrumentul principal este gradul Leray - Schauder aplicat unor operatori de punct fix asociati problemelor considerate.

In capitolul 5, studiem existenta si multiplicitate solutiilor radiale pentru probleme Neumann in bile si domenii circulare, asociate unor perturbatii de tip pendul fortat ale operatorilor curburii medii in spatii euclidieni si Minkowski. Utilizam gradul Leray - Schauder si metoda sub - supra solutiilor.

In capitolul 6, aratam ca daca $\mathcal{A} \subset \mathbb{R}^N$ este un domeniu circular sau o bila centrata in origine, atunci problema Neumann omogena pe $\mathcal{A}$ pentru ecuatia cu date continue

$$\operatorname{div}\left(\frac{\nabla v}{\sqrt{1-|\nabla v|^2}}\right) = g(|x|, v) + h(|x|)$$

are cel putin o solutie radiala cand $g(|x|, \cdot)$ are o primitiva periodica si $\int_{\mathcal{A}} h(|x|)\, dx = 0$. Se folosesc metoda directa a calculului variational, teoria gradului topologic si unele inegalitati variationale.

In capitolul 7, motivati de existenta solutiilor radiale pentru probleme de tipul

$$\operatorname{div}\left(\frac{\nabla v}{\sqrt{1-|\nabla v|^2}}\right) = g(|x|, v) \quad \text{in} \quad \mathcal{A}, \quad \frac{\partial v}{\partial \nu} = 0 \quad \text{pe} \quad \partial \mathcal{A},$$

unde $0 \leq R_1 < R_2$, $\mathcal{A} = \{x \in \mathbb{R}^N : R_1 \leq |x| \leq R_2\}$ si $g : [R_1, R_2] \times \mathbb{R} \to \mathbb{R}$ este continua, studiem probleme mai generale de tipul

$$[r^{N-1}\phi(u')]' = r^{N-1}g(r,u), \quad u'(R_1) = 0 = u'(R_2),$$

unde $\phi := \Phi' : (-a, a) \to \mathbb{R}$ este un homeomorfism crescator cu $\phi(0) = 0$ si functia continua $\Phi : [-a, a] \to \mathbb{R}$ este de clasa $C^1$ pe $(-a, a)$. Functionala asociata problemei de mai sus este definita pe spatiul functiilor continue pe $[R_1, R_2]$ si este suma unei functionale de clasa $C^1$ cu o functionala semicontinua inferior si convexa. Utilizand teoria punctului critica Szulkin, obtinem diferite rezultate de existenta pentru anumite clase de neliniaritati. Se discuta si problema periodica.

In capitolul urmator, utilizam teoria punctului critic pentru perturbatii semicontinue si convexe ale $C^1$-functionalelor pentru a demonstra multiplicitatea solutiilor radiale ale unor probleme Neumann cu parametru ce contin operatorul $v \mapsto \mathrm{div}\left(\frac{\nabla v}{\sqrt{1-|\nabla v|^2}}\right)$. Se demonstreaza si rezultate similare pentru problema periodica.

In capitolul 9 analizam o clasa de functionale $I$ pe un spatiu Banach $X$ de tipul $I = \Psi + \mathcal{G}$, cu $\Psi : X \to (-\infty, +\infty]$ proprie, convexa, semicontinua inferior si $\mathcal{G} : X \to \mathbb{R}$ de clasa $C^1$. In plus, $I$ este $G$-invarianta in raport cu un subgrup discret $G \subset X$ cu dim (span $G$) $= N$. Daca unele conditii aditionale sunt satisfacute, atunci aratam ca $I$ are $N+1$ orbite critice. In particular, rezulta ca pendulul relativist $N$-dimensional are cel putin $N+1$ solutii periodice distincte geometric.

In ultimul capitol indicam unele linii de cercetare ce pot fi dezvoltate in conexiune cu rezultatele din capitolele anterioare.

**Abstract.** In this work we present some existence, uniqueness and multiplicity results for some nonlinear boundary value problems with $\phi$-Laplacians. Those problems originate from differential geometry or special relativity.

In the first chapter one has an abstract and in the second one an introduction.

In Chapter 3, using Schauder fixed point theorem, we prove existence results of radial solutions for Dirichlet problems in the unit ball and in an annular domain, associated to mean curvature operators in Euclidian and Minkowski spaces.

In Chapter 4, we study the existence of radial solutions for Neumann problems in a ball and in an annular domain, associated to mean curvature operators in Euclidian and Minkowski spaces. The main tool is Leray-Schuader degree together with some fixed point reformulations of ours nonlinear Neumann boundary value problems.

In Chapter 5, we study the existence and multiplicity of radial solutions for Neumann problems in a ball and in an annular domain, associated to pendulum-like perturbations of mean curvature operators in Euclidean and Minkowski spaces and of the $p$-Laplacian operator. Our approach relies on the Leray-Schauder degree and the upper and lower solutions method.

In Chapter 6, we show that if $\mathcal{A} \subset \mathbb{R}^N$ is an annulus or a ball centered at 0, the homogeneous Neumann problem on $\mathcal{A}$ for the equation with continuous data

$$\operatorname{div}\left(\frac{\nabla v}{\sqrt{1-|\nabla v|^2}}\right) = g(|x|, v) + h(|x|)$$

has at least one radial solution when $g(|x|, \cdot)$ has a periodic indefinite integral and $\int_{\mathcal{A}} h(|x|)\, dx = 0$. The proof is based upon the direct method of the calculus of variations, variational inequalities and degree theory.

In Chapter 7, motivated by the existence of radial solutions to the Neumann problem involving the mean extrinsic curvature operator in Minkowski space

$$\operatorname{div}\left(\frac{\nabla v}{\sqrt{1-|\nabla v|^2}}\right) = g(|x|, v) \quad \text{in} \quad \mathcal{A}, \quad \frac{\partial v}{\partial \nu} = 0 \quad \text{on} \quad \partial\mathcal{A},$$

where $0 \le R_1 < R_2$, $\mathcal{A} = \{x \in \mathbb{R}^N : R_1 \le |x| \le R_2\}$ and $g : [R_1, R_2] \times \mathbb{R} \to \mathbb{R}$ is continuous, we study the more general problem

$$[r^{N-1}\phi(u')]' = r^{N-1}g(r, u), \quad u'(R_1) = 0 = u'(R_2),$$

where $\phi := \Phi' : (-a, a) \to \mathbb{R}$ is an increasing homeomorphism with $\phi(0) = 0$ and the continuous function $\Phi : [-a, a] \to \mathbb{R}$ is of class $C^1$ on $(-a, a)$. The associated functional in the space of continuous functions over $[R_1, R_2]$ is the sum of a convex lower semicontinuous functional and of a functional of class $C^1$. Using the critical point theory of Szulkin, we obtain various existence and multiplicity results for several classes of nonlinearities. We also discuss the case of the periodic problem.

In the next Chapter, we use the critical point theory for convex, lower semicontinuous perturbations of $C^1-$functionals to establish existence of multiple radial solutions for some one parameter Neumann problems involving the operator $v \mapsto \operatorname{div}\left(\frac{\nabla v}{\sqrt{1-|\nabla v|^2}}\right)$. Similar results for periodic problems are also provided.

In Chapter 9 we deal with a class of functionals $I$ on a Banach space $X$, having the structure $I = \Psi + \mathcal{G}$, with $\Psi : X \to (-\infty, +\infty]$ proper, convex, lower semicontinuous and $\mathcal{G} : X \to \mathbb{R}$ of class $C^1$. Also, $I$ is $G$-invariant with respect to a discrete subgroup $G \subset X$ with dim (span $G$) $= N$. Under some appropriate additional assumptions we prove that $I$ has at least $N + 1$ critical orbits. As a consequence, we obtain that the periodically perturbed $N$-dimensional relativistic pendulum equation has at least $N + 1$ geometrically distinct periodic solutions.

In the last Chapter we give some possible developments of the results given in this work.

**Notice that this is a collection of recent papers of the author jointly with Petru Jebelean and Jean Mawhin. I thank both of them for our fruitful collaboration.**

**This habilitation thesis is dedicated to Dana and Petru.**

# Chapter 2

# Introduction

The aim of this work is to present some existence and multiplicity results of radial solutions for Dirichlet and Neumann problems in a ball or an annular domain, associated to mean curvature operator in the flat Minkowski space

$$\mathbb{L}^{N+1} := \{(x,t) : x \in \mathbb{R}^N, t \in \mathbb{R}\}$$

endowed with the Lorentzian metric

$$\sum_{j=1}^{N}(dx_j)^2 - (dt)^2,$$

where $(x,t)$ are the canonical coordinates in $\mathbb{R}^{N+1}$. The Euclidean situation is also considered.

Those problems originate from studying, in differential geometry or relativity, maximal or constant mean curvature hypersurfaces, i.e. spacelike submanifolds of codimension one in $\mathbb{L}^{N+1}$, having the property that their mean extrinsic curvature (trace of its second fundamental form) is respectively zero or constant (see e.g. [2, 35, 117]). More specifically, let $M$ be a spacelike hypersurfaces of codimension one in $\mathbb{L}^{N+1}$ and assume that $M$ is the graph of a smooth function $v : \Omega \to \mathbb{R}$ with $\Omega$ a domain in $\{(x,t) : x \in \mathbb{R}^N, t = 0\} \simeq \mathbb{R}^N$. The spacelike condition implies $|\nabla v| < 1$, and the mean curvature $H$ at the point $(x, v(x))$, $x \in \Omega$ verifies the equation

$$\operatorname{div}\left(\frac{\nabla v}{\sqrt{1-|\nabla v|^2}}\right) = NH(x,v) \quad \text{in} \quad \Omega.$$

A first essential result concerning the above PDE was proved by E. Calabi [32] in the case $\Omega = \mathbb{R}^N$ and $N \leq 4$. This was later extended to arbitrary dimension by S.Y. Cheng and S.T. Yau in [35]. In the Euclidian situation similar results have been obtained trough the efforts of Bernstein, Federer, Fleming, de Giorgio, Almgren, Simons, Bombieri and Giusti. On the other hand, if $H \equiv c > 0$

and $\Omega = \mathbb{R}^N$, then A. Treibergs [117] obtained an existence result about entire solutions for the above PDE in the presence of a pair of well ordered upper and lower-solutions. If $H$ is bounded, then it has been shown in [9] that the above equation has at least one solution $u \in C^1(\Omega) \cap W^{2,2}(\Omega)$ and $u = 0$ on $\partial\Omega$.

**Chapter 2** [12]
The problems we consider here are of the type

$$\operatorname{div}(\phi_N(\nabla v)) = f(|x|, v, \frac{dv}{dr}) \quad \text{in} \quad \Omega, \quad v = 0 \quad \text{on} \quad \partial\Omega, \tag{2.1}$$

where

$$\phi_N(y) = \frac{y}{\sqrt{1 \pm |y|^2}} \quad (y \in \mathbb{R}^N),$$

with the $+$ sign in the Euclidian case, the $-$ sign in the Minkowski case, $\Omega$ denotes the unit ball $\mathcal{B} \subset \mathbb{R}^N$ or an annular domain $\mathcal{A} = \{x \in \mathbb{R}^N : 1 < |x| < 2\}$, the function $f$ is continuous, $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^N$ and $\frac{dv}{dr}$ stands for the radial derivative of $v$. Setting $|x| = r$ and $v(x) = u(r)$, the above Dirichlet problem becomes

$$(r^{N-1}\phi_1(u'))' = r^{N-1} f(r, u, u'), \quad b(u, u') = 0, \tag{2.2}$$

where $b(u, u') = 0$ denotes the mixed boundary condition $u'(0) = 0 = u(1)$ or the Dirichlet boundary condition $u(1) = 0 = u(2)$, according to $\Omega$ equals to $\mathcal{B}$, respectively $\mathcal{A}$. Notice that (2.1) needs not to be Euler-Lagrange equations of a variational problem.

For this reason, (2.2) is transformed into a fixed point problem to which we apply Schauder fixed point theorem. Notice that in the Euclidian space situation, the corresponding fixed point operator is not defined on the whole space and we overcome this difficulty by using a cutting method introduced in [25]. For other results concerning the Euclidian situation, see for example [33]. In the Minkowski setting, we prove that the problem is solvable for an arbitrary continuous right-hand member $f$. The case where $N = 1$ was already considered in [22].

**Chapter 3** [15]
In Chapter 2 (see also [12]) we studied the existence of radial solutions for nonlinear Dirichlet problems in the unit ball and in an annular domain from $\mathbb{R}^N$, associated with the *mean curvature operator in Euclidian space*

$$\mathcal{E}v = \operatorname{div}\left(\frac{\nabla v}{\sqrt{1 + |\nabla v|^2}}\right)$$

and with the *mean extrinsic curvature operator in Minkowski space*

$$\mathcal{M}v = \operatorname{div}\left(\frac{\nabla v}{\sqrt{1 - |\nabla v|^2}}\right).$$

To formulate these problems, let $R_1, R_2 \in \mathbb{R}$, $0 \leq R_1 < R_2$ and let us denote by $\mathcal{A}$ the annular domain $\{x \in \mathbb{R}^N : R_1 \leq |x| \leq R_2\}$. For $f : [R_1, R_2] \times \mathbb{R}^2 \to \mathbb{R}$ a given continuous function, we consider the following Neumann boundary-value problems:

$$\mathcal{M}v = f(|x|, v, \frac{dv}{dr}) \quad \text{in} \quad \mathcal{A}, \quad \frac{\partial v}{\partial \nu} = 0 \quad \text{on} \quad \partial\mathcal{A} \tag{2.3}$$

and

$$\mathcal{E}v = f(|x|, v, \frac{dv}{dr}) \quad \text{in} \quad \mathcal{A}, \quad \frac{\partial v}{\partial \nu} = 0 \quad \text{on} \quad \partial\mathcal{A}. \tag{2.4}$$

As usual, we have denoted by $\frac{dv}{dr}$ the radial derivative and by $\frac{\partial v}{\partial \nu}$ the outward normal derivative of $v$. It should be noticed that for $R_1 = 0$ one has Neumann problems in the ball of radius $R_2$.

Setting $r = |x|$ and $v(x) = u(r)$, the above problems (3.1) and (2.4) become

$$\left(r^{N-1}\frac{u'}{\sqrt{1-|u'|^2}}\right)' = r^{N-1}f(r, u, u'), \quad u'(R_1) = 0 = u'(R_2), \tag{2.5}$$

respectively,

$$\left(r^{N-1}\frac{u'}{\sqrt{1+|u'|^2}}\right)' = r^{N-1}f(r, u, u'), \quad u'(R_1) = 0 = u'(R_2). \tag{2.6}$$

Clearly, the solutions of (2.5) and (2.6) are classical radial solutions of (3.1), respectively (2.4).

Our approach for problem (2.5) relies upon a Leray-Schauder type continuation theorem, that we recall here for the convenience of the reader (see e.g. [92] and references therein). Let $(X, \|\cdot\|)$ be a real normed space, $\Omega$ be a bounded open subset of $X$ and $S : \overline{\Omega} \to X$ be a compact operator such that $0 \notin (I - S)(\partial\Omega)$. The Leray-Schauder degree of $I - S$ with respect to $\Omega$ and 0 is denoted by $d_{LS}[I - S, \Omega, 0]$ (see e.g. [47]). We set $B_\rho = \{x \in X : \|x\| < \rho\}$.

**Lemma 1** *Let $S : \mathbb{R} \times \overline{B}_\rho \to X$ be a compact operator such that*

$$x \neq S(\lambda, x) \quad \text{for all} \quad (\lambda, x) \in \mathbb{R} \times \partial B_\rho$$

*and such that*

$$d_{LS}[I - S(\lambda_0, \cdot), B_\rho, 0] \neq 0 \quad \text{for some} \quad \lambda_0 \in \mathbb{R}.$$

*Then the set $\mathcal{S}$ of solutions $(\lambda, x) \in \mathbb{R} \times \overline{B}_\rho$ of problem*

$$x = S(\lambda, x)$$

*contains a continuum (closed and connected) $\mathcal{C}$ whose projection on $\mathbb{R}$ is $\mathbb{R}$.*

The existence result obtained for (3.1) is then employed, via a cutting method, to derive the existence of solutions for problem (2.4). In particular, we extend the method of (not necessarily ordered) lower and upper solutions to problem of the type (3.1), and give some applications and several examples. In the last section of this chapter we deal with pendulum-like nonlinearities.

For interesting results concerning radial solutions for Dirichlet boundary value problems associated to some nonlinear perturbations of the operators $\mathcal{E}$ and $p$-Laplacian the reader can consult [38, 60, 61, 69]. The Neumann problem associated to some nonlinear perturbations of the $p$-Laplacian is considered for example in papers [46, 118].

**Chapter 4** [16]
In this Chapter we present existence and multiplicity results for the Neumann problem

$$\mathcal{T}(v) + \mu \sin v = l(|x|) \quad \text{in} \quad \mathcal{A}, \qquad \frac{\partial v}{\partial \nu} = 0 \quad \text{on} \quad \partial \mathcal{A}, \tag{2.7}$$

where $\mathcal{T}$ is in one of the following situations:

$$\mathcal{T}(v) = \operatorname{div}\left(\frac{\nabla v}{\sqrt{1-|\nabla v|^2}}\right) \quad (\textit{mean extrinsic curvature in Minkowski space}),$$

$$\mathcal{T}(v) = \operatorname{div}\left(\frac{\nabla v}{\sqrt{1+|\nabla v|^2}}\right) \quad (\textit{mean curvature in Euclidean space}),$$

$$\mathcal{T}(v) = \operatorname{div}(|\nabla v|^{p-2}\nabla v) \quad (\textit{p-Laplacian}).$$

Here, $\mu > 0$ is a constant, $\mathcal{A} = \{x \in \mathbb{R}^N : R_1 < |x| < R_2\}$ $(0 \le R_1, < R_2)$, $l : [R_1, R_2] \to \mathbb{R}$ is a given continuous function.

Our approach relies upon the idea that setting $r = |x|$ and $v(x) = u(r)$, problem (2.7) reduces to

$$(r^{N-1}\phi(u'))' + r^{N-1}\mu \sin u = r^{N-1}l(r), \quad u'(R_1) = 0 = u'(R_2), \tag{2.8}$$

where $\phi(v) = \frac{v}{\sqrt{1-v^2}}$ in the Minkowski case, $\phi(v) = \frac{v}{\sqrt{1+v^2}}$ in the Euclidean case and $\phi(v) = |v|^{p-2}v$ $(p > 1)$ in the $p$-Laplacian case. Actually, in what follows $\phi$ will be a general increasing homeomorphism with $\phi(0) = 0$ and which is in one of the following situations:

$\phi : (-a, a) \to \mathbb{R}$ (*singular*),

$\phi : \mathbb{R} \to (-a, a)$ (*bounded*),

$\phi : \mathbb{R} \to \mathbb{R}$ (*classical*).

We prove (Corollary 8) using degree arguments that the problem

$$\operatorname{div}\left(\frac{\nabla v}{\sqrt{1-|\nabla v|^2}}\right) + \mu \sin v = l(|x|) \quad \text{in} \quad \mathcal{A}, \quad \frac{\partial v}{\partial \nu} = 0 \quad \text{on} \quad \partial \mathcal{A}, \tag{2.9}$$

has at least two classical radial solutions not differing by a a multiple of $2\pi$ if

$$2(R_2 - R_1) < \pi$$

and

$$\left| \frac{N}{R_2^N - R_1^N} \int_{R_1}^{R_2} r^{N-1} l(r) dr \right| < \mu \cos\left(R_2 - R_1\right).$$

Moreover, if

$$2(R_2 - R_1) = \pi,$$

then problem (2.9) has at least one classical radial solution provided that

$$\int_{R_1}^{R_2} r^{N-1} l(r) dr = 0. \tag{2.10}$$

Note that in Theorem 5.1 from [15] we have proved that if condition (2.10) is fulfilled and if $2(R_2 - R_1) \leq 1$ then one has existence of at least one classical radial solution. On the other hand for the $p$-Laplacian, we prove for example (Corollary 9) that problem

$$\operatorname{div}(|\nabla v|^{p-2}\nabla v) + \mu \sin v = l(|x|) \quad \text{in} \quad \mathcal{A}, \quad \frac{\partial v}{\partial \nu} = 0 \quad \text{on} \quad \partial\mathcal{A}, \tag{2.11}$$

has at least two classical radial solutions not differing by a a multiple of $2\pi$ if (2.10) holds and $R_2$ is sufficiently small (or $N$ sufficiently large). Moreover the same type of result holds true for the Neumann problem

$$\operatorname{div}\left(\frac{\nabla v}{\sqrt{1 + |\nabla v|^2}}\right) + \mu \sin v = l(|x|) \quad \text{in} \quad \mathcal{A}, \quad \frac{\partial v}{\partial \nu} = 0 \quad \text{on} \quad \partial\mathcal{A}. \tag{2.12}$$

In the case

$$R_1 > 0$$

(i.e., $\mathcal{A}$ is an annular domain) we show (Corollary 10) using again degree arguments and the upper and lower solutions method that (2.9) and (2.11) have at least two classical radial solution not differing by a multiple of $2\pi$ if $||l||_\infty < \mu$ and have at least one classical radial solution if $||l||_\infty = \mu$. Moreover, if

$$\frac{2\mu R_2}{N} < 1$$

holds, then we prove (Corollary 10) that one has the same result for the Neumann problem (2.12).

In is worth to point out that corresponding results for the periodic problem and $N = 1$ have been proved in [17, 22]. For existence and multiplicity results concerning periodic solutions of the classical pendulum equation see for example [51, 56, 73, 100] and for other results concerning boundary value problems associated to singular or bounded $\phi$-Laplacians see [10] - [54], [93, 115, 116].

The Chapter is organized as follows. In Section 2 we introduce the function spaces and the operators which are needed in the sequel. Section 3 present a fixed point operator and some degree computations in the singular case. Existence and multiplicity results for problem (2.8) are given in Sections 4 and 5 under conditions on the radius and the mean value of the forcing term or on the norm of the forcing term.

**Chapter 5** [20]
In Chapter 4 (see also [16]), we have used topological degree techniques to obtain existence and multiplicity results for the radial solutions of the Neumann problem

$$\operatorname{div}\left(\frac{\nabla v}{\sqrt{1-|\nabla v|^2}}\right) + \mu \sin v = h(|x|) \quad \text{in} \quad \mathcal{A}, \quad \frac{\partial v}{\partial \nu} = 0 \quad \text{on} \quad \partial\mathcal{A}, \tag{2.13}$$

on the ball or annulus

$$\mathcal{A} = \{x \in \mathbb{R}^N : R_1 \le |x| \le R_2\} \quad (0 \le R_1 < R_2)$$

i.e. for the equivalent one-dimensional problem

$$\left(r^{N-1}\frac{u'}{\sqrt{1-u'^2}}\right)' + r^{N-1}\mu \sin u = r^{N-1}h(r), \quad u'(R_1) = 0 = u'(R_2).$$

We have proved the existence of at least two radial solutions not differing by a multiple of $2\pi$ when

$$2(R_2 - R_1) < \pi \quad \text{and} \quad \left|\frac{N}{R_2^N - R_1^N}\int_{R_1}^{R_2} h(r)\, r^{N-1} dr\right| < \mu \cos(R_2 - R_1),$$

and the existence of at least one radial solution when $2(R_2 - R_1) = \pi$ and

$$\int_{R_1}^{R_2} h(r)\, r^{N-1} dr = 0. \tag{2.14}$$

Condition (2.14) is easily seen to be necessary for the existence of a radial solution to (2.13) for any $\mu > 0$ and a natural question is to know if condition

$$2(R_2 - R_1) \le \pi \tag{2.15}$$

can be dropped.

In the analogous problem of the forced pendulum equation

$$u'' + \mu \sin u = h(t)$$

with periodic or Neumann homogeneous boundary conditions on $[0, T]$, it has been shown that the corresponding necessary condition

$$\int_0^T h(t)\, dt = 0 \tag{2.16}$$

is also sufficient for the existence of at least two solutions not differing by a multiple of $2\pi$. But, in this case, all the known proofs are of variational or symplectic nature (see e.g. the survey [92]).

Recently, it has been proved in [29] that the "relativistic forced pendulum equation"

$$\left(\frac{u'}{\sqrt{1-u'^2}}\right)' + \mu \sin u = h(t)$$

has at least one T-periodic solution for any $\mu > 0$ when the (necessary) condition (2.16) is satisfied. The approach is essentially variational, but combined with some topological arguments. The aim of this chapter is to adapt the methodology introduced in [29] to the radial Neumann problem for (2.13) and prove that, for the existence part, condition (2.15) can be dropped.

Actually, in this chapter we consider problems of type

$$[r^{N-1}\phi(u')]' = r^{N-1}[g(r,u) + h(r)], \quad u'(R_1) = 0 = u'(R_2) \tag{2.17}$$

where $\phi : (-a, a) \to \mathbb{R}$ is a suitable homeomorphism and $g$ belongs to some class of functions $2\pi$-periodic with respect to its second variable.

**Chapter 6** [18]

The study in this chapter is essentially motivated by the existence of radial solutions to the Neumann problem involving the *mean extrinsic curvature operator in Minkowski space* (see e.g. [15]) :

$$\operatorname{div}\left(\frac{\nabla v}{\sqrt{1-|\nabla v|^2}}\right) = g(|x|, v) \quad \text{in} \quad \mathcal{A}, \quad \frac{\partial v}{\partial \nu} = 0 \quad \text{on} \quad \partial\mathcal{A}, \tag{2.18}$$

where $0 \le R_1 < R_2$, $\mathcal{A} = \{x \in \mathbb{R}^N : R_1 \le |x| \le R_2\}$ and $g : [R_1, R_2] \times \mathbb{R} \to \mathbb{R}$ is a continuous function. Setting $r = |x|$ and $v(x) = u(r)$, the above problem (2.18) becomes

$$\left[r^{N-1}\left(\frac{u'}{\sqrt{1-u'^2}}\right)\right]' = r^{N-1}g(r,u), \quad u'(R_1) = 0 = u'(R_2), \tag{2.19}$$

and the solutions of (2.19) are classical radial solutions of (2.18).

In this chapter we obtain existence results for the more general problem

$$[r^{N-1}\phi(u')]' = r^{N-1}g(r,u), \quad u'(R_1) = 0 = u'(R_2), \tag{2.20}$$

where $\phi := \Phi' : (-a, a) \to \mathbb{R}$ is an increasing homeomorphism with $\phi(0) = 0$ and the continuous function $\Phi : [-a, a] \to \mathbb{R}$ is of class $C^1$ on $(-a, a)$ and, without loss of generality, we can assume that $\Phi(0) = 0$. This kind of $\phi$ is called *singular $\phi$-Laplacian.* Note that for $\phi(s) = \dfrac{s}{\sqrt{1-s^2}}$ one takes $\Phi(s) = 1 - \sqrt{1-s^2}$.

Our approach is a variational one and relies on Szulkin's critical point theory [112]. Using a strategy inspired from [20, 29], we show in Proposition 8 that $u$

is a solution of (2.20) provided that $u$ is a critical point of the energy functional $I : C[R_1, R_2] \to (-\infty, +\infty]$ defined by

$$I(u) = \begin{cases} \displaystyle\int_{R_1}^{R_2} r^{N-1}\Phi(u')\,dr + \int_{R_1}^{R_2} r^{N-1}G(r,u)\,dr, & \text{if } u \in K, \\ +\infty, & \text{otherwise}, \end{cases}$$

where $G : [R_1, R_2] \times \mathbb{R} \to \mathbb{R}$ is the primitive of $g$ with respect to the second variable and $K = \{u \in W^{1,\infty}[R_1, R_2] : |u'| \leq a \text{ a.e. on } [R_1, R_2]\}$. The functional $I$ has the structure required by Szulkin's critical point theory, i.e., it is the sum of a proper convex, lower semicontinuous functional and of a $C^1$ functional. In this context, a critical point of $I$ means a function $u \in K$ such that

$$\int_{R_1}^{R_2} r^{N-1}[\Phi(v') - \Phi(u')]\,dr + \int_{R_1}^{R_2} r^{N-1}g(r,u)(v-u)\,dr \geq 0 \qquad \text{for all } v \in K.$$

In Section 2 we introduce some notations and definitions and we prove the above mentioned Proposition 8. Notice that, in contrast to [20], we replace some auxiliary result based upon Leray-Schauder theory by an elementary argument (Lemma 15) and obtain in this way a purely variational treatment of our problem. A similar methodology can be applied to obtain pure variational proofs of the results on periodic solutions in [29, 30, 96].

Section 3 deals with minimization problems for $I$ based upon the fact that if there exists $\rho > 0$ such that

$$\inf\left\{ I(u) : u \in K, \left| \int_{R_1}^{R_2} r^{N-1}u\,dr \right| \leq \rho \right\} = \inf_K I,$$

then $I$ is bounded from below and attains its infimum at some $u$, which solves problem (2.20) (Lemma 16). Theorem 1 from [20] is then an immediate consequence of this result (Corrollary 13). We also prove (Theorem 22) that if $g$ is such that

$$\liminf_{|x|\to\infty} G(r,x) > 0, \qquad \text{uniformly in } r \in [R_1, R_2],$$

then (2.20) has at least one solution $u$ which minimizes $I$ on $C$.

The same is also true if $g$ is bounded and

$$\lim_{|x|\to\infty} \int_{R_1}^{R_2} r^{N-1}G(r,x)\,dr = +\infty$$

(Theorem 23). On the other hand, if $G(r,\cdot)$ is convex for any $r \in [R_1, R_2]$, then (2.20) has at least one solution if and only if the function

$$x \mapsto \int_{R_1}^{R_2} r^{N-1}g(r,x)\,dr$$

has at least one zero, or, equivalently, the real convex function

$$x \mapsto \int_{R_1}^{R_2} r^{N-1} G(r,x)\, dr$$

has a minimum (Theorem 24).

In Section 4 we derive some properties of the (PS)–sequences (Lemma 17) and we show that if $g$ is bounded and

$$\lim_{|x|\to\infty} \int_{R_1}^{R_2} r^{N-1} G(r,x)\, dr = -\infty,$$

then (2.20) has at least one solution $u$ which is a saddle point of $I$ (Theorem 25). As in Section 3, if $g$ is not necessarily bounded but the above condition upon $G$ is replaced with the following more restrictive assumption

$$\lim_{|x|\to\infty} G(r,x) = -\infty, \quad \text{uniformly in } r \in [R_1, R_2],$$

then the same result holds true (Theorem 26).

In Section 5 we consider the problem

$$[r^{N-1}\phi(u')]' = r^{N-1}[\lambda |u|^{m-2}u - f(r,u)], \quad u'(R_1) = 0 = u'(R_2), \tag{2.21}$$

where $\lambda > 0$ and $m \geq 2$ are fixed real numbers and $f : [R_1, R_2] \times \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying the classical Ambrosetti–Rabinowitz condition : there exists $\theta > m$ and $x_0 > 0$ such that

$$0 < \theta F(r,x) \leq x f(r,x) \quad \text{for all } r \in [R_1, R_2] \text{ and } |x| \geq x_0.$$

We also assume that

$$\limsup_{|x|\to 0} \frac{mF(r,x)}{|x|^m} < \lambda \quad \text{uniformly in } r \in [R_1, R_2],$$

and prove that under these assumptions, problem (2.21) has at least one solution $u$ which is a mountain pass critical point of the corresponding $I$ (Theorem 27).

Section 6 is devoted to the periodic problem

$$[\phi(u')]' = g(r,u), \quad u(R_1) - u(R_2) = 0 = u'(R_1) - u'(R_2), \tag{2.22}$$

Here we discuss the manner in which the above results for problems (2.20) and (2.21) can be transposed for problem (2.22).

**Chapter 7** [19]

This chapter is motivated by the existence of nontrivial solutions for the Neumann problems:

$$\begin{aligned} -\mathrm{div}\left(\frac{\nabla v}{\sqrt{1-|\nabla v|^2}}\right) + \alpha |v|^{p-2} v = f(|x|, v) + \lambda b(|x|)|v|^{q-2} v \quad &\text{in } \mathcal{A}, \\ \frac{\partial v}{\partial \nu} = 0 \quad \text{on} \quad \partial\mathcal{A}&, \end{aligned} \tag{2.23}$$

$$-\mathrm{div}\left(\frac{\nabla v}{\sqrt{1-|\nabla v|^2}}\right)+\lambda|v|^{m-2}v=f(|x|,v)+h(|x|)\quad\text{in }\mathcal{A},$$
$$\frac{\partial v}{\partial\nu}=0\quad\text{on}\quad\partial\mathcal{A}\tag{2.24}$$

and for the periodic problems:

$$-\left(\frac{u'}{\sqrt{1-|u'|^2}}\right)'+\alpha|u|^{p-2}u=f(r,u)+\lambda b(r)|u|^{q-2}u\quad\text{in }[R_1,R_2],$$
$$u(R_1)-u(R_2)=0=u'(R_1)-u'(R_2),\tag{2.25}$$

$$-\left(\frac{u'}{\sqrt{1-|u'|^2}}\right)'+\lambda|u|^{m-2}u=f(r,u)+h(r)\quad\text{in }[R_1,R_2],$$
$$u(R_1)-u(R_2)=0=u'(R_1)-u'(R_2),\tag{2.26}$$

where $0\le R_1<R_2$ and $\mathcal{A}=\{x\in I\!R^N : R_1\le|x|\le R_2\}$.

We assume the following hypothesis on the data.

$(H_f)$ *The functions $f:[R_1,R_2]\times I\!R\to I\!R$, $b,h:[R_1,R_2]\to I\!R$ are continuous; the constants $\alpha>0$, $p>q\ge 2$, $m\ge 2$ are fixed and $\lambda$ is a real positive parameter.*

Viewing the radial symmetry, we shall look for radial solutions of problems (2.23) and (2.24). So, letting $r=|x|$ and $v(x)=u(r)$, we reduce (2.23) and (2.24) to the one-dimensional Neumann problems

$$[r^{N-1}\phi(u')]'=r^{N-1}[\alpha|u|^{p-2}u-f(r,u)-\lambda b(r)|u|^{q-2}u]\text{ in }[R_1,R_2],$$
$$u'(R_1)=0=u'(R_2),\tag{2.27}$$

and

$$[r^{N-1}\phi(u')]'=r^{N-1}[\lambda|u|^{m-2}u-f(r,u)-h(r)]\text{ in }[R_1,R_2],$$
$$u'(R_1)=0=u'(R_2),\tag{2.28}$$

where $\phi(y)=\dfrac{y}{\sqrt{1-y^2}}$, $\forall y\in(-1,1)$. Also, it is clear that problems (2.25) and (2.26) can be rewritten as

$$[\phi(u')]'=\alpha|u|^{p-2}u-f(r,u)-\lambda b(r)|u|^{q-2}u\quad\text{in }[R_1,R_2],$$
$$u(R_1)-u(R_2)=0=u'(R_1)-u'(R_2),\tag{2.29}$$

and

$$[\phi(u')]'=\lambda|u|^{m-2}u-f(r,u)-h(r)\quad\text{in }[R_1,R_2],$$
$$u(R_1)-u(R_2)=0=u'(R_1)-u'(R_2),\tag{2.30}$$

with the same choice of $\phi$.

More generally, in this chapter the mapping $\phi : (-a, a) \to I\!R$ entering in the above boundary value problems will be an increasing homeomorphism with $\phi(0) = 0$. Following [22], this type of $\phi$ is called *singular*. Precisely, we assume the following hypothesis on $\phi$ :

$(H_\Phi)$ *$\Phi : [-a, a] \to \mathbb{R}$ is continuous, of class $C^1$ on $(-a, a)$, $\Phi(0) = 0$ and $\phi := \Phi' : (-a, a) \to I\!R$ is an increasing homeomorphism such that $\phi(0) = 0$.*

Denoting by $F$ the indefinite integral of $f$ with respect to the second variable, it is easy to see that if $F$ satisfies

$$\limsup_{|x|\to 0} \frac{pF(r, x)}{|x|^p} < \alpha \quad \text{uniformly in } r \in [R_1, R_2], \tag{2.31}$$

then $f(r, 0) = 0$ for all $r \in [R_1, R_2]$, meaning that problems (2.27) – (2.30) admit the trivial solution $u = 0$ provided that $h \equiv 0$. If, in addition, $F$ satisfies the Ambrosetti–Rabinowitz type condition [7] :

*(AR) there exists $\theta > p$ and $x_0 > 0$ such that*

$$0 < \theta F(r, x) \le x f(r, x) \quad \textit{for all} \quad r \in [R_1, R_2] \quad \textit{and} \quad |x| \ge x_0, \tag{2.32}$$

then problems (2.27) and (2.29) with $\lambda = 0$ or problems (2.28) and (2.30) with $h \equiv 0$ have at least one nontrivial solution (see [18]).

We prove in Theorem 31 and Theorem 32 that if, in addition to (2.31) and (2.32) we assume :

*(i) either*

$$\liminf_{x\to 0-} \frac{F(r, x)}{|x|^p} \ge 0 \quad \textit{uniformly in } r \in [R_1, R_2] \tag{2.33}$$

*or*

$$\liminf_{x\to 0+} \frac{F(r, x)}{x^p} \ge 0 \quad \textit{uniformly in } r \in [R_1, R_2]; \tag{2.34}$$

*(ii) it holds*

$$\int_{R_1}^{R_2} r^{N-1} b(r)\, dr > 0,$$

then problems (2.27) and (2.29) have at least two nontrivial solutions for sufficiently small values of the parameter $\lambda$. It is easy to see that those assumptions correspond to problems with *convex-concave nonlinearities* initiated in 1994 for semilinear Dirichlet problems by Ambrosetti, Brezis and Cerami [5], extended to quasilinear Dirichlet problems involving the p-Laplacian by Ambrosetti, Garcia Azorero, Peral [6], Garcia Azorero, Peral, Manfredi [59]. Radial solutions with Dirichlet conditions have been considered independently by Kormann [76], using bifurcation theory and by Tang [114] using ordinary differential equations methods.

One the other hand, under the hypotheses :

*(i)′* *there exists* $k_1, k_2 > 0$ *and* $0 < \sigma < m$ *such that*

$$-l(r) \leq F(r,x) \leq k_1|x|^\sigma + k_2, \quad \textit{for all} \quad (r,x) \in [R_1, R_2] \times \mathbb{R}, \quad (2.35)$$

*where* $l \geq 0$ *is measurable and* $\int_{R_1}^{R_2} r^{N-1} l(r) dr < +\infty$*;*

*(ii)′* *one has that either*

$$\lim_{|x|\to\infty} \int_{R_1}^{R_2} r^{N-1} F(r,x) dr = +\infty, \quad (2.36)$$

*or the limits* $F_\pm(r) = \lim_{x\to\pm\infty} F(r,x)$ *exist for all* $r \in [R_1, R_2]$ *and*

$$\begin{aligned} F(r,x) &< F_+(r), \quad \forall r \in [R_1, R_2], \ x \geq 0, \\ F(r,x) &< F_-(r), \quad \forall r \in [R_1, R_2], \ x \leq 0; \end{aligned} \quad (2.37)$$

*(iii)′* *it holds*

$$\int_{R_1}^{R_2} r^{N-1} h(r) dr = 0,$$

we prove in Theorem 33 (see also Theorem 34 for the periodic case) that problem (2.28) has at least three nontrivial solutions for sufficiently small values of the parameter $\lambda$. Results of this type in the classical case, called *multiplicity results near resonance*, have been initiated in [99] (for $N = 1$), using bifurcation from infinity and Leray-Schauder degree theory. A variational approach was introduced by Sanchez in [109] to attack such multiplicity problems, and conditions of type (i)' and (ii)' were introduced by Ma, Ramos and Sanchez in [108, 83] for semilinear and quasilinear Dirichlet problems involving the $p$-Laplacian. See also [84, 82, 102, 44, 106] for a similar variational treatment of various semilinear or quasilinear equations, systems or inequalities with Dirichlet bonditions, [103] for perturbations of p-Laplacian with Neumann boundary conditions, and [81] for periodic solutions of perturbations of the one-dimensional $p$-Laplacian. The existence of at least two solutions near resonance at a non-principal eigenvalue have been first obtained in [98] using a topological approach and then for semilinear or quasilinear problems using critical point theory in [45, 75, 111], but this question seems to be meaningless for the singular $\phi$ considered here because resonance only occurs at 0.

The main used tools are some abstract local minimization results combined with mountain pass techniques in the frame of the Szulkin's critical point theory [112]. The rest of the paper is organized as follows. In Section 2 we give some abstract results (Proposition 10 and Proposition 11) which we need in the sequel. The concrete functional framework and the variational setting, employed in the treatment of the above problems, are described in Section 3. Section 4 and Section 5 are devoted to the proofs of the main multiplicity results.

**Chapter 8** [11]
In the recent paper [29], Brezis and Mawhin show that the forced pendulum like problem

$$(\phi(u'))' = f(t,u) + h(t), \quad u(0) - u(T) = 0 = u'(0) - u'(T), \tag{2.38}$$

has at least one solution, provided that $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function for which there exists $\omega > 0$ such that

$$F(t,u) = F(t, u+\omega), \qquad \forall (t,u) \in [0,T] \times \mathbb{R},$$

where $F : [0,T] \times \mathbb{R} \to \mathbb{R}$ is defined by

$$F(t,u) = \int_0^u f(t,\xi) d\xi, \qquad \forall (t,u) \in [0,T] \times \mathbb{R},$$

$h : [0,T] \to \mathbb{R}$ is a continuous function satisfying

$$\int_0^T h(t) dt = 0,$$

and $\phi : (-a,a) \to \mathbb{R}$ $(0 < a < \infty)$ is an increasing homeomorphism with $\phi(0) = 0$ and there exists $\Phi : [-a,a] \to \mathbb{R}$ a continuous function with $\Phi(0) = 0$, $\Phi$ of class $C^1$ on $(-a,a)$ and $\Phi' = \phi$. They consider the action functional $\mathcal{I} : K_\# \to \mathbb{R}$ associated to (2.38), given by

$$\mathcal{I}(u) = \int_0^T \{\Phi(u') + F(t,u) + hu\} dt, \qquad (u \in K_\#),$$

where

$$K_\# = \{u \in Lip(\mathbb{R}) : \ |u'(t)| \leq a \text{ for a.e. } t \in \mathbb{R}, \ u \text{ is } T-\text{periodic}\},$$

and prove that $\mathcal{I}$ has at least one minimizer $u$ in $K_\#$ satisfying the variational inequality

$$\int_0^T [\Phi(v') - \Phi(u')] + \int_0^T [f(t,u) + h][v-u] \geq 0, \qquad \forall v \in K_\#. \tag{2.39}$$

Then, using (2.39) and a topological result from [22], they show that any minimizer of $\mathcal{I}$ on $K_\#$ is a solution of (2.38). Hence, (2.38) has at least one solution. Notice that the corresponding classical result ($\phi = id_{\mathbb{R}}$) was proved by Hamel [70] and rediscovered independently by Dancer [41] and Willem [120]. Also, Brezis and Mawhin extend their result from [29] to systems in their subsequent paper [30].

In [18] it is emphasized that Szulkin's critical point theory [112] is an appropriate functional framework for problems of this type. More precisely, set

$$\widehat{K} = \{u \in W^{1,\infty}(0,T) \ : \ \|u'\|_\infty \leq a, \ u(0) = u(T)\}$$

and let $\Psi : C[0,T] \to (-\infty, +\infty]$,

$$\Psi(u) = \int_0^T \Phi(u') \text{ if } u \in \widehat{K}, \qquad \Psi(u) = +\infty \text{ if } u \in C[0,T] \setminus \widehat{K},$$

and $\mathcal{F} : C[0,T] \to \mathbb{R}$,

$$\mathcal{F}(u) = \int_0^T \{F(t,u) + hu\}dt, \qquad (u \in C[0,T]).$$

Then, $\Psi$ is a lower semicontinuous, convex functional and $\mathcal{F}$ is of class $C^1$. Hence, the action $\widehat{\mathcal{I}} : C[0,T] \to (-\infty, +\infty]$ defined by $\widehat{\mathcal{I}} = \Psi + \mathcal{F}$, has the structure required by Szulkin's critical point theory. In this context, a critical point of $\widehat{\mathcal{I}}$ means a function $u \in \widehat{K}$ such that (2.39) holds true. Then, using some ideas from [29], it is shown that any critical point of $\widehat{\mathcal{I}}$ is a solution of (2.38). Note that $C[0,T]$ is not reflexive, so the direct method in the calculus of variations cannot be applied. Nevertheless, a substitute for this is provided, namely, it is shown that if there exists $\rho > 0$ such that $\inf_{\widehat{K}_\rho} \widehat{\mathcal{I}} = \inf_{\widehat{K}} \widehat{\mathcal{I}}$, where

$$\widehat{K}_\rho = \{u \in \widehat{K} : \left| \int_0^T u \right| \leq \rho\},$$

then $\widehat{\mathcal{I}}$ is bounded from below on $C[0,T]$ and attains its infimum at some $u \in \widehat{K}_\rho$ which solves (2.38). The Brezis-Mawhin result is an immediate consequence of this result.

Another proof of Brezis-Mawhin result is given by Manásevich and Ward in [85]. The main idea is to introduce the change of variable $\phi(u') = v$. Then, problem (2.38) becomes

$$u' = \phi^{-1}(v), \quad v' = f(t,u) + h(t), \quad u(0) - u(T) - 0 = v(0) - v(T). \quad (2.40)$$

Letting

$$\widehat{\phi}(v) = \int_0^v \phi^{-1}(s)ds, \quad w = (u,v),$$

with the Hamiltonian function $H(t,w) = -\widehat{\phi}(v) + F(t,u) + h(t)u$, system (2.40) takes the Hamiltonian form

$$w' = J\nabla_w H(t,w), \qquad w(0) = w(T),$$

where $J$ is the standard symplectic matrix. The classical saddle point theorem of Rabinowitz is then applied to a sequence of approximating problems, obtaining a sequence of critical points. A subsequence of these critical points converges to a solution. Notice that the action functional associated to the above Hamiltonian system is strongly indefinite and the classical saddle point theorem does not apply to it.

A second geometrically distinct solution of problem (2.38) is obtained in [27] using the functional framework introduced in [18] and a mountain pass type argument (Corollary 3.3 from [112]). We note that the corresponding classical result was proved by Mawhin and Willem in [100] using a modified version of the Mountain Pass Theorem. Another proof of the Mawhin-Willem result was given by Franks [58] using a generalization of the Poincaré -Birkhoff theorem. Very recently Fonda and Toader [55] prove the results from [27, 100] in a unified way, using Ding's version of the Poincaré -Birkhoff theorem (see [50]). Using Franks's generalization of the Poincaré -Birkhoff theorem, Maró [86] give another proof of the main result from [27].

In the very recent paper [95], Mawhin obtains multiplicity of solutions for the $N$-dimensional analogous of (2.38):

$$(\phi(u'))' = \nabla_u F(t,u) + h(t), \quad u(0) - u(T) = 0 = u'(0) - u'(T), \tag{2.41}$$

under the following hypotheses:

$(H_\phi)$ $\phi$ is a homeomorphism from $B(a) \subset \mathbb{R}^N$ onto $\mathbb{R}^N$ such that $\phi(0) = 0$, $\phi = \nabla\Phi$, with $\Phi : \overline{B(a)} \to \mathbb{R}$ of class $C^1$ on $B(a)$, continuous, strictly convex on $\overline{B(a)}$, and such that $\Phi(0) = 0$;

$(H_F)$ $F : [0,T] \times \mathbb{R}^N \to \mathbb{R}$ is continuous, $\omega_i-$periodic ($\omega_i > 0$) with respect to each $u_i$ ($1 \leq i \leq N$) and $\nabla_u F$ exists and is continuous on $[0,T] \times \mathbb{R}^N$;

$(H_h)$ $h : [0,T] \to \mathbb{R}^N$ is continuous and

$$\int_0^T h(t)dt = 0.$$

Under the above assumptions, Mawhin shows that (2.41) has a Hamiltonian formulation, then applies a generalized saddle point theorem for strongly indefinite functionals due to Szulkin [113] (see also [53, 79]) in order to prove that (2.41) has at least $N + 1$ geometrically distinct solutions. The corresponding classical result has been proved independently, using Lusternik-Schnirelman theory in Hilbert manifolds or variants of it, by Chang [34], Mawhin [90] and Rabinowitz [107]. The case $N = 2$ has been discussed by Fournier and Willem in [57]. It is interesting to note that the Hamiltonian system associated to (2.41) is spatially periodic like in [53], but the results in [53] cannot be applied to it because the superlinearity condition $(H_3)$ in [53] with respect to the spatial variable is not satisfied in this relativistic case.

For a nice presentation of the classical forced pendulum equation we refer the reader to the paper [89].

The aim of this chapter is to give a different proof of Mawhin's result in [95], based upon a Lusternik-Schnirelman type approach for Szulkin functionals. More precisely, we will consider functionals $I : X \to (-\infty, +\infty]$ in a Banach space $X$ such that $I = \Psi + \mathcal{G}$, $\Psi$ is proper, convex, lower semicontinuous and $\mathcal{G}$ is of class $C^1$. Also, $I$ will be $G$-invariant with respect to a discrete subgroup $G$ with dim (span $G$) $= N$ and bounded from below. Under some additional assumptions, which are automatically satisfied by the Lagrangian action associated to (2.41), we prove that $I$ has $N + 1$ critical orbits (Theorem 35). With this aim, we use a Deformation Lemma (Proposition 16) together with Ekeland's variational principle and the classical Lusternik-Schnirelman category in order to prove that one has critical value at the levels (introduced in [107] for $C^1$-functionals),

$$c_j = \inf_{A\in\mathcal{A}_j} \sup_A I \qquad (1 \le j \le N+1),$$

where

$$\mathcal{A}_j = \{A \subset X : \ A \text{ is compact and } \mathrm{cat}_{\pi(X)}(\pi(A)) \ge j\},$$

and $\pi : X \to X/G$ denotes the canonical projection. The corresponding abstract result for $C^1$-functionals is proved in [101]. We point out that we use also some ideas from the proof of Theorem 4.3 in [112], but the deformation obtained in Proposition 2.3 from [112] can not be employed in our case because it is not "$G$-invariant" (see Proposition 16 (ii)).

The chapter is organized as follows. In Section 2 we show that the action functional associated to problem (2.41) has the structure required by Szulkin's critical point theory and present the main properties involved in the proof of the existence of at least $N+1$ geometrically distinct solutions for (2.41). In Section 3 we introduce some notations and the hypotheses. In Section 4 we prove a technical result (Proposition 15); this is the key ingredient in the proof of the deformation lemma (Proposition 16) which is given in Section 5. The next Section is a resume of the main tools of the proof of the main result: Ekeland's variational principle and the classical Lusternik-Schnirelman category. In the last Section we prove the main result of the paper (Theorem 35).

# Chapter 3

# Dirichlet problems with $\phi$-Laplacians

## 3.1 Radial solutions in the unit ball

In this Section, $\mathcal{B}$ denotes the open unit ball in $\mathbb{R}^N$ and $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ is a continuous function. The first main result concerns the existence of classical radial solutions of the nonlinear Dirichlet problem associated with the *mean extrinsic curvature operator in Minkowski space*

$$\operatorname{div}\left(\frac{\nabla v}{\sqrt{1-|\nabla v|^2}}\right) = f(|x|, v, \frac{dv}{dr}) \quad \text{in} \quad \mathcal{B}, \quad v = 0 \quad \text{on} \quad \partial\mathcal{B}. \tag{3.1}$$

We have for (3.1) the following 'universal' existence result.

**Theorem 1** *Problem (3.1) has at least one classical radial solution for any continuous right-hand member $f$.*

Notice that, when $f(r,u,v) = H$ (case of constant mean extrinsic curvature), the radial solution of (3.1) is unique and explicitely given, for any $H \in \mathbb{R}$, by

$$u(r) = 0 \quad (H = 0), \quad u(r) = \frac{N}{H}\left[\sqrt{1 + \frac{H^2}{N^2}r^2} - \sqrt{1 + \frac{H^2}{N^2}}\right] \quad (H \neq 0).$$

The second main result of the section deals with the existence of classical radial solutions of the nonlinear Dirichlet problem associated with the *mean curvature operator in Euclidian space*

$$\operatorname{div}\left(\frac{\nabla v}{\sqrt{1+|\nabla v|^2}}\right) = f(|x|, v, \frac{dv}{dr}) \quad \text{in} \quad \mathcal{B}, \quad v = 0 \quad \text{on} \quad \partial\mathcal{B}. \tag{3.2}$$

**Theorem 2** *Assume that there exists $\alpha > 0$ such that $\frac{\alpha}{N} < 1$ and*

$$|f(r,u,v)| \leq \alpha \quad \textit{for all} \quad (r,u,v) \in [0,1] \times R_\alpha,$$

*where $R_\alpha$ is the square given by $R_\alpha = \left[-\frac{\alpha/N}{\sqrt{1-(\alpha/N)^2}}, \frac{\alpha/N}{\sqrt{1-(\alpha/N)^2}}\right]^2$. Then, problem (3.2) has at least one classical radial solution.*

Notice that, when $f(r,u,v) = H$ (case of constant mean curvature), the radial solution of (3.2) only exists if $|H| < N$, is unique and is explicitely given by

$$u(r) = 0 \quad (H = 0), \quad u(r) = \frac{N}{H}\left[\sqrt{1 - \frac{H^2}{N^2}} - \sqrt{1 - \frac{H^2}{N^2}r^2}\right] \quad (0 < |H| < N).$$

Hence condition $\frac{\alpha}{N} < 1$ in Theorem 2 is sharp.

When dealing with the radial solutions for (3.1) or (3.2), one is led to study (setting $|x| = r$ and $v(x) = u(r)$) the mixed boundary-value problem

$$(r^{N-1}\phi(u'))' = r^{N-1}f(r,u,u'), \quad u'(0) = 0 = u(1), \tag{3.3}$$

where $\phi(y) = \frac{y}{\sqrt{1-y^2}}$ in the Minkowski case and $\phi(y) = \frac{y}{\sqrt{1+y^2}}$ in the Euclidian case ($y \in \mathbb{R}$).

We first reformulate (3.3) as a fixed point problem, for a general class of $\phi$ containing the two examples above as special cases, namely $\phi : (-a,a) \to \mathbb{R}$ an increasing homeomorphism such that $\phi(0) = 0$ and $0 < a \leq \infty$. In this section $C$ stands for the Banach space of continuous functions defined on $[0,1]$ endowed with the usual sup-norm $||\cdot||_\infty$ and $C^1$ denotes the Banach space of continuously differentiable functions on $[0,1]$ equipped with the norm $||u|| = ||u||_\infty + ||u'||_\infty$. The subspaces of $C^1$ defined by

$$C^1_M = \{u \in C^1 : u'(0) = 0 = u(1)\}$$

and

$$C_0 = \{u \in C : u(0) = 0\}$$

are closed. Then, setting

$$\gamma(r) = \frac{1}{r^{N-1}} \quad (r > 0),$$

consider the linear operators

$$S : C \to C_0, \quad Su(r) = \gamma(r)\int_0^r t^{N-1}u(t)dt \quad (r \in (0,1]),$$

$$K : C \to C^1, \quad Ku(r) = \int_1^r u(t)dt \quad (r \in [0,1]).$$

It is easy to see that $K$ is a bounded operator and standard arguments, invoking the Arzela-Ascoli theorem, show that $S$ is compact. Now, let $N_f : C^1 \to C$ be the Nemytskii operator associated to $f$, defined by

$$N_f(u) = f(\cdot, u(\cdot), u'(\cdot)) \quad \forall u \in C^1.$$

Note that $N_f$ is continuous and takes bounded sets into bounded sets. The following result has been proved in [63] if $a = \infty$. The proof for the case $a < \infty$ is completely similar to the one given in [63] and, actually, can be easily deduced from the properties of the above operators.

**Lemma 2** *The nonlinear operator*

$$\mathcal{M} : C^1_M \to C^1_M, \quad \mathcal{M} = K \circ \phi^{-1} \circ S \circ N_f.$$

*is well defined, compact, and* $u \in C^1_M$ *is a solution of (3.3) if and only if* $\mathcal{M}(u) = u$.

**Proposition 1** *Assume that* $0 < a < \infty$ *and* $\phi : (-a, a) \to \mathbb{R}$ *is an increasing homeomorphism such that* $\phi(0) = 0$. *Then, problem (3.3) has at least one solution.*

*Proof.* Let $u \in C^1_M$ and $v = \mathcal{M}(u)$. It follows that

$$||v'||_\infty = ||\phi^{-1} \circ S \circ N_f(u)||_\infty < a. \tag{3.4}$$

From (3.4) and

$$||v||_\infty = ||K(v')||_\infty,$$

it follows that

$$||v||_\infty < a.$$

Hence,

$$||v|| < 2a.$$

From the above estimate and Schauder fixed point theorem, we deduce that there exist $u \in C^1_M$ such that $u = \mathcal{M}(u)$. Using Lemma 2, it follows that $u$ is also a solution of (3.3). ■

**Proposition 2** *Let* $0 < a \le \infty$ *and* $\phi : \mathbb{R} \to (-a, a)$ *be an increasing homeomorphism such that* $\phi(0) = 0$. *If there exists* $\alpha > 0$ *such that* $\frac{\alpha}{N} < a$ *and*

$$|f(r, u, v)| \le \alpha \quad \text{for all} \quad (r, u, v) \in [0, 1] \times R_\alpha(\phi), \tag{3.5}$$

*where* $R_\alpha(\phi)$ *is the rectangle given by*

$$R_\alpha(\phi) = [-\phi^{-1}(\alpha/N)), -\phi^{-1}(-\alpha/N)] \times [\phi^{-1}(-\alpha/N), \phi^{-1}(\alpha/N)],$$

*then problem (3.3) has at least one solution* $u \in \Omega_\alpha(\phi)$, *where*

$$\Omega_\alpha(\phi) = \{u \in C^1_M : (u(r), u'(r)) \in R_\alpha(\phi), \quad \forall r \in [0, 1]\}.$$

*Proof.* We distinguish two cases.

*The case* $a = \infty$. We show that

$$\mathcal{M}(\Omega_\alpha(\phi)) \subset \Omega_\alpha(\phi), \tag{3.6}$$

where $\mathcal{M}$ is the fixed point operator associated to (3.3) (see Lemma 2). Let $u \in \Omega_\alpha(\phi)$ and $v = \mathcal{M}(u)$. Using (3.5), it results that

$$|\phi(v'(r))| = \left| \frac{1}{r^{N-1}} \int_0^r t^{N-1} f(t, u(t), u'(t)) dt \right| \leq \frac{\alpha}{N}$$

for all $r \in (0,1]$, and because $\phi(v'(0)) = 0$, the homeomorphic character of $\phi$ implies that

$$v'(r) \in [\phi^{-1}(-\alpha/N), \phi^{-1}(\alpha/N)] \quad \text{for all} \quad r \in [0,1].$$

Hence, using $v = K(v')$, we deduce that

$$v(r) \in [-\phi^{-1}(\alpha/N)), -\phi^{-1}(-\alpha/N)] \quad \text{for all} \quad r \in [0,1].$$

Consequently, $v \in \Omega_\alpha(\phi)$ and (3.6) is proved. Now, using the fact that $\Omega_\alpha(\phi)$ is a closed convex set in $C^1_M$ invariant for the compact operator $\mathcal{M}$, it follows by Schauder fixed point theorem that that there exists $u \in \Omega_\alpha(\phi)$ such that $\mathcal{M}(u) = u$, which is also a solution of (3.3).

*The case* $a < \infty$. Since $\frac{\alpha}{N} < a$, we can construct an increasing homeomorphism $\psi : \mathbb{R} \to \mathbb{R}$ such that

$$\psi(u) = \phi(u) \quad \text{for all} \quad u \in [\phi^{-1}(-\alpha/N), \phi^{-1}(\alpha/N)].$$

It is clear that $R_\alpha(\phi) = R_\alpha(\psi)$ and $\Omega_\alpha(\phi) = \Omega_\alpha(\psi)$. Hence, by the first step, problem

$$(r^{N-1}\psi(u'))' = r^{N-1} f(r, u, u'), \quad u'(0) = 0 = u(1),$$

has at least one solution $u \in \Omega_\alpha(\psi)$, which is also a solution of (3.3). ∎

**The proofs of Theorem 1 and 2.** Taking $v(x) = u(|x|)$ for all $x \in \mathcal{B}$, we have that Theorem 1 follows from Proposition 1 (with $\phi(u) = \frac{u}{\sqrt{1-u^2}}$) and Theorem 2 follows from Proposition 2 (with $\phi(u) = \frac{u}{\sqrt{1+u^2}}$).

**Remark 1** Taking in Proposition 2, $\phi(u) = |u|^{p-2}u$ $(p > 1)$ and $v(x) = u(|x|)$ for all $x \in \overline{\mathcal{B}}$, we recover an existence result already proved in [48].

When $f$ is independent of $\frac{du}{dr}$, it is easy to formulate simple uniqueness conditions for the solution of (3.3), and hence for the radial solution of (3.1) and of (3.2).

**Theorem 3** *If $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is such that $f(r,\cdot)$ is non decreasing for each fixed $r \in [0,1]$, then problem*

$$(r^{N-1}\phi(u'))' = r^{N-1} f(r,u), \quad u'(0) = 0 = u(1)$$

*has at most one solution, and the same is true for the radial solutions of problems*

$$div\left(\frac{\nabla v}{\sqrt{1-|\nabla v|^2}}\right) = f(|x|, v) \quad in \quad \mathcal{B}, \quad v = 0 \quad on \quad \partial\mathcal{B}$$

*and*

$$div\left(\frac{\nabla v}{\sqrt{1+|\nabla v|^2}}\right) = f(|x|, v) \quad in \quad \mathcal{B}, \quad v = 0 \quad on \quad \partial\mathcal{B}$$

*Proof.* Assume that $u$ and $w$ are solutions of (3.3), and that $u \neq w$. It follows from the boundary conditions that $E := \{r \in [0,1] : u'(r) \neq w'(r)\}$ has positive measure. Now, multiplying identity

$$\left[r^{N-1}(\phi(u') - \phi(w'))\right]' = r^{N-1}[f(r,u) - f(r,w)]$$

by $u - w$, integrating over $[0,1]$, integrating by parts and using the boundary conditions and the increasing character of $\phi$, we get

$$\begin{aligned} 0 &> -\int_E [\phi(u'(r)) - \phi(w'(r))][u'(r) - w'(r)] r^{N-1}\, dr \\ &= \int_0^1 [f(r,u(r)) - f(r,w(r))][u(r) - w(r)]\, dr \geq 0, \end{aligned} \tag{3.7}$$

a contradiction. ∎

## 3.2 Radial solutions in an annular domain

In this section, $\mathcal{A}$ denotes the annular domain $\{x \in R^N : 1 < |x| < 2\}$ and $f : [1,2] \times \mathbb{R}^2 \to \mathbb{R}$ is a continuous function. The first main result concerns the existence of classical radial solutions of the nonlinear Dirichlet problem associated with the *mean extrinsic curvature operator in Minkowski space*

$$\operatorname{div}\left(\frac{\nabla v}{\sqrt{1-|\nabla v|^2}}\right) = f(|x|, v, \frac{dv}{dr}) \quad \text{in} \quad \mathcal{A}, \quad v = 0 \quad \text{on} \quad \partial\mathcal{A}. \tag{3.8}$$

We have the following 'universal' existence result.

**Theorem 4** *Problem (3.8) has at least one classical radial solution for any continuous right-hand member $f$.*

The following result concerns the existence of radial solutions for the nonlinear Dirichlet problem associated with the *mean curvature operator in Euclidian space*

$$\operatorname{div}\left(\frac{\nabla v}{\sqrt{1+|\nabla v|^2}}\right) = f(|x|, v, \frac{dv}{dr}) \quad \text{in} \quad \mathcal{A}, \quad v = 0 \quad \text{on} \quad \partial\mathcal{A}. \tag{3.9}$$

**Theorem 5** *Assume that there exists $\alpha > 0$ such that $\alpha_N := \frac{2\alpha(2^N-1)}{N2^{N-1}} < 1$ and*

$$|f(r,u,v)| \le \alpha \quad \textit{for all} \quad (r,u,v) \in [1,2] \times P_\alpha,$$

*where $P_\alpha$ is the square given by $P_\alpha = \left[-\frac{\alpha_N}{\sqrt{1-\alpha_N^2}}, \frac{\alpha_N}{\sqrt{1-\alpha_N^2}}\right]^2$. Then, problem (3.9) has at least one classical radial solution.*

When dealing with the radial solutions for problems (3.8) or (3.9), we are led to consider the nonlinear Dirichlet problem

$$(r^{N-1}\phi(u'))' = r^{N-1}f(r,u,u'), \quad u(1) = 0 = u(2), \tag{3.10}$$

where $\phi(u) = \frac{u}{\sqrt{1-u^2}}$ or $\phi(u) = \frac{u}{\sqrt{1+u^2}}$, respectively.

As in the preceding section we start with the case where $\phi : (-a,a) \to \mathbb{R}$ is an increasing homeomorphism such that $\phi(0) = 0$ and $0 < a \le \infty$. In this situation we reformulate (3.10) as a fixed point problem.

In this section $C$ stands for the Banach space of continuous functions defined on $[1,2]$ endowed with the norm $||\cdot||_\infty$. On the other hand, $C^1$ denotes the Banach space of continuously differentiable functions on $[1,2]$ equipped with the norm $||u|| = ||u||_\infty + ||u'||_\infty$ and $C_D^1$ denotes the closed subspace of $C^1$ defined by

$$C_D^1 = \{u \in C^1 : u(1) = 0 = u(2)\}.$$

Consider the linear operators

$$L : C \to C, \quad Lu(r) = \gamma(r)\int_1^r t^{N-1}u(t)dt \quad (r \in [1,2]),$$

$$H : C \to C^1, \quad Hu(r) = \int_1^r u(t)dt \quad (r \in [1,2]).$$

It is not difficult to prove that $K$ is a bounded operator and $L$ is compact. Then, let $N_f$ be the Nemitskii operator associated to $f$ defined like in the previous section. The following lemma is the key ingredient used in the construction of the fixed point operator associated to (3.10).

**Lemma 3** *For each $h \in C$ there exists an unique $\alpha := Q_\phi(h) \in \mathbb{R}$ such that*

$$\int_1^2 \phi^{-1}(h(r) - Q_\phi(h)\gamma(r))dr = 0. \tag{3.11}$$

*Moreover, the function* $Q_\phi : C \to \mathbb{R}$ *is continuous and satisfies*

$$|Q_\phi(h)| \leq ||h/\gamma||_\infty \quad \textit{for all} \quad h \in C. \tag{3.12}$$

*Proof.* Let $h \in C$. We first prove uniqueness. Let $\alpha_i \in \mathbb{R}$ be such that

$$\int_1^2 \phi^{-1}(h(r) - \alpha_i \gamma(r))dr = 0 \quad (i = 1, 2).$$

It follows that there exists $r_0 \in [1, 2]$ such that

$$\phi^{-1}(h(r_0) - \alpha_1 \gamma(r_0)) = \phi^{-1}(h(r_0) - \alpha_2 \gamma(r_0)),$$

and using the injectivity of $\phi^{-1}$ we deduce that $\alpha_1 = \alpha_2$. For the existence, it is clear that the function

$$F : [-||h/\gamma||_\infty, ||h/\gamma||_\infty] \to \mathbb{R}, \quad t \mapsto \int_1^2 \phi^{-1}(h(r) - t\gamma(r))dr$$

is continuous and $F(-||h/\gamma||_\infty)F(||h/\gamma||_\infty) \leq 0$. Hence, there exists an unique $\alpha := Q_\phi(h) \in [-||h/\gamma||_\infty, ||h/\gamma||_\infty]$ such that $F(\alpha) = 0$, which means that (3.11) and (3.12) hold. The continuity of $Q_\phi$ follows immediately from the dominated convergence theorem. ■

The following result is a fixed point reformulation of (3.10) when $\phi : (-a, a) \to \mathbb{R}$ is an increasing homeomorphism such that $\phi(0) = 0$ and $0 < a \leq \infty$. In the case $a = \infty$, a different fixed point operator associated to (3.10) has been used in [62] in order to obtain a multiplicity result.

**Lemma 4** *Consider the nonlinear operator*

$$\mathcal{D} : C_D^1 \to C_D^1, \quad \mathcal{D} = H \circ \phi^{-1} \circ (I - \gamma Q_\phi) \circ L \circ N_f.$$

*Then,* $\mathcal{D}$ *is well defined, compact and* $u \in C_D^1$ *is a solution of (3.10) if and only if* $\mathcal{D}(u) = u$.

*Proof.* Let $u \in C_D^1$. It is clear that $\mathcal{D}(u)(1) = 0$. On the other hand, applying Lemma 3 with $h = (L \circ N_f)(u)$, it follows that $\mathcal{D}(u)(2) = 0$. Hence, $\mathcal{D}$ is well defined. Now, we know that the operators which compose $\mathcal{D}$ are continuous and take bounded sets into bounded sets. Moreover, the linear operator $H$ is compact. This implies the compactness of $\mathcal{D}$.

Let $u \in C_D^1$ be such that $\mathcal{D}(u) = u$. This implies that $u$ satisfies the Dirichlet boundary condition on $[1, 2]$, $||u'||_\infty < a$ and

$$\phi(u')(r) = \frac{1}{r^{N-1}} \int_1^r t^{N-1} N_f(u)(t)dt - \frac{1}{r^{N-1}} Q_\phi[(L \circ N_f)(u))]$$

for all $r \in [1, 2]$. This implies that $u$ satisfies the differential equation in (3.10). The remaining part of the proof is obvious. ■

The following result is an immediate consequence of the above fixed point reduction and Schauder fixed point theorem.

**Proposition 3** *Assume that* $0 < a < \infty$ *and* $\phi : (-a, a) \to \mathbb{R}$ *is an increasing homeomorphism such that* $\phi(0) = 0$. *Then, problem (3.10) has at least one solution.*

*Proof.* See the proof of Proposition 1. ■

**Proposition 4** *Let* $\phi : \mathbb{R} \to (-a, a)$ *be an increasing homeomorphism such that* $\phi(0) = 0$ *and* $0 < a \leq \infty$. *If there exists* $\alpha > 0$ *such that* $\alpha_N := \frac{2\alpha(2^N-1)}{N2^{N-1}} < a$ *and*

$$|f(r, u, v)| \leq \alpha \quad \text{for all} \quad (r, u, v) \in [1, 2] \times R_\alpha(\phi), \tag{3.13}$$

*where* $R_\alpha(\phi)$ *is the square given by*

$$R_\alpha(\phi) = [\phi^{-1}(-\alpha_N), \phi^{-1}(\alpha_N)]^2,$$

*then problem (3.10) has at least one solution* $u \in \Omega_\alpha(\phi)$, *where*

$$\Omega_\alpha(\phi) = \{u \in C_D^1 : (u(r), u'(r)) \in R_\alpha(\phi) \quad \forall r \in [1, 2]\}.$$

*Proof.* We distinguish two cases.

*The case* $a = \infty$. We show that

$$\mathcal{D}(\Omega_\alpha(\phi)) \subset \Omega_\alpha(\phi), \tag{3.14}$$

where $\mathcal{D}$ is the fixed point operator associated to (3.10) (see Lemma 4). Let $u \in \Omega_\alpha(\phi)$ and $v = \mathcal{D}(u)$. It follows that

$$\phi(v') = (L \circ N_f)(u) - \gamma Q_\phi[(L \circ N_f)(u)].$$

Using Lemma 3 and (3.13), we infer

$$||\gamma Q_\phi[(L \circ N_f)(u)]||_\infty \leq ||(L \circ N_f)(u)||_\infty \leq \frac{\alpha_N}{2}.$$

Hence

$$||\phi(v')||_\infty \leq \alpha_N.$$

This implies that $v \in \Omega_\alpha(\phi)$ and (3.14) holds. The result follows now from Schauder fixed point theorem.

*The case* $a < \infty$. To prove the result in this case, use the first step and similar arguments as in the proof of Proposition 2. ■

**The proofs of Theorem 4 and 5.** Taking $v(x) = u(|x|)$ for all $x \in \overline{\mathcal{A}}$, we have that Theorem 4 follows from Proposition 3 (with $\phi(u) = \frac{u}{\sqrt{1-u^2}}$) and Theorem 5 follows from Proposition 4 (with $\phi(u) = \frac{u}{\sqrt{1+u^2}}$).

When $f$ is independent of $\frac{du}{dr}$, proceeding exactly like in Section 2, we can prove the following simple uniqueness conditions for the solution of (3.10), and hence for the radial solution of (3.8) and of (3.9).

**Theorem 6** *If $f : [1,2] \times \mathbb{R} \to \mathbb{R}$ is such that $f(r,\cdot)$ is non decreasing for each fixed $r \in [1,2]$, then problem*

$$(r^{N-1}\phi(u'))' = r^{N-1}f(r,u), \quad u(1) = 0 = u(2)$$

*has at most one solution, and the same is true for the radial solutions in $\mathcal{A}$ of problems*

$$div\left(\frac{\nabla v}{\sqrt{1-|\nabla v|^2}}\right) = f(|x|,v) \quad in \quad \mathcal{A}, \quad v = 0 \quad on \quad \partial\mathcal{A}$$

*and*

$$div\left(\frac{\nabla v}{\sqrt{1+|\nabla v|^2}}\right) = f(|x|,v) \quad in \quad \mathcal{A}, \quad v = 0 \quad on \quad \partial\mathcal{A}$$

# Chapter 4

# Neumann problems with $\phi$-Laplacians

## 4.1 A class of Neumann boundary-value problems

Consider the Neumann boundary-value problem (BVP)

$$(r^{N-1}\phi(u'))' = r^{N-1}f(r,u,u'), \quad u'(R_1) = 0 = u'(R_2), \tag{4.1}$$

where $\phi$ is a homeomorphism such that $\phi(0) = 0$, belonging to one of the following classes ($0 < a < \infty$):

$$\begin{aligned}
\phi &: (-a,a) \to \mathbb{R} \quad (\textit{singular}),\\
\phi &: \mathbb{R} \to \mathbb{R} \quad (\textit{classical}),\\
\phi &: \mathbb{R} \to (-a,a) \quad (\textit{bounded}),
\end{aligned}$$

and $f : [R_1, R_2] \times \mathbb{R}^2 \to \mathbb{R}$ is continuous. By a *solution* of (4.1) we mean a continuously differentiable function $u$ such that $u' \in \mathrm{dom}(\phi)$, $r \mapsto r^{N-1}\phi(u')$ is differentiable and (4.1) is satisfied.

We denote by $C$ the Banach space of continuous functions defined on $[R_1, R_2]$ endowed with the usual norm $||\cdot||_\infty$, by $C^1$ the Banach space of continuously differentiable functions defined on $[R_1, R_2]$ endowed with the norm

$$||u|| = ||u||_\infty + ||u'||_\infty,$$

and by $C^1_\dagger$ the closed subspace of $C^1$ defined by

$$C^1_\dagger = \{u \in C^1 : u'(R_1) = 0 = u'(R_2)\}.$$

The corresponding open ball with center in 0 and radius $\rho$ is denoted by $B_\rho$. For any continuous function $w : [R_1, R_2] \to \mathbb{R}$, we write

$$w_L := \min_{[R_1,R_2]} w, \quad w_M := \max_{[R_1,R_2]} w.$$

Let us introduce the continuous projector

$$Q : C \to C, \quad Qu = \frac{N}{R_2^N - R_1^N} \int_{R_1}^{R_2} r^{N-1} u(r) dr,$$

the continuous function

$$\gamma : (0, \infty) \to \mathbb{R}, \quad \gamma(r) = \frac{1}{r^{N-1}},$$

and the linear operators

$$\begin{aligned} L &: C \to C, \quad Lu(r) = \gamma(r) \int_{R_1}^{r} t^{N-1} u(t) dt \quad (r \in (R_1, R_2]), \\ H &: C \to C^1, \quad Hu(r) = \int_{R_1}^{r} u(t) dt \quad (r \in [R_1, R_2]). \end{aligned}$$

It is not difficult to prove that $L$ is compact (Arzelà-Ascoli) and $H$ is bounded. Finally, we associate to $f$ its Nemytskii operator

$$N_f : C^1 \to C, \quad N_f(u) = f(\cdot, u(\cdot), u'(\cdot)).$$

It is known that $N_f$ is continuous and takes bounded sets into bounded sets.

Let us decompose any function $u \in C^1_\dagger$ in the form

$$u = \overline{u} + \widetilde{u} \quad (\overline{u} = u(R_1), \quad \widetilde{u}(R_1) = 0),$$

and let

$$\widetilde{C}^1_\dagger = \{u \in C^1_\dagger : u(R_1) = 0\}.$$

We first study an associated modified problem.

**Lemma 5** *If $\phi$ is singular, the set $\mathcal{S}$ of the solutions $(\overline{u}, \widetilde{u}) \in \mathbb{R} \times \widetilde{C}^1_\dagger$ of problem*

$$(r^{N-1} \phi(\widetilde{u}'))' = r^{N-1}[N_f(\overline{u} + \widetilde{u}) - Q \circ N_f(\overline{u} + \widetilde{u})] \tag{4.2}$$

*contains a continuum $\mathcal{C}$ whose projection on $\mathbb{R}$ is $\mathbb{R}$ and whose projection on $\widetilde{C}^1_\dagger$ is contained in the ball $B_{\rho(a)}$ where $\rho(a) = a(1 + R_2 - R_1)$.*

*Proof.* Consider the nonlinear operator

$$\widetilde{M} : \mathbb{R} \times \widetilde{C}^1_\dagger \to \widetilde{C}^1_\dagger, \quad \widetilde{M}(\overline{u}, \widetilde{u}) = [H \circ \phi^{-1} \circ L \circ (I - Q) \circ N_f](\overline{u} + \widetilde{u}).$$

Let $(\overline{u}, \widetilde{u}) \in \mathbb{R} \times \widetilde{C}^1_\dagger$ and $\widetilde{v} = \widetilde{M}(\overline{u}, \widetilde{u})$. It follows that $\widetilde{v} \in C^1$, $\widetilde{v}(R_1) = 0$, $||\widetilde{v}'||_\infty < a$ and

$$\phi(\widetilde{v}'(r)) = \gamma(r) \int_{R_1}^{r} t^{N-1}[N_f(\overline{u} + \widetilde{u})(t) - QN_f(\overline{u} + \widetilde{u})] dt \quad (r \in (R_1, R_2]). \tag{4.3}$$

Moreover $\phi(\widetilde{v}'(R_1)) = 0$ and

$$\begin{aligned}\phi(\widetilde{v}'(R_2)) &= \gamma(R_2)\int_{R_1}^{R_2} t^{N-1}[N_f(\overline{u}+\widetilde{u})(t) - QN_f(\overline{u}+\widetilde{u})]dt \\ &= \gamma(R_2)\left[\int_{R_1}^{R_2} t^{N-1}N_f(\overline{u}+\widetilde{u})(t)dt - QN_f(\overline{u}+\widetilde{u})\int_{R_1}^{R_2} t^{N-1}dt\right] \\ &= 0.\end{aligned}$$

Hence, $\widetilde{M}$ is well defined and it is clear that $\widetilde{M}$ is compact. Now, using (4.3) we infer that $(\overline{u}, \widetilde{u}) \in \mathbb{R} \times \widetilde{C}^1_\dagger$ is a solution of (4.2) if and only if

$$\widetilde{u} = \widetilde{M}(\overline{u}, \widetilde{u}). \tag{4.4}$$

So, it suffices to prove that the set of solution of the above problem contains a continuum of solutions whose projection on $\mathbb{R}$ is $\mathbb{R}$ and whose projection on $\widetilde{C}^1_\dagger$ is contained in the ball $B_{\rho(a)}$. Note that if $(\overline{u}, \widetilde{u}) \in \mathbb{R} \times \widetilde{C}^1_\dagger$ satisfies (4.4), then

$$||\widetilde{u}'||_\infty < a, \quad ||\widetilde{u}||_\infty < a(R_2 - R_1).$$

We deduce that

$$\widetilde{u} \neq \widetilde{M}(\overline{u}, \widetilde{u}) \quad \text{for all} \quad (\overline{u}, \widetilde{u}) \in \mathbb{R} \times \partial B_{\rho(a)}. \tag{4.5}$$

Consider the compact homotopy

$$\widetilde{\mathcal{M}} : [0,1] \times \widetilde{C}^1_\dagger \to \widetilde{C}^1_\dagger, \quad \widetilde{\mathcal{M}}(\lambda, \widetilde{u}) = [H \circ \phi^{-1} \circ \lambda L \circ (I - Q) \circ N_f](\widetilde{u}).$$

Note that

$$\widetilde{\mathcal{M}}(0, \cdot) = 0, \quad \widetilde{\mathcal{M}}(1, \cdot) = \widetilde{M}(0, \cdot).$$

It is clear that

$$\widetilde{u} \neq \widetilde{\mathcal{M}}(\lambda, \widetilde{u}) \quad \text{for all} \quad (\lambda, \widetilde{u}) \in [0,1] \times \partial B_{\rho(a)}.$$

Hence from the invariance under a homotopy of the Leray-Schauder degree [47] we deduce that

$$d_{LS}[I - \widetilde{M}(0, \cdot), B_{\rho(a)}, 0] = d_{LS}[I, B_{\rho(a)}, 0] = 1. \tag{4.6}$$

The result follows now from Lemma 1, (4.5) and (4.6). ∎

**Remark 2** Assume that $\phi$ is *classical* or *singular* and let us consider the nonlinear operator

$$\mathcal{N} : C^1_\dagger \to C^1_\dagger, \quad \mathcal{N} = P + QN_f + H \circ \phi^{-1} \circ L \circ (I - Q) \circ N_f,$$

where $P : C \to C$ is the continuous projector defined by $Pu = u(R_1)$. Using the same strategy as above, it is not difficult to prove that $\mathcal{N}$ is well defined, compact and for any $u \in C^1_\dagger$ one has that $u$ is a solution of (4.1) iff $u$ is a fixed point of $\mathcal{N}$.

In the *singular* case we have the following existence result.

**Theorem 7** *Assume that $\phi$ is singular and there exist $\varepsilon \in \{-1,1\}$ and $\rho > 0$ such that*

$$\varepsilon\,(sgn\ u)QN_f(u) \geq 0 \tag{4.7}$$

*for any $u \in C^1_\dagger$ satisfying $|u|_L \geq \rho$ and $||u'||_\infty < a$. Then the BVP (4.1) has at least one solution.*

*Proof.* Let $\mathcal{C}$ be the continuum given in Lemma 5 and $\widetilde{u}_1 \in \widetilde{C}^1_\dagger$ be such that $(\rho + \rho(a), \widetilde{u}_1) \in \mathcal{C}$. Taking $u_1 = \rho + \rho(a) + \widetilde{u}_1$, one has that $u_1 \geq 0$, $|u_1|_L > \rho$ and $||u_1'||_\infty < a$. Hence, from (4.7) it follows that $\varepsilon QN_f(u_1) \geq 0$. On the other hand, let $\widetilde{u}_2 \in \widetilde{C}^1_\dagger$ be such that $(-\rho - \rho(a), \widetilde{u}_2) \in \mathcal{C}$. Taking $u_2 = -\rho - \rho(a) + \widetilde{u}_2$, one has that $u_2 \leq 0$, $|u_2|_L > \rho$ and $||u_2'||_\infty < a$. Hence, from (4.7) it follows that $\varepsilon QN_f(u_2) \leq 0$. Using the intermediate value theorem, we infer that there exists $(\overline{u}, \widetilde{u}) \in \mathcal{C}$ such that $QN_f(\overline{u} + \widetilde{u}) = 0$. This implies that $u = \overline{u} + \widetilde{u}$ is a solution of (4.1). ■

The following very useful result is a direct consequence of the above theorem.

**Corollary 1** *Assume that $\phi$ is singular and let $h : [R_1, R_2] \times \mathbb{R}^2 \to \mathbb{R}$ and $g : [R_1, R_2] \times \mathbb{R} \to \mathbb{R}$ be continuous, with $h$ bounded on $[R_1, R_2] \times \mathbb{R} \times (-a, a)$, and $g$ such that*

$$\begin{array}{rrcl} & \lim\limits_{u \to -\infty} g(r,u) & = & +\infty, \quad \lim\limits_{u \to +\infty} g(r,u) = -\infty \\ (resp. & \lim\limits_{u \to -\infty} g(r,u) & = & -\infty, \quad \lim\limits_{u \to +\infty} g(r,u) = +\infty) \end{array} \tag{4.8}$$

*uniformly in $r \in [R_1, R_2]$. Then the BVP*

$$(r^{N-1}\phi(u'))' + r^{N-1}g(r,u) = r^{N-1}h(r,u,u'), \quad u'(R_1) = 0 = u'(R_2)$$

*has at least one solution.*

*In particular, the problem*

$$(r^{N-1}\phi(u'))' + \mu r^{N-1}u = r^{N-1}h(r,u,u'), \quad u'(R_1) = 0 = u'(R_2)$$

*has at least one solution for each $\mu \neq 0$.*

**Remark 3** Consider the non-homogeneous Neumann problem with a singular $\phi$-Laplacian

$$\left(\frac{ru'}{\sqrt{1-u'^2}}\right)' = r(\kappa u + \lambda), \qquad u'(0) = 0,\ u'(R) = \gamma,$$

where $R > 0$, $\kappa, \lambda \in \mathbb{R}$ and $\gamma \in (-1,1)$. If $\kappa \neq 0$ (the case $\kappa = 0$ follows immediately by a direct integration), then it is proved, by a shooting argument, in Theorem 4.2 and Theorem 6.9 from [80] that the above Neumann problem has at least one solution. Now, it follows from Corollary 6 [14] that the Neumann problem

$$\left(\frac{r^{N-1}u'}{\sqrt{1-u'^2}}\right)' = r^{N-1}(\kappa u + h(r,u,u')), \qquad u'(0)=0,\ u'(R)=\gamma,$$

has at least one solution if $N \geq 1$ is an integer and the continuous perturbation $h : [0,R]\times\mathbb{R}^2 \to \mathbb{R}$ is bounded on $[0,R]\times\mathbb{R}\times(-1,1)$. For the geometric motivation of the above problems see the paper [80].

We now consider the *bounded* and *classical* cases.

**Lemma 6** *Let $\psi : (-a,a) \to (-b,b)$ be a homeomorphism such that $\psi(0)=0$ and $0 < a, b \leq \infty$. Assume that there exists a constant $k \geq 0$ such that $\frac{kR_2}{N} < b$ and*

$$|f(r,u,v)| \leq k \quad for \quad all \quad (r,u,v) \in [R_1,R_2]\times\mathbb{R}^2. \tag{4.9}$$

*If $u$ is a possible solution of the Neumann BVP*

$$(r^{N-1}\psi(u'))' = r^{N-1}f(r,u,u'), \quad u'(R_1) = 0 = u'(R_2), \tag{4.10}$$

*then*

$$||u'||_\infty \leq \max(|\psi^{-1}(\pm kR_2/N)|) =: \rho_1(\psi). \tag{4.11}$$

*Proof.* If $u \in C^1_\dagger$ solves (4.10) then

$$u'(r) = \psi^{-1}\left(\gamma(r)\int_{R_1}^r t^{N-1}f(t,u(t),u'(t))dt\right) \quad (r \in [R_1,R_2]). \tag{4.12}$$

Using (4.9) we get

$$\left|\gamma(r)\int_{R_1}^r t^{N-1}f(t,u(t),u'(t))dt\right| \leq \frac{kR_2}{N},$$

which, together with (4.12), gives (4.11). ∎

**Theorem 8** *Let $\phi : \mathbb{R} \to (-b,b)$ be a homeomorphism such that $\phi(0)=0$ with $0 < b \leq \infty$ and let $f$ be like in Lemma 6. Assume that there exist $a \in (\rho_1(\phi),\infty)$, $\varepsilon \in \{-1,1\}$ and $\rho > 0$ such that (4.7) holds for any $u \in C^1_\dagger$ satisfying $|u|_L \geq \rho$ and $||u'||_\infty < a$. Then the BVP (4.1) has at least one solution.*

*Proof.* Let $d \in (\rho_1(\phi),a)$ and $\psi : (-a,a) \to \mathbb{R}$ be a homeomorphism which coincides with $\phi$ on $[-d,d]$. Then, $\rho_1(\phi) = \rho_1(\psi)$ and using Lemma 6 one has that the solutions of (4.10) coincide with the solutions of (4.1). Now the result follows from Theorem 7. ∎

**Lemma 7** *Let* $R_1 > 0$ *and* $\psi : (-a, a) \to (-b, b)$ *be a homeomorphism such that* $\psi(0) = 0$ *and* $0 < a, b \leq \infty$. *Assume that there exists* $c \in C$ *such that* $2R_1^{1-N}||c^-/\gamma||_{L^1} < b$ *and*

$$f(r, u, v) \geq c(r), \quad for \quad all \quad (r, u, v) \in [R_1, R_2] \times \mathbb{R}^2. \tag{4.13}$$

*If* $u$ *is a possible solution of the Neumann BVP (4.10) then*

$$||u'||_\infty \leq \max(|\psi^{-1}(\pm 2R_1^{1-N}||c^-/\gamma||_{L^1})|) =: \rho_2(\psi). \tag{4.14}$$

*Proof.* First of all, let us note that

$$|r^{N-1}f(r, u, v)| \leq r^{N-1}f(r, u, v) + 2\frac{c^-(r)}{\gamma(r)} \tag{4.15}$$

for all $(r, u, v) \in [R_1, R_2] \times \mathbb{R}^2$. If $u$ solves (4.10) then

$$QN_f(u) = 0. \tag{4.16}$$

From (4.15) and (4.16) we get

$$\int_{R_1}^{R_2} |r^{N-1}f(r, u(r), u'(r))|dr \leq 2||c^-/\gamma||_{L^1}. \tag{4.17}$$

Now the result follows from (4.12) and (4.17). ∎

**Theorem 9** *Let* $R_1 > 0$ *and* $\phi : \mathbb{R} \to (-b, b)$ *be a homeomorphism such that* $\phi(0) = 0$ *with* $0 < b \leq \infty$ *and* $f$ *be like in Lemma 7. Assume that there exist* $a \in (\rho_2(\phi), \infty)$, $\varepsilon \in \{-1, 1\}$ *and* $\rho > 0$ *such that (4.7) holds for any* $u \in C^1_\dagger$ *satisfying* $|u|_L \geq \rho$ *and* $||u'||_\infty < a$. *Then the BVP (4.1) has at least one solution.*

*Proof.* See the proof of Theorem 8. ∎

**Remark 4** In the particular case $N = 1$ Theorem 7 was proved in [93], while Theorems 8 and 9 were obtained in [21].

## 4.2 Existence of radial solutions

The results of the previous section can be used to derive the existence of radial solutions for the Neumann problems (3.1) and (2.4).

**Theorem 10** *Assume that there exist* $\varepsilon \in \{-1, 1\}$ *and* $\rho > 0$ *such that*

$$\varepsilon(sgn\ u)\int_{R_1}^{R_2} r^{N-1}f(r, u(r), u'(r))dr \geq 0 \tag{4.18}$$

*for all* $u \in C^1_\dagger$ *such that* $|u|_L \geq \rho$ *and* $\|u'\|_\infty < 1$. *Then problem (3.1) has at least one classical radial solution.*

*Proof.* Theorem 7 applies with

$$\phi : (-1, 1) \to \mathbb{R}, \quad \phi(y) = \frac{y}{\sqrt{1 - y^2}}. \tag{4.19}$$

■

**Corollary 2** *Let $h : [R_1, R_2] \times \mathbb{R}^2 \to \mathbb{R}$ and $g : [R_1, R_2] \times \mathbb{R} \to \mathbb{R}$ be continuous, with $h$ bounded on $[R_1, R_2] \times \mathbb{R} \times (-1, 1)$, and $g$ such that condition (4.8) holds. Then the Neumann BVP*

$$\mathcal{M}v + g(|x|, v) = h(|x|, v, \frac{dv}{dr}) \quad in \quad \mathcal{A}, \quad \frac{\partial v}{\partial \nu} = 0 \quad on \quad \partial\mathcal{A}$$

*has at least one radial solution.*

**Example 1** *For any $p > 1$ and any $l \in C$, the Neumann problems*

$$\mathcal{M}v \pm |v|^{p-1}v = l(|x|) \quad in \quad \mathcal{A}, \quad \frac{\partial v}{\partial \nu} = 0 \quad on \quad \partial\mathcal{A}$$

*have at least one radial solution.*

As another example of application, let us consider the Neumann problem

$$\mathcal{M}v + g(v) = l(|x|) \quad \text{in} \quad \mathcal{A}, \quad \frac{\partial v}{\partial \nu} = 0 \quad \text{on} \quad \partial\mathcal{A}, \tag{4.20}$$

where $g : \mathbb{R} \to \mathbb{R}$ is continuous and $l \in C$. It is not difficult to check that Theorem 10 with $f(r, u, u') = l(r) - g(u)$ yields the following *Landesman-Lazer-type existence condition.*

**Corollary 3** *If either*

$$\limsup_{v \to -\infty} g(v) < \frac{N}{R_2^N - R_1^N} \int_{R_1}^{R_2} r^{N-1} l(r) dr < \liminf_{v \to +\infty} g(v)$$

*or*

$$\limsup_{v \to +\infty} g(v) < \frac{N}{R_2^N - R_1^N} \int_{R_1}^{R_2} r^{N-1} l(r) dr < \liminf_{v \to -\infty} g(v),$$

*then problem (4.20) has at least one classical radial solution.*

**Example 2** *The Neumann problem*

$$\mathcal{M}v + \arctan v + \sin v = l(|x|) \quad in \quad \mathcal{A}, \quad \frac{\partial v}{\partial \nu} = 0 \quad on \quad \partial\mathcal{A}$$

*has a radial solution if* $l \in C$ *is such that*

$$1 - \frac{\pi}{2} < \frac{N}{R_2^N - R_1^N} \int_{R_1}^{R_2} r^{N-1} l(r) dr < \frac{\pi}{2} - 1.$$

Let $\phi$ be defined by (4.19). For a constant $q \in [0,1)$, we set

$$\tilde{\rho}(q) := \phi(q).$$

**Theorem 11** *Assume that there is some* $k \geq 0$ *such that* $kR_2 < N$ *and (4.9) holds true. If there exist constants* $a > \tilde{\rho}(kR_2/N)$, $\varepsilon \in \{-1, 1\}$ *and* $\rho > 0$ *such that the sign conditions condition (4.18) are fulfilled for all* $u \in C^1_\dagger$ *such that* $|u|_L \geq \rho$ *and* $\|u'\|_\infty < a$, *then problem (2.4) has at leat one classical radial solution.*

*Proof.* Theorem 8 applies with

$$\phi : \mathbb{R} \to (-1,1), \quad \phi(y) = \frac{y}{\sqrt{1+y^2}}.$$

■

**Theorem 12** *Let* $R_1 > 0$ *and assume that there is some* $c \in C$ *such that* $k := 2R_1^{1-N} ||c^-/\gamma||_{L^1} < 1$ *and (4.13) holds true. If there exist constants* $a > \tilde{\rho}(k)$, $\varepsilon \in \{-1, 1\}$ *and* $\rho > 0$ *such that the sign conditions (4.18) are fulfilled for all* $u \in C^1_\dagger$ *such that* $|u|_L \geq \rho$ *and* $\|u'\|_\infty < a$, *then problem (2.4) has at leat one classical radial solution.*

*Proof.* It follows from Theorem 9. ■

**Remark 5** It is worth to point out that Theorems 8 and 9 also can be employed to derive existence results of radial solutions for Neumann problems in an annular domain, associated to $p$-Laplacian operator.

## 4.3 Upper and lower solutions in the singular case

In this section, we extend the method of upper and lower solutions (see e.g. [43]) to the Neumann boundary value problem (4.1).

**Definition 1** *A* lower solution $\alpha$ *(resp.* upper solution $\beta$*) of (4.1) is a function* $\alpha \in C^1$ *such that* $\|\alpha'\|_\infty < a$, $r^{N-1}\phi(\alpha') \in C^1$, $\alpha'(R_1) \geq 0 \geq \alpha'(R_2)$ *(resp.* $\beta \in C^1$, $\|\beta'\|_\infty < a$, $r^{N-1}\phi(\beta') \in C^1$, $\beta'(R_1) \leq 0 \leq \beta'(R_2)$*) and*

$$\begin{array}{rrcl} & (r^{N-1}\phi(\alpha'(r)))' & \geq & r^{N-1}f(r,\alpha(r),\alpha'(r)) \\ (resp. & (r^{N-1}\phi(\beta'(r)))' & \leq & r^{N-1}f(r,\beta(r),\beta'(r))) \end{array} \tag{4.21}$$

*for all* $r \in [R_1, R_2]$.

Below we shall invoke the hypothesis:

$(H)$ $R_1 > 0$ and $\phi : (-a, a) \to \mathbb{R}$ is an increasing homeomorphism such that $\phi(0) = 0$ with $0 < a < \infty$.

**Theorem 13** *Assume that (H) holds true. If (4.1) has a lower solution* $\alpha$ *and a upper solution* $\beta$ *such that* $\alpha(r) \leq \beta(r)$ *for all* $r \in [R_1, R_2]$, *then problem (4.1) has a solution* $u$ *such that* $\alpha(r) \leq u(r) \leq \beta(r)$ *for all* $r \in [R_1, R_2]$.

*Proof.* Let $\gamma : [R_1, R_2] \times \mathbb{R} \to \mathbb{R}$ be the continuous function defined by

$$\gamma(r,u) = \begin{cases} \beta(r) & if \quad u > \beta(r) \\ u & if \quad \alpha(r) \leq u \leq \beta(r) \\ \alpha(r) & if \quad u < \alpha(r), \end{cases}$$

and define $F : [R_1, R_2] \times \mathbb{R}^2 \to \mathbb{R}$ by $F(r, u, v) = f(r, \gamma(r, u), v)$. We consider the modified problem

$$(r^{N-1}\phi(u'))' = r^{N-1}(F(r,u,u') + u - \gamma(r,u)), \quad u'(R_1) = 0 = u'(R_2) \tag{4.22}$$

and first show that if $u$ is a solution of (4.22) then $\alpha(r) \leq u(r) \leq \beta(r)$ for all $r \in [R_1, R_2]$, so that $u$ is a solution of (4.1). Suppose by contradiction that there is some $r_0 \in [R_1, R_2]$ such that $[\alpha - u]_M = \alpha(r_0) - u(r_0) > 0$. If $r_0 \in (R_1, R_2)$ then $\alpha'(r_0) = u'(r_0)$ and there are sequences $(r_k)$ in $[r_0 - \varepsilon, r_0)$ and $(r'_k)$ in $(r_0, t_0 + \varepsilon]$ converging to $r_0$ such that $\alpha'(r_k) - u'(r_k) \geq 0$ and $\alpha'(r'_k) - u'(r'_k) \leq 0$. As $\phi$ is an increasing homeomorphism, this implies

$$\begin{array}{rcl} r_k^{N-1}\phi(\alpha'(r_k)) - r_0^{N-1}\phi(\alpha'(r_0)) & \geq & r_k^{N-1}\phi(u'(r_k)) - r_0^{N-1}\phi(u'(r_0)) \\ {r'}_k^{N-1}\phi(\alpha'(r'_k)) - r_0^{N-1}\phi(\alpha'(r_0)) & \leq & {r'}_k^{N-1}\phi(u'(r'_k)) - r_0^{N-1}\phi(u'(r_0)) \end{array}$$

and hence

$$(r^{N-1}\phi(\alpha'(r)))'_{r=r_0} \leq (r^{N-1}\phi(u'(r)))'_{r=r_0}.$$

Hence, because $\alpha$ is a lower solution of (4.1) we obtain

$$\begin{array}{rcl} (r^{N-1}\phi(\alpha'(r)))'_{r=r_0} & \leq & (r^{N-1}\phi(u'(r)))'_{r=r_0} \\ & = & r_0^{N-1}(f(r_0,\alpha(r_0),\alpha'(r_0)) + u(r_0) - \alpha(r_0)) \\ & < & r_0^{N-1}f(r_0,\alpha(r_0),\alpha'(r_0)) \\ & \leq & (r^{N-1}\phi(\alpha'(r)))'_{r=r_0}, \end{array}$$

a contradiction. If $[\alpha - u]_M = \alpha(R_1) - u(R_1) > 0$, then $\alpha'(R_1) - u'(R_1) = \alpha'(R_1) \leq 0$. Using the fact that $\alpha'(R_1) \geq 0$, we deduce that $\alpha'(R_1) = \alpha'(R_1) - u'(R_1) = 0$. This implies that $\phi(\alpha'(R_1)) = \phi(u'(R_1))$. On the other hand, $[\alpha - u]_M = \alpha(R_1) - u(R_1)$ implies, reasoning in a similar way as for $r_0 \in (R_1, R_2)$, that

$$(r^{N-1}\phi(\alpha'(r)))'_{r=R_1} \leq (r^{N-1}\phi(u'(r)))'_{r=R_1}.$$

Using the inequality above and $\alpha'(R_1) = u'(R_1)$, since $R_1 > 0$, we can proceed as in the case $r_0 \in (R_1, R_2)$ to obtain again a contradiction. The case where $r_0 = R_2$ is similar. In consequence we have that $\alpha(r) \leq u(r)$ for all $r \in [R_1, R_2]$. Analogously, using the fact that $\beta$ is a upper solution of (4.1), we can show that $u(r) \leq \beta(r)$ for all $r \in [R_1, R_2]$.

We now apply Corollary 1 to the modified problem (4.22) to obtain the existence of a solution. ■

**Remark 6** (i) We remark that if in theorem above $\alpha, \beta$ are strict, then $\alpha(r) < u(r) < \beta(r)$ for all $r \in [R_1, R_2]$. Moreover, let us consider the open set

$$\Omega_{\alpha,\beta} = \{u \in C^1_\dagger : \alpha(r) < u(r) < \beta(r) \quad \text{for all} \quad r \in [R_1, R_2], \quad ||u'||_\infty < a\}.$$

Then, arguing as in the proof of Lemma 3 [22] one has that

$$d_{LS}[I - \mathcal{N}, \Omega_{\alpha,\beta}, 0] = -1,$$

where $\mathcal{N}$ is the fixed point operator associated to (4.1) introduced in Remark (2)

(ii) In contrast to the classical $p$-Laplacian or Euclidean mean curvature cases, no Nagumo-type condition is required upon $f$ in Theorem 13.

We now show, adapting an argument introduced by Amann-Ambrosetti-Mancini [4] in semilinear elliptic problems, that the existence conclusion in Theorem 13 also holds when the lower and upper solutions are not ordered. See [22] for the case where $N = 1$.

**Theorem 14** *Assume that (H) holds true. If (4.1) has a lower solution $\alpha$ and an upper solution $\beta$, then problem (4.1) has at least one solution.*

*Proof.* Let $\mathcal{C}$ be given by Lemma 5. If there is some $(\overline{u}, \widetilde{u}) \in \mathcal{C}$ such that

$$\int_{R_1}^{R_2} r^{N-1} f(r, \overline{u} + \widetilde{u}(r), \widetilde{u}'(r))\, dr = 0,$$

then $\overline{u} + \widetilde{u}$ solves (4.1). If

$$\int_{R_1}^{R_2} r^{N-1} f(r, \overline{u} + \widetilde{u}(r), \widetilde{u}'(r))\, dr > 0$$

for all $(\overline{u}, \widetilde{u}) \in \mathcal{C}$, then, using (4.2), $\overline{u} + \widetilde{u}$ is an upper solution for (4.1) for each $(\overline{u}, \widetilde{u}) \in \mathcal{C}$. Then, for $(\alpha_M + a(1+R_2-R_1), \widetilde{u}) \in \mathcal{C}$, $\alpha_M + a(1+R_2-R_1) + \widetilde{u}(r) \geq \alpha(r)$ for all $r \in [R_1, R_2]$ is an upper solution and the existence of a solution to (4.1) follows from Theorem 13. Similarly, if

$$\int_{R_1}^{R_2} r^{N-1} f(r, \overline{u} + \widetilde{u}(r), \widetilde{u}'(r))\, dr < 0$$

for all $(\overline{u}, \widetilde{u}) \in \mathcal{C}$, then $(\beta_L - a(1 + R_2 - R_1), \widetilde{u}) \in \mathcal{C}$ gives the lower solution $\beta_L - a(1 + R_2 - R_1) + \widetilde{u}(r) \leq \beta(r)$ for all $r \in [R_1, R_2]$ and the existence of a solution. ■

**Remark 7** Assume that $\phi : \mathbb{R} \to \mathbb{R}$ is a increasing homeomorphism with $\phi(0) = 0$ and the nonlinearity $f$ is bounded from below by a continuous function $c \in C$, that is (4.13) holds. Using Lemma 6 and the same strategy as in the proof of Theorem 8, it follows that Theorem 14 holds also for *classical* homeomorphisms under the additional condition (4.13).

The choice of constant lower and upper solutions in Theorems 13 and 14 leads to the following simple existence condition.

**Corollary 4** *If (H) holds true then problem (4.1) has at least one solution if there exist constants $A$ and $B$ such that*

$$f(r, A, 0) \cdot f(r, B, 0) \leq 0$$

*for all $r \in [R_1, R_2]$.*

A simple application of Theorem 14 provides a necessary and sufficient condition of existence of a solution of (4.1) when $f = f(r, u)$ and $f(r, \cdot)$ is monotone. We adapt an argument first introduced for semilinear Dirichlet problems by Kazdan-Warner [74].

**Corollary 5** *Assume that (H) holds true. If $f : [R_1, R_2] \times \mathbb{R} \to \mathbb{R}$ is continuous and $f(r, \cdot)$ is either non decreasing or non increasing for every $r \in [R_1, R_2]$, then problem*

$$(r^{N-1}\phi(u'))' = r^{N-1} f(r, u), \quad u'(R_1) = 0 = u'(R_2) \tag{4.23}$$

*is solvable if and only if there exists $c \in \mathbb{R}$ such that*

$$\int_{R_1}^{R_2} r^{N-1} f(r, c)\, dr = 0. \tag{4.24}$$

*Proof.* *Necessity.* If problem (4.23) has a solution $u$, then, integrating both members of the differential equation in (4.23) and using the boundary condition, it follows that

$$\int_{R_1}^{R_2} r^{N-1} f(r, u(r))\, dr = 0. \tag{4.25}$$

Assuming for example that $f(r, \cdot)$ is non decreasing for every $r \in [R_1, R_2],$, we deduce from (4.25) that

$$\int_{R_1}^{R_2} r^{N-1} f(r, u_L)\, dr \leq 0 \leq \int_{R_1}^{R_2} r^{N-1} f(r, u_M)\, dr,$$

so that, by the intermediate value theorem, there exists some $c \in [u_L, u_M]$ satisfying (4.24). The reasoning is similar when $f(r, \cdot)$ is non decreasing for every $r \in [R_1, R_2],$.

*Sufficiency.* If $c \in \mathbb{R}$ satisfies (4.24), then the problem

$$(r^{N-1}\phi(u'))' = r^{N-1} f(r, c), \quad u'(R_1) = 0 = u'(R_2) \tag{4.26}$$

has a one-parameter family of solutions of the form $d + \widetilde{w}(r)$ with $\widetilde{w} \in \widetilde{C}^1_\dagger$. There exists $d_1 \leq d_2$ such that, for all $r \in [R_1, R_2]$,

$$\alpha(r) := d_1 + \widetilde{w}(r) \leq c \leq d_2 + \widetilde{w}(r) =: \beta(r).$$

Hence, if $f(r, \cdot)$ is non decreasing for each $r \in [R_1, R_2]$, then

$$(r^{N-1}\phi(\alpha'(r)))' = (r^{N-1}\phi(\widetilde{w}'(r)))' = r^{N-1} f(r, c) \geq r^{N-1} f(r, \alpha(r))$$

and $\alpha$ is a lower solution for (4.23). Similarly $\beta$ is an upper solution for (4.23). A similar argument shows that, if $f(r, \cdot)$ is non increasing for every $r \in [R_1, R_2]$, $\alpha$ is an upper solution and $\beta$ a lower solution for (4.23). So the result follows from Theorem 14. ∎

The results above can be applied to classical radial solutions of the Neumann problem (3.1).

**Corollary 6** *Let $R_1 > 0$. Problem (3.1) has at least one classical radial solution if there exist constants $A$ and $B$ such that*

$$f(r, A, 0) \cdot f(r, B, 0) \leq 0$$

*for all $r \in [R_1, R_2]$.*

**Corollary 7** *Let $R_1 > 0$. If $f : [R_1, R_2] \times \mathbb{R} \to \mathbb{R}$ is continuous and $f(r, \cdot)$ is either non decreasing or non increasing for every $r \in [R_1, R_2]$, then problem*

$$\mathcal{M}v = f(|x|, v) \quad in \quad \mathcal{A}, \quad \frac{\partial v}{\partial \nu} = 0 \quad on \quad \partial\mathcal{A}$$

*has a classical radial solution if and only if there exists $c \in \mathbb{R}$ such that (4.24) holds.*

**Example 3** *Let $f(v) = e^v$ or $f(v) = |v|^{p-1}v^+$ with $p > 1$. If $R_1 > 0$ then the Neumann problem*

$$\begin{array}{rcl} \mathcal{M}v + f(v) & = & l(|x|) \quad in \quad \mathcal{A}, \quad \dfrac{\partial v}{\partial \nu} = 0 \quad on \quad \partial\mathcal{A} \\ (resp. \quad \mathcal{M}v - f(v) & = & l(|x|) \quad in \quad \mathcal{A}, \quad \dfrac{\partial v}{\partial \nu} = 0 \quad on \quad \partial\mathcal{A}) \end{array}$$

*has a classical radial solution if and only if $l \in C$ is such that*

$$\int_{R_1}^{R_2} r^{N-1} l(r)\, dr > 0 \quad (resp. \quad \int_{R_1}^{R_2} r^{N-1} l(r)\, dr < 0).$$

*The same result holds true if we replace the operator $\mathcal{M}$ by the p-Laplacian operator.*

**Remark 8** Multiplicity results of the Ambrosetti-Prodi type similar to the ones obtained in [**?**] for $N = 1$ can be deduced in a similar way from Theorem 13 and Remark 6 (i).

## 4.4 Pendulum-like nonlinearities

Consider the Neumann problem

$$\mathcal{M}v + b\sin(v) = l(|x|) \quad \text{in} \quad \mathcal{A}, \quad \frac{\partial v}{\partial \nu} = 0 \quad \text{on} \quad \partial\mathcal{A}, \tag{4.27}$$

where $b > 0$ and $l \in C$.

Let us suppose also that $R_1 > 0$. Then, using Theorem 13, Remark 6 (i) and the method introduced in [100], one can prove that (4.27) has at least one radial solution if $||l||_\infty \le b$ and at least two radial solutions if $||l||_\infty < b$.

The following result shows that one has existence even in the case $R_1 = 0$ for any $l$ with $Ql = 0$, under an additional condition concerning the distance between $R_1$ and $R_2$. We adapt to our situation an argument used in [115].

**Theorem 15** *If $Ql = 0$ and $2(R_2 - R_1) \le 1$, then (4.27) has at least one classical radial solution.*

*Proof.* It is clear that it is sufficient to prove the existence of at least one solution for the Neumann problem

$$(r^{N-1}\phi(u'))' + r^{N-1} b\sin(u) = r^{N-1} l(r), \quad u'(R_1) = 0 = u'(R_2), \tag{4.28}$$

with $\phi$ given in (4.19). Let us make the change of variable

$$u = \arcsin(w).$$

Then, we obtain the Neumann problem

$$\left(r^{N-1}\phi\left(\frac{w'}{\sqrt{1-w^2}}\right)\right)' + r^{N-1} bw = r^{N-1} l(r), \quad w'(R_1) = 0 = w'(R_2). \tag{4.29}$$

Consider the closed subspace of $C$ defined by

$$\widehat{C} = \{w \in C : Qw = 0\}$$

and denotes by $N$ the Nemytskii operator $N_f$ associated to $f(r, w) = l(r) - bw$. Consider also the nonlinear operator

$$\mathcal{T} : \widehat{K} \to \widehat{C}, \quad \mathcal{T}(w) = (I - Q) \circ H \circ \sqrt{1 - w^2}\phi^{-1} \circ L \circ N(w),$$

where

$$\widehat{K} = \{w \in \widehat{C} : ||w||_\infty \leq 2(R_2 - R_1)\}.$$

One has that $\mathcal{T}$ is well defined and compact. It is clear that

$$||\mathcal{T}(w)||_\infty < 2(R_2 - R_1) \leq 1 \quad \text{for all} \quad w \in \widehat{K}.$$

Using the Schauder's fixed point theorem, we infer that there exist $w \in \widehat{K}$ such that $w = \mathcal{T}(w)$, and because $||w||_\infty < 1$, it follows that $w$ is a solution of (4.29) and $u = \arcsin(w)$ is a solution of (4.28). ■

# Chapter 5

# Pendulum-like nonlinearities - 1

## 5.1 Notation, function spaces and operators

Let $0 \leq R_1 < R_2$. We denote by $C$ the Banach space of continuous functions defined on $[R_1, R_2]$ endowed with the usual norm $||\cdot||_\infty$, by $C^1$ the Banach space of continuously differentiable functions defined on $[R_1, R_2]$ endowed with the norm

$$||u|| = ||u||_\infty + ||u'||_\infty,$$

and by $C^1_\dagger$ the closed subspace of $C^1$ defined by

$$C^1_\dagger = \{u \in C^1 : u'(R_1) = 0 = u'(R_2)\}.$$

The corresponding open ball with center in 0 and radius $\rho$ is denoted by $B_\rho$. For any continuous function $w : [R_1, R_2] \to \mathbb{R}$, we write

$$w_L = \min_{[R_1,R_2]} w, \quad w_M = \max_{[R_1,R_2]} w.$$

Let us introduce the continuous projectors

$$Q : C \to C, \quad \underline{u} = Qu = \frac{N}{R_2^N - R_1^N} \int_{R_1}^{R_2} r^{N-1} u(r) dr,$$

$$P : C \to C, \quad Pu = u(R_1),$$

the continuous function

$$\gamma : (0, \infty) \to \mathbb{R}, \quad \gamma(r) = \frac{1}{r^{N-1}},$$

and the linear operators

$$\begin{aligned} L &: C \to C, \quad Lu(r) = \gamma(r)\int_{R_1}^{r} t^{N-1}u(t)dt \quad (r \in (R_1, R_2]), \quad Lu(0) = 0, \\ H &: C \to C^1, \quad Hu(r) = \int_{R_1}^{r} u(t)dt \quad (r \in [R_1, R_2]). \end{aligned}$$

It is not difficult to prove that $L$ is compact (Arzelà-Ascoli) and $H$ is bounded. Finally, we denote by $\widehat{C}^1_\dagger$ the closed subspace of $C^1_\dagger$ defined by

$$\widehat{C}^1_\dagger = \{u \in C^1_\dagger : \underline{u} = 0\},$$

and note that

$$C^1_\dagger = \mathbb{R} \oplus \widehat{C}^1_\dagger,$$

so, any $u \in C^1_\dagger$ can be uniquely written as $u = \underline{u} + \widehat{u}$, with $\underline{u} \in \mathbb{R}$, $\widehat{u} \in \widehat{C}^1_\dagger$.

## 5.2 A fixed point operator and degree computations

Throughout this section we assume that $\phi$ is *singular*. We note that the case where $N = 1$ and $R_1 = 0$ in the results of this section is considered in [93].

**Proposition 5** *Assume that $F : C^1_\dagger \to C$ is continuous and takes bounded sets into bounded sets. The function $u \in C^1_\dagger$ is a solution of the abstract Neumann problem*

$$(r^{N-1}\phi(u'))' = r^{N-1}F(u), \quad u'(R_1) = 0 = u'(R_2) \tag{5.1}$$

*if and only if it is a fixed point of the compact operator $M_F$ defined on $C^1_\dagger$ by*

$$M_F = P + QF + H \circ \phi^{-1} \circ L \circ (I - Q) \circ F.$$

*Furthermore, one has $\|(M_F(u))'\|_\infty < a$ for all $u \in C^1_\dagger$.*

*Proof.* Let $u \in C^1_\dagger$ and $v = M_F(u)$. One has that $v \in C^1$ and

$$\phi(v') = L \circ (I - Q) \circ F(u).$$

So, $\phi(v'(R_1)) = 0$ and because $QF(u)$ is constant,

$$\phi(v'(R_2)) = \frac{1}{R_2^{N-1}}\int_{R_1}^{R_2} t^{N-1}F(u)(t)dt - \frac{1}{R_2^{N-1}}QF(u)\int_{R_1}^{R_2} t^{N-1}dt = 0.$$

It follows that $M_F$ is well defined. Its compactness follows very easily taking into account the properties of the operators composing $M_F$. From the above computation and since $\phi$ is singular, we get $||v'||_\infty < a$.

Now, let $u \in C^1_\dagger$ be such that $u = M_F(u)$. It follows

$$QF(u) = 0, \tag{5.2}$$

implying that

$$u = Pu + H \circ \phi^{-1} \circ L \circ F(u), \quad u' = \phi^{-1} \circ L \circ F(u).$$

Then

$$\phi(u'(r)) = \gamma(r) \int_{R_1}^{r} t^{N-1} F(u)(t)dt \quad (r \in (R_1, R_2]),$$

and $u$ verifies the differential equation in (5.1).

Reciprocally, let $u$ be a solution of (5.1). Then, taking into account the fact that $u$ verifies (5.2), after two integrations we deduce that $u$ is a fixed point of $M_F$. ■

**Lemma 8** *Let the continuous function $h : [R_1, R_2] \times \mathbb{R}^2 \to \mathbb{R}$ be bounded on $[R_1, R_2] \times \mathbb{R} \times (-a, a)$, $\mu \neq 0$ and consider the Neumann problem*

$$(r^{N-1}\phi(u'))' + \mu r^{N-1}u = r^{N-1}h(r, u, u'), \quad u'(R_1) = 0 = u'(R_2). \tag{5.3}$$

*If $M_\mu$ is the fixed point operator associated to (5.3), then there exists $\rho > 0$ such that any possible fixed point of $M_\mu$ is contained in $B_\rho$ and*

$$d_{LS}[I - M_\mu, B_\rho, 0] = sign(\mu).$$

*Proof.* Let us consider say, the case where $\mu > 0$, the other one being similar. We can find a constant $R > 0$ such that

$$sign(u)[-\mu u + h(r, u, v)] < 0 \tag{5.4}$$

for all $r \in [R_1, R_2]$, $v \in (-a, a)$ and $|u| \geq R$.

One the other hand, consider the compact homotopy

$$\mathcal{M} : [0, 1] \times C^1_\dagger \to C^1_\dagger, \quad \mathcal{M}(\lambda, \cdot) = P + QF_\mu + H \circ \phi^{-1} \circ \lambda L \circ (I - Q) \circ F_\mu,$$

where

$$F_\mu : C^1_\dagger \to C, \quad F_\mu(u) = -\mu u + h(\cdot, u, u').$$

Let $(\lambda, u) \in [0, 1] \times C^1_\dagger$ be such that

$$u = \mathcal{M}(\lambda, u).$$

It follows that

$$u' = \phi^{-1} \circ \lambda L \circ (I - Q) \circ F_\mu(u)$$

and

$$||u'||_\infty < a. \tag{5.5}$$

Note also that

$$QF_\mu(u) = 0. \tag{5.6}$$

Assume that $u_L \geq R$. Then, using (5.4) and (5.5) we have

$$F_\mu(u)(r) < 0 \quad \text{for all} \quad r \in [R_1, R_2].$$

This implies that

$$QF_\mu(u) < Q(0) = 0,$$

contradiction with (5.6). It follows that $u_L < R$, and analogously $u_M > -R$. Then, from

$$u_M \leq u_L + \int_{R_1}^{R_2} |u'(r)|dr$$

and (5.5), we deduce that

$$-R - a(R_2 - R_1) < u_L \leq u_M < R + a(R_2 - R_1),$$

which together with (5.5) give

$$||u|| < R + a(R_2 - R_1 + 1) =: \rho_0.$$

Since

$$\mathcal{M}(1, \cdot) = M_\mu \quad \text{and} \quad \mathcal{M}(0, \cdot) = P + QF_\mu,$$

the homotopy invariance of the Leray-Schauder degree implies that

$$d_{LS}[I - M_\mu, B_\rho, 0] = d_{LS}[I - (P + QF_\mu), B_\rho, 0],$$

for any $\rho \geq \rho_0$. The range of $P + QF_\mu$ is contained in the subspace of constant functions. Using the reduction property of the Leray Schauder degree we have

$$d_{LS}[I - (P + QF_\mu), B_\rho, 0] = d_B[I - (P + QF_\mu)|_{\mathbb{R}}, (-\rho, \rho), 0],$$

where $d_B$ denotes the Brouwer degree. But,

$$I - (P + QF_\mu)|_{\mathbb{R}} = -QF_\mu|_{\mathbb{R}},$$

and we can take $\rho$ sufficiently large such that

$$QF_\mu(-\rho) > 0 > QF_\mu(\rho),$$

implying that

$$d_B[-QF_\mu|_{\mathbb{R}}, (-\rho, \rho), 0] = 1 = sign(\mu).$$

Consequently,

$$d_{LS}[I - M_\mu, B_\rho, 0] = sign(\mu).$$

■

Now, consider the Neumann boundary-value problem (BVP)

$$(r^{N-1}\phi(u'))' = r^{N-1}f(r,u,u'), \quad u'(R_1) = 0 = u'(R_2), \tag{5.7}$$

where $f : [R_1, R_2] \times \mathbb{R}^2 \to \mathbb{R}$ is continuous.

**Definition 2** *A* strict lower solution $\alpha$ *(resp.* strict upper solution $\beta$*) of (5.7) is a function* $\alpha \in C^1$ *such that* $\|\alpha'\|_\infty < a$, $r^{N-1}\phi(\alpha') \in C^1$, $\alpha'(R_1) \geq 0 \geq \alpha'(R_2)$ *(resp.* $\beta \in C^1$, $\|\beta'\|_\infty < a$, $r^{N-1}\phi(\beta') \in C^1$, $\beta'(R_1) \leq 0 \leq \beta'(R_2)$*) and*

$$\begin{array}{rcl} (r^{N-1}\phi(\alpha'(r)))' & > & r^{N-1}f(r,\alpha(r),\alpha'(r)) \\ (resp. \quad (r^{N-1}\phi(\beta'(r)))' & < & r^{N-1}f(r,\beta(r),\beta'(r))) \end{array} \tag{5.8}$$

*for all* $r \in [R_1, R_2]$.

**Lemma 9** *Assume that (5.7) has a strict lower solution $\alpha$ and a strict upper solution $\beta$ such that*

$$\alpha(r) < \beta(r) \quad \textit{for all} \quad r \in [R_1, R_2],$$

*and if $N \geq 2$ assume also that $R_1 > 0$. Then*

$$d_{LS}[I - M_f, \Omega_{\alpha,\beta}, 0] = -1,$$

*where*

$$\Omega_{\alpha,\beta} = \{u \in C^1_\dagger : \alpha(r) < u(r) < \beta(r) \quad \textit{for all} \quad r \in [R_1, R_2], \quad ||u'||_\infty < a\}$$

*and $M_f$ is the fixed point operator associated to (5.7).*

*Proof.* Let $\gamma : [R_1, R_2] \times \mathbb{R} \to \mathbb{R}$ be the continuous function given by

$$\gamma(r,u) = \begin{cases} \beta(r) & if \quad u > \beta(r) \\ u & if \quad \alpha(r) \leq u \leq \beta(r) \\ \alpha(r) & if \quad u < \alpha(r), \end{cases}$$

and define $f_1 : [R_1, R_2] \times \mathbb{R}^2 \to \mathbb{R}$ by $f_1(r,u,v) = f(r,\gamma(r,u),v)$. We consider the modified problem

$$(r^{N-1}\phi(u'))' = r^{N-1}(f_1(r,u,u') + u - \gamma(r,u)), \quad u'(R_1) = 0 = u'(R_2), \tag{5.9}$$

and let $M_{f_1}$ be the associated fixed point operator of (5.9). Then, arguing exactly as in the proof of Theorem 4.2 from [15], one has that if $u$ is a solution of (5.9) then $\alpha(r) < u(r) < \beta(r)$ for all $r \in [R_1, R_2]$. It follows that any fixed point of $M_{f_1}$ is contained in $\Omega_{\alpha,\beta}$, and using the excision property of the Leray-Schauder degree and Lemma 8 we infer that

$$d_{LS}[I - M_{f_1}, \Omega_{\alpha,\beta}, 0] = d_{LS}[I - M_{f_1}, B_\rho, 0] = -1,$$

for any $\rho$ sufficiently large. On the other hand

$$M_f(u) = M_{f_1}(u) \quad \text{for all} \quad u \in \overline{\Omega}_{\alpha,\beta}.$$

Consequently,

$$d_{LS}[I - M_f, \Omega_{\alpha,\beta}, 0] = -1.$$

■

## 5.3 Conditions on the mean value of the forcing term

We consider the Neumann boundary value problem

$$(r^{N-1}\phi(u'))' + r^{N-1}g(u) = r^{N-1}l(r), \quad u'(R_1) = 0 = u'(R_2), \tag{5.10}$$

where $g : \mathbb{R} \to \mathbb{R}$ is continuous and $l \in C$.

The idea of the following lemma comes from Theorem 2 in [100].

**Lemma 10** *Assume that $\phi$ is singular and that there exist $t < s$ and $A < B$ such that either*

$$Qg(t + \widehat{u}) \leq A \quad \textit{and} \quad Qg(s + \widehat{u}) \geq B \tag{5.11}$$

*or*

$$Qg(t + \widehat{u}) \geq B \quad \textit{and} \quad Qg(s + \widehat{u}) \leq A \tag{5.12}$$

*for any $\widehat{u} \in \widehat{C}^1_\dagger$ satisfying $||\widehat{u}||_\infty < a(R_2 - R_1)$. If*

$$A < \underline{l} < B, \tag{5.13}$$

*then problem (5.10) has at least one solution $u$ such that $t < \underline{u} < s$.*

*Proof.* Let us assume that (5.11) holds true and let $\varepsilon > 0$ be fixed. For any $\lambda \in [0, 1]$, consider the Neumann problem

$$\begin{gathered}(r^{N-1}\phi(u'))' + \lambda r^{N-1}g(u) + (1-\lambda)\varepsilon r^{N-1}\left(u - \frac{t+s}{2}\right) = \lambda r^{N-1}l(r) \\ u'(R_1) = 0 = u'(R_2).\end{gathered} \tag{5.14}$$

Let also

$$\mathcal{M}(\lambda, \cdot) : C^1_\dagger \to C^1_\dagger \quad (\lambda \in [0, 1])$$

be the fixed point operator associated to (5.14) (see Proposition 5). We will show that

$$u - \mathcal{M}(\lambda, u) \neq 0 \quad \text{for any} \quad (\lambda, u) \in (0, 1] \times \partial\Omega, \tag{5.15}$$

and

$$u - \mathcal{M}(0, u) = 0 \quad \text{implies} \quad u \in \Omega, \tag{5.16}$$

where

$$\Omega = \{u \in C^1_\dagger : \quad t < \underline{u} < s, \quad ||\widehat{u}||_\infty < a(R_2 - R_1), \quad ||u'||_\infty < a\}.$$

Then, using the invariance by homotopy, the excision property of the Leray-Schauder degree and Lemma 8, one has that

$$d_{LS}[I - \mathcal{M}(1,\cdot), \Omega, 0] = d_{LS}[I - \mathcal{M}(0,\cdot), \Omega, 0] = 1.$$

Hence, the existence property of the Leray-Schauder degree implies the existence of some $u \in \Omega$ (in particular $t < \underline{u} < s$) with $u = \mathcal{M}(1,u)$ which is also a solution of (5.10).

So, let us consider $(\lambda, u) \in (0,1] \times C^1_\dagger$ such that $u = \mathcal{M}(\lambda, u)$. It follows that (5.5) holds true and $u = \underline{u} + \widehat{u} \in \mathbb{R} \oplus \widehat{C}^1_\dagger$ is a solution of (5.14). As $Q\widehat{u} = 0$, there exists $r_0 \in [R_1, R_2]$ such that $\widehat{u}(r_0) = 0$, yielding

$$||\widehat{u}||_\infty \leq \int_{r_0}^{R_2} |\widehat{u}'(r)|dr < a(R_2 - R_1). \tag{5.17}$$

Integrating (5.14) over $[R_1, R_2]$ we obtain

$$(1-\lambda)\varepsilon\left(\underline{u} - \frac{t+s}{2}\right) + \lambda(Qg(\underline{u} + \widehat{u}) - \underline{l}) = 0. \tag{5.18}$$

On the other hand, from (5.11), (5.13) and (5.17) it follows that

$$\begin{aligned} (1-\lambda)\varepsilon\left(t - \frac{t+s}{2}\right) + \lambda(Qg(t+\widehat{u}) - \underline{l}) &\leq (1-\lambda)\varepsilon\frac{t-s}{2} + \lambda(A - \underline{l}) < 0; \\ (1-\lambda)\varepsilon\left(s - \frac{t+s}{2}\right) + \lambda(Qg(s+\widehat{u}) - \underline{l}) &\geq (1-\lambda)\varepsilon\frac{s-t}{2} + \lambda(B - \underline{l}) > 0. \end{aligned} \tag{5.19}$$

Moreover, if $u \in \partial\Omega$, from (5.5) and (5.17) one has $\underline{u} = t$ or $\underline{u} = s$. But $\underline{u}$ verifies (5.18), contradiction with (5.19). Consequently, (5.15) is proved.

Now, let $u \in C^1_\dagger$ be such that $u = \mathcal{M}(0,u)$. We deduce that $u$ verifies (5.5), (5.17) and (5.14) with $\lambda = 0$. Hence, $\underline{u} = \dfrac{t+s}{2}$ and $u \in \Omega$.

If (5.12) holds true then one takes $\varepsilon < 0$. ■

**Remark 9** From the proof above it can be seen that if the assumption "$A < B$" is replaced by "$A \leq B$" then problem (5.10) has at least one solution $u$ such that $t \leq \underline{u} \leq s$, provided that $A \leq \underline{l} \leq B$.

**Theorem 16** *If $\phi$ is singular, $l \in C$, $\mu > 0$ and*

$$2a(R_2 - R_1) < \pi,$$

*then, the Neumann problem (2.8) has at least two solutions not differing by a multiple of $2\pi$, provided that*

$$|\underline{l}| < \mu \cos\left[a(R_2 - R_1)\right].$$

*Proof.* We apply Lemma 10 with $g(u) = \mu \sin(u)$ and

$$A = -\mu \cos\left[a(R_2 - R_1)\right] = -B.$$

Taking $t = -\pi/2$, $s = \pi/2$, condition (5.11) is fulfilled and so, we get the existence of a solution $u_1$ which satisfies $-\pi/2 < \underline{u}_1 < \pi/2$. Then, setting $t = \pi/2$, $s = 3\pi/2$, condition (5.12) is accomplished and we obtain a second solution $u_2$ with $\pi/2 < \underline{u}_2 < 3\pi/2$.

If we assume that there is some $j \in \mathbb{Z}$ such that $u_2 = u_1 + 2j\pi$ then necessarily one has $0 < 2j\pi < 2\pi$, a contradiction. ■

**Remark 10** On account of Remark 9, if in Theorem 16 one has $2a(R_2 - R_1) = \pi$, then problem (2.8) has at least one solution for any $l \in C$ with $\underline{l} = 0$.

**Corollary 8** *Let $\mu > 0$ and $l \in C$. If $2(R_2 - R_1) < \pi$, then the Neumann problem (2.9) has at least two classical radial solutions not differing by a multiple of $2\pi$, provided that $|\underline{l}| < \mu \cos(R_2 - R_1)$. Moreover, if $2(R_2 - R_1) = \pi$, then (2.9) has at least one classical radial solution for any $l \in C$ with $\underline{l} = 0$.*

Bellow we give a second proof of the Theorem 16 and we consider also the classical case. The main idea of our proof comes from [4] and has been used for the classical forced pendulum in [51].

Let $f : [R_1, R_2] \times \mathbb{R}^2$ be a continuous function and $N_f : C^1 \to C$ be the Nemytskii operator associated to $f$. We first consider the modified problem of finding $(\underline{u}, \widehat{u}) \in \mathbb{R} \times \widehat{C}^1_\dagger$ such that

$$(r^{N-1}\phi(\widehat{u}'))' = r^{N-1}[N_f(\underline{u} + \widehat{u}) - Q \circ N_f(\underline{u} + \widehat{u})]. \tag{5.20}$$

**Lemma 11** *If $\phi$ is singular or classical and there exists $\alpha > 0$ such that*

$$|f(r, u, v)| \leq \alpha \quad \text{for all} \quad (r, u, v) \in [R_1, R_2] \times \mathbb{R}^2,$$

*then the set of the solutions of problem (5.20) contains a continuum $\mathcal{C}$ whose projection on $\mathbb{R}$ is $\mathbb{R}$ and whose projection on $\widehat{C}^1_\dagger$ is contained in $B_\phi = \{\widehat{u} \in \widehat{C}^1_\dagger : ||\widehat{u}'||_\infty \leq c_\phi, ||\widehat{u}||_\infty \leq c_\phi(R_2 - R_1)\}$, where $c_\phi = \max(|\phi^{-1}(\pm 2\alpha R_2/N)|)$.*

*Proof.* Let us consider

$$\widehat{M} : \mathbb{R} \times \widehat{C}^1_\dagger \to \widehat{C}^1_\dagger, \quad \widehat{M}(\underline{u}, \widehat{u}) = (I - Q) \circ H \circ \phi^{-1} \circ L \circ (I - Q) \circ N_f(\underline{u} + \widehat{u}).$$

It is not difficult to prove that $\widehat{M}$ is well defined and compact. Moreover, if $(\underline{u}, \widehat{u}) \in \mathbb{R} \times \widehat{C}^1_\dagger$ satisfies $\widehat{u} = \widehat{M}(\underline{u}, \widehat{u})$, then $(\underline{u}, \widehat{u})$ is a solution of (5.20). On the other hand a simple computation shows that the range of $\widehat{M}$ is contained in $B_\phi$ (in both of the two cases) and the proof follows now exactly like the proof of Lemma 2.1 in [15]. ■

**Remark 11** The assumption concerning the boundedness of $f$ can be dropped in the singular case but then $B_\phi = \{\widehat{u} \in \widehat{C}^1_\dagger : ||\widehat{u}'||_\infty < a, ||\widehat{u}||_\infty < a(R_2 - R_1)\}$.

Let $\psi : (-b, b) \to (-c, c)$ be a homeomorphism such that $\psi(0) = 0$ and $0 < b, c \leq \infty$. For $l \in C$ and $\mu > 0$ such that $2(||l||_\infty + \mu)R_2/N < c$ we introduce the notation

$$\rho(\psi) = \max(|\psi^{-1}(\pm 2(||l||_\infty + \mu)R_2/N)|).$$

**Theorem 17** *Assume that $\phi$ is singular or classical, $l \in C$, $\mu > 0$ and*

$$2\rho(\phi)(R_2 - R_1) < \pi, \tag{5.21}$$

*then, the Neumann problem (2.8) has at least two solutions not differing by a multiple of $2\pi$, provided that $|\underline{l}| < \mu \cos[\rho(\phi)(R_2 - R_1)]$.*

*Proof.* Consider the continuous function

$$\Gamma : \mathbb{R} \times \mathcal{C} \to \mathbb{R}, \quad \Gamma(\underline{u}, \widehat{u}) = Q \circ N_f(\underline{u} + \widehat{u}).$$

For any $\widehat{u}_1, \widehat{u}_2$ such that $(-\frac{\pi}{2}, \widehat{u}_1), (\frac{\pi}{2}, \widehat{u}_2) \in \mathcal{C}$, one has that

$$\Gamma(-\frac{\pi}{2}, \widehat{u}_1) > 0 > \Gamma(\frac{\pi}{2}, \widehat{u}_2).$$

Hence, using that $\mathcal{C}$ is a continuum and the continuity of $\Gamma$, we deduce the existence of $(\underline{u}, \widehat{u}) \in \mathcal{C}$ such that $-\frac{\pi}{2} < \underline{u} < \frac{\pi}{2}$ and $\Gamma(\underline{u}, \widehat{u}) = 0$. Then, $u = \underline{u} + \widehat{u}$ is a solution of (2.8). Analogously, (2.8) has a solution $w$ satisfying $\frac{\pi}{2} < \underline{w} < \frac{3\pi}{2}$. Clearly, $u - w$ is not a multiple of $2\pi$. ■

**Remark 12** (i) If in (5.21) one has equality, then in the above theorem we have only existence.

(ii) In the above theorem, if $\phi$ is singular, then $\rho(\phi) < a$ and Theorem 16 follows from Theorem 17.

The following result is a direct consequence of Lemma 2.5 from [15].

**Lemma 12** *Let $\psi : (-b, b) \to (-c, c)$ be a homeomorphism such that $\psi(0) = 0$ and $0 < b, c \leq \infty$. Let $\mu > 0$ and $l \in C$ be such that $(||l||_\infty + \mu)R_2/N < c$. If $u$ is a possible solution of the Neumann problem*

$$(r^{N-1}\psi(u'))' + r^{N-1}\mu \sin u = r^{N-1}l(r), \quad u'(R_1) = 0 = u'(R_2) \tag{5.22}$$

*then*

$$||u'||_\infty \leq \max(|\psi^{-1}(\pm(||l||_\infty + \mu)R_2/N)|).$$

**Theorem 18** *Assume that* $\psi : \mathbb{R} \to (-c, c)$ *is a homeomorphism such that* $\psi(0) = 0$ *and* $0 < c \leq \infty$. *If* $l \in C$, $\mu > 0$, $2(||l||_\infty + \mu)R_2/N < c$ *and*

$$2\rho(\psi)(R_2 - R_1) < \pi,$$

*then, the Neumann problem (5.22) has at least two solutions not differing by a multiple of* $2\pi$, *provided that*

$$|\underline{l}| < \mu \cos[\rho(\psi)(R_2 - R_1)]$$

*is satisfied.*

*Proof.* Let $d = \rho(\psi)+1$ and $b = \rho(\psi)+2$. Consider $\phi : (-b, b) \to \mathbb{R}$ a singular homeomorphism which coincides with $\psi$ on $[-d, d]$. Then $\rho(\psi) = \rho(\phi)$, so using Lemma 12 we infer that the solutions of (2.8) coincide with the solutions of (5.22). Now the result follows from Theorem 17 (the singular case). ∎

**Corollary 9** *If (5.21) is satisfied with* $\phi(u) = |u|^{p-2}u$ $(p > 1)$, *(resp.* $\phi(u) = \frac{u}{\sqrt{1+u^2}}$*) then the Neumann problem (2.11) (resp. (2.12)) has at least two classical radial solutions not differing by a multiple of* $2\pi$ *for any* $l \in C$ *with* $\underline{l} = 0$ *(resp.* $l \in C$ *with* $\underline{l} = 0$ *and* $2(||l||_\infty + \mu)R_2/N < 1$*).*

## 5.4 Norm conditions on the forcing term

In the proof of the following theorem we adapt to our situation a strategy introduced in Theorem 3 from [100].

**Theorem 19** *Assume that* $\phi$ *is singular and let* $\mu > 0$, $R_1 > 0$ *in the case* $N \geq 2$ *and assume that* $l \in C$ *satisfies*

$$||l||_\infty < \mu.$$

*Then problem (2.8) has at least two solutions not differing by a multiple of* $2\pi$. *Moreover, if*

$$||l||_\infty = \mu,$$

*then problem (2.8) has at least one solution.*

*Proof.* Assume that $||l||_\infty \leq \mu$. Then $\alpha = -\frac{3\pi}{2}$ is a constant lower solution for (2.8) and $\beta = -\frac{\pi}{2}$ is a constant upper solution for (2.8) such that $\alpha < \beta$. Hence, using Theorem 4.2 from [15], it follows that (2.8) has a solution $u_1$ such that $\alpha \leq u_1 \leq \beta$. Note that if $||l||_\infty < \mu$, then $\alpha, \beta$ are strict and $\alpha < u_1 < \beta$.

Now, let us assume that $||l||_\infty < \mu$, let $M_\mu$ be the fixed point operator associated to (2.8), and let

$$\Omega = \Omega_{-\frac{3\pi}{2},\frac{3\pi}{2}} \setminus (\overline{\Omega}_{-\frac{3\pi}{2},-\frac{\pi}{2}} \cup \overline{\Omega}_{\frac{\pi}{2},\frac{3\pi}{2}}). \quad \text{(see Lemma 9)}$$

Then using the additivity property of the Leray-Schauder degree and Lemma 9, we deduce that

$$d_{LS}[I - M_\mu, \Omega, 0] = 1.$$

Hence, the existence property of the Leray-Schauder degree yields the existence of a solution $u_2 \in \Omega$ of (2.8). If we assume that $u_2 = u_1 + 2j\pi$ for some $j \in \mathbb{Z}$ then, as $-3\pi/2 < u_1 < -\pi/2$, one has

$$-\frac{3\pi}{2} + 2j\pi < u_2 = u_1 + 2j\pi < -\frac{\pi}{2} + 2j\pi.$$

This leads to one of the contradictions: $u_2 \in \Omega_{\frac{\pi}{2},\frac{3\pi}{2}}$ if $j = 1$ or $u_2 = u_1 \in \Omega_{-\frac{3\pi}{2},-\frac{\pi}{2}}$ for $j = 0$. ■

Using Lemma 12, Theorem 19 and arguing exactly as in the proof of Theorem 18 with $\rho(\psi)$ replaced by $\max(|\psi^{-1}(\pm 2\mu R_2/N)|)$ we obtain the following result.

**Theorem 20** *Let $\psi : \mathbb{R} \to (-c, c)$ be an increasing homeomorphism such that $\psi(0) = 0$ and $0 < c \leq \infty$. Let also $\mu > 0$, $R_1 > 0$ in the case $N \geq 2$ and $l \in C$ be such that $\frac{2\mu R_2}{N} < c$. If $||l||_\infty < \mu$, then (5.22) has at least two solutions not differing by a multiple of $2\pi$. If $||l||_\infty = \mu$, then (5.22) has at least one solution.*

**Corollary 10** *Let $\mu > 0$, $R_1 > 0$ and $l \in C$ be such that $\frac{2\mu R_2}{N} < 1$. If $||l||_\infty < \mu$, then the Neumann problem (2.12) has at least two classical radial solutions not differing by a multiple of $2\pi$. Moreover, if $||l||_\infty = \mu$, then (2.12) has at least one classical radial solution. The same conclusion holds also for (2.9) and (2.11) without the assumption $\frac{2\mu R_2}{N} < 1$.*

# Chapter 6

# Pendulum-like nonlinearities - 2

## 6.1 Hypotheses and function spaces

In what follows, we assume that $\Phi : [-a, a] \to \mathbb{R}$ satisfies the following hypothesis :

$(H_\Phi)$ *$\Phi$ is continuous, of class $C^1$ on $(-a, a)$, with $\phi := \Phi' : (-a, a) \to \mathbb{R}$ an increasing homeomorphism such that $\phi(0) = 0$.*

Consequently, $\Phi : [-a, a] \to \mathbb{R}$ is strictly convex.

Given $0 \le R_1 < R_2$, the function $g : [R_1, R_2] \times \mathbb{R} \to \mathbb{R}$ satisfies the following hypothesis :
$(H_g)$ *$g$ is continuous and its indefinite integral*

$$G(r, x) := \int_0^x g(r, \xi)d\xi, \quad (r, x) \in [R_1, R_2] \times \mathbb{R}$$

*is $2\pi$–periodic for each $r \in [R_1, R_2]$.*

We set $C := C[R_1, R_2]$, $L^1 := L^1(R_1, R_2)$, $L^\infty := L^\infty(R_1, R_2)$ and $W^{1,\infty} := W^{1,\infty}(R_1, R_2)$. The usual norm $\|\cdot\|_\infty$ is considered on $L^\infty$ and $W^{1,\infty}$ is endowed with the norm

$$\|v\| = \|v\|_\infty + \|v'\|_\infty \quad (v \in W^{1,\infty}).$$

Each $v \in L^1$ can be written $v(r) = \overline{v} + \tilde{v}(r)$, with

$$\overline{v} := \frac{N}{R_2^N - R_1^N} \int_{R_1}^{R_2} v(r)\, r^{N-1}dr, \quad \int_{R_1}^{R_2} \tilde{v}(r)\, r^{N-1}dr = 0.$$

If $v \in W^{1,\infty}$ then $\tilde{v}$ vanishes at some $r_0 \in (R_1, R_2)$ and

$$|\tilde{v}(r)| = |\tilde{v}(r) - \tilde{v}(r_0)| \le \int_{R_1}^{R_2} |v'(t)|dt \le (R_2 - R_1)\|v'\|_\infty. \tag{6.1}$$

We set

$$K = \{v \in W^{1,\infty} \ : \ \|v'\|_\infty \leq a\}.$$

$K$ is closed in $W^{1,\infty}$ and convex.

**Lemma 13** *If $\{u_n\} \subset K$ and $u \in C$ are such that $u_n(r) \to u(r)$ for all $r \in [R_1, R_2]$, then*

*(i) $u \in K$;*

*(ii) $u'_n \to u'$ in the $w^*$–topology $\sigma(L^\infty, L^1)$.*

*Proof.* From the relation

$$|u_n(r_1) - u_n(r_2)| = \left| \int_{r_2}^{r_1} u'_n(r)\, dr \right| \leq a|r_1 - r_2|,$$

letting $n \to \infty$, we get

$$|u(r_1) - u(r_2)| \leq a|r_1 - r_2| \quad (r_1, r_2 \in [R_1, R_2]),$$

which yields $u \in K$.

Next, we show that that if $\{u'_k\}$ is a subsequence of $\{u'_n\}$ with $u'_k \to v \in L^\infty$ in the $w^*$–topology $\sigma(L^\infty, L^1)$ then

$$v = u' \quad \text{a.e. on } [R_1, R_2]. \tag{6.2}$$

Indeed, as

$$\int_{R_1}^{R_2} u'_k(r) f(r)\, dr \to \int_{R_1}^{R_2} v(r) f(r)\, dr \quad \text{for all} \quad f \in L^1,$$

taking $f \equiv \chi_{r_1, r_2}$, the characteristic function of the interval having the endpoints $r_1, r_2 \in [R_1, R_2]$, it follows

$$\int_{r_1}^{r_2} u'_k(r)\, dr \to \int_{r_1}^{r_2} v(r)\, dr \quad (r_1, r_2 \in [R_1, R_2]).$$

Then, letting $k \to \infty$ in

$$u_k(r_2) - u_k(r_1) = \int_{r_1}^{r_2} u'_k(r)\, dr$$

we obtain

$$u(r_2) - u(r_1) = \int_{r_1}^{r_2} v(r)\, dr \quad (r_1, r_2 \in [R_1, R_2])$$

which, clearly implies (6.2).

Now, to prove *(ii)* it suffices to show that if $\{u'_j\}$ is an arbitrary subsequence of $\{u'_n\}$, then it contains itself a subsequence $\{u'_k\}$ such that $u'_k \to u'$ in the $w^*$–topology $\sigma(L^\infty, L^1)$. Since $L^1$ is separable and $\{u'_j\}$ is bounded in $L^\infty = (L^1)^*$, we know that it has a subsequence $\{u'_k\}$ convergent to some $v \in L^\infty$ in the $w^*$–topology $\sigma(L^\infty, L^1)$. Then, as shown before (see (6.2)), we have $v = u'$. ■

## 6.2 A minimization problem

Let $h \in C$ and $\mathcal{F} : K \to \mathbb{R}$ be given by

$$\mathcal{F}(v) = \int_{R_1}^{R_2} \{\Phi[v'(r)] + G(r, v(r)) + h(r)v(r)\}\, r^{N-1} dr \quad (v \in K).$$

On account of hypotheses $(H_\Phi)$ and $(H_g)$ the functional $\mathcal{F}$ is well defined.

**Proposition 6** *If $\overline{h} = 0$ then $\mathcal{F}$ has a minimum over $K$.*

*Proof. Step I.* We prove that if $\{u_n\} \subset K$ is a sequence which converges uniformly on $[R_1, R_2]$ to some $u \in K$, then

$$\liminf_{n\to\infty} \int_{R_1}^{R_2} \Phi[u_n'(r)]\, r^{N-1} dr \geq \int_{R_1}^{R_2} \Phi[u'(r)]\, r^{N-1} dr. \tag{6.3}$$

By virtue of $(H_\Phi)$ the function $\Phi$ is convex, hence for all $y \in [-a, a]$ and $z \in (-a, a)$ one has

$$\Phi(y) - \Phi(z) \geq \phi(z)(y - z). \tag{6.4}$$

This implies that for any $\lambda \in [0, 1)$ it holds

$$\begin{aligned} \int_{R_1}^{R_2} \Phi[u_n'(r)]\, r^{N-1} dr \quad &\geq \quad \int_{R_1}^{R_2} \Phi[\lambda u'(r)]\, r^{N-1} dr \\ &+ \quad \int_{R_1}^{R_2} \phi[\lambda u'(r)][u_n'(r) - \lambda u'(r)]\, r^{N-1} dr. \end{aligned} \tag{6.5}$$

From Lemma 13 we have that $u_n' \to u'$ in the $w^*$–topology $\sigma(L^\infty, L^1)$. Since the map $r \mapsto r^{N-1}\phi[\lambda u'(r)]$ belongs to $L^\infty \subset L^1$, using (6.5) we infer that

$$\begin{aligned} \liminf_{n\to\infty} \int_{R_1}^{R_2} \Phi[u_n'(r)]\, r^{N-1} dr \quad &\geq \quad \int_{R_1}^{R_2} \Phi[\lambda u'(r)]\, r^{N-1} dr \\ &+ \quad (1 - \lambda) \int_{R_1}^{R_2} \phi[\lambda u'(r)] u'(r)\, r^{N-1} dr. \end{aligned}$$

As $\phi(t)t \geq 0$, for all $t \in (-a, a)$, we get

$$\liminf_{n\to\infty} \int_{R_1}^{R_2} \Phi[u_n'(r)]\, r^{N-1} dr \geq \int_{R_1}^{R_2} \Phi[\lambda u'(r)]\, r^{N-1} dr,$$

which, using Lebesgue's dominated convergence theorem, gives (6.3) by letting $\lambda \to 1$.

*Step II.* Due to the $2\pi$–periodicity of $G(r, \cdot)$ (see $(H_g)$) and because of $\overline{h} = 0$, we have

$$\mathcal{F}(v + 2\pi) = \mathcal{F}(v), \quad \forall v \in K.$$

Therefore, if $u$ minimizes $\mathcal{F}$ over $K$, then the same is true for $u + 2k\pi$ for any $k \in \mathbb{Z}$. This means that we can search, without loss of generality, a minimizer $u \in K$ with $\overline{u} \in [0, 2\pi]$. Thus, the problem reduces to minimize $\mathcal{F}$ over

$$\hat{K} = \{v \in K \ : \ \overline{v} \in [0, 2\pi]\}.$$

If $v \in \hat{K}$ then, using (6.1) we obtain

$$|v(r)| \leq |\overline{v}| + |\tilde{v}(r)| \leq 2\pi + (R_2 - R_1)a.$$

This, together with $\|v'\|_\infty \leq a$ shows that $\hat{K}$ is bounded in $W^{1,\infty}$ and, by the compactness of the embedding $W^{1,\infty} \subset C$, the set $\hat{K}$ is relatively compact in $C$. Let $\{u_n\} \subset \hat{K}$ be a minimizing sequence for $\mathcal{F}$. Passing to a subsequence if necessary and using Lemma 13, we may assume that $\{u_n\}$ converges uniformly to some $u \in K$. It is easily seen that actually $u \in \hat{K}$. By *Step I* we obtain

$$\inf_{\hat{K}} \mathcal{F} = \lim_{n\to\infty} \mathcal{F}(u_n) \geq \mathcal{F}(u),$$

showing that $u$ minimizes $\mathcal{F}$ over $\hat{K}$. ■

**Remark 13** If $\{u_n\} \subset K$ and $u \in C$ are such that $u_n(r) \to u(r)$ for all $r \in [R_1, R_2]$, then by Lemma 13 and the reasoning in *Step I* of the above proof we have that $u \in K$ and (6.3) still holds true.

**Lemma 14** *If $u$ minimizes $\mathcal{F}$ over $K$ then $u$ satisfies the variational inequality*

$$\int_{R_1}^{R_2} \left(\Phi[v'(r)] - \Phi[u'(r)] + \{g[r, u(r)] + h(r)\}[v(r) - u(r)]\right) r^{N-1} dr \geq 0$$

*for all $v \in K$.*

*Proof.* The argument is standard. See for example Lemma 2 in [29]. ■

## 6.3 An existence result

We show that the minimizers of $\mathcal{F}$ provide classical solutions for the Neumann boundary value problem

$$[r^{N-1}\phi(u')]' = r^{N-1}[g(r, u) + h(r)], \quad u'(R_1) = 0 = u'(R_2), \tag{6.6}$$

under the basic assumptions $(H_\Phi)$ and $(H_g)$. Recall that by a *solution* of (6.6) we mean a function $u \in C^1[R_1, R_2]$, such that $\|u'\|_\infty < a$, $r \mapsto r^{N-1}\phi(u')$ is differentiable and (6.6) is satisfied.

Let us begin with the simpler problem

$$[r^{N-1}\phi(u')]' = r^{N-1}[u + f(r)], \quad u'(R_1) = 0 = u'(R_2). \tag{6.7}$$

**Proposition 7** *For any $f \in C$, problem (6.7) has a unique solution $\widehat{u}_f$ and $\widehat{u}_f$ satisfies the variational inequality*

$$\int_{R_1}^{R_2} \left(\Phi[v'(r)] - \Phi[\widehat{u}_f'(r)] + \{\widehat{u}_f(r) + f(r)\}[v(r) - \widehat{u}_f(r)]\right) r^{N-1} dr \geq 0 \quad (6.8)$$

*for all $v \in K$.*

*Proof.* The existence part follows from Corollary 2.4 in [15]. If $u$ and $v$ are two solutions of (6.7), then

$$\int_{R_1}^{R_2} \{r^{N-1}[\phi(u'(r)) - \phi(v'(r))]\}'[u(r) - v(r)]\, dr = \int_{R_1}^{R_2} [u(r) - v(r)]^2\, r^{N-1} dr$$

and hence, integrating the first term by parts and using the boundary conditions we obtain

$$\int_{R_1}^{R_2} \{[\phi(u'(r)) - \phi(v'(r))][u'(r) - v'(r)] + [u(r) - v(r)]^2\}\, r^{N-1} dr = 0.$$

The monotonicity of $\phi$ yields $u = v$.

From (6.4) we have

$$\begin{aligned} & \int_{R_1}^{R_2} \{\Phi[v'(r)] - \Phi[\widehat{u}_f'(r)]\}\, r^{N-1} dr \\ \geq\ & \int_{R_1}^{R_2} \phi[\widehat{u}_f'(r)][v'(r) - \widehat{u}_f'(r)]\, r^{N-1} dr \\ =\ & -\int_{R_1}^{R_2} \{r^{N-1}\phi[\widehat{u}_f'(r)]\}'[v(r) - \widehat{u}_f(r)]\, dr \\ =\ & -\int_{R_1}^{R_2} [\widehat{u}_f(r) + f(r)][v(r) - \widehat{u}_f(r)]\, r^{N-1} dr, \end{aligned}$$

showing that (6.8) holds for all $v \in K$. ■

**Theorem 21** *If hypotheses $(H_\Phi)$ and $(H_g)$ hold true, then, for any $h \in C$ with $\overline{h} = 0$, problem (6.6) has at least one solution which minimizes $\mathcal{F}$ over $K$.*

*Proof.* For any $w \in K$ we set

$$f_w := g(\cdot, w) + h - w \in C.$$

By Proposition 7, the unique solution $\widehat{u}_{f_w}$ of problem (6.7) with $f = f_w$ satisfies the variational inequality

$$\int_{R_1}^{R_2} \{\Phi[v'(r)] - \Phi[\widehat{u}_{f_w}'(r)] + [\widehat{u}_{f_w}(r) + f_w(r)][v(r) - \widehat{u}_{f_w}(r)]\}\, r^{N-1} dr \geq 0 \quad (6.9)$$

for all $v \in K$. Let $u \in K$ be a minimizer of $\mathcal{F}$ over $K$; we know that it exists by Proposition 6. From Lemma 14, $u$ satisfies the variational inequality

$$\int_{R_1}^{R_2} \{\Phi[v'(r)] - \Phi[u'(r)] + [u(r) + f_u(r)][v(r) - u(r)]\} \, r^{N-1} dr \geq 0 \qquad (6.10)$$

for all $v \in K$. Taking $v = \widehat{u}_{f_u}$ in (6.10) and $w = v = u$ in (6.9) and adding the resulting inequalities, we get

$$\int_{R_1}^{R_2} [u(r) - \widehat{u}_{f_u}(r)]^2 \, r^{N-1} dr \leq 0.$$

It follows that $u = \widehat{u}_{f_u}$. Consequently, the minimizer $u$ solves (6.6). ■

**Corollary 11** *For any $\mu \in \mathbb{R}$ and $h \in C$ with $\overline{h} = 0$ the problem*

$$\left( r^{N-1} \frac{u'}{\sqrt{1 - u'^2}} \right)' + r^{N-1} \mu \sin u = r^{N-1} h(r), \quad u'(R_1) = 0 = u'(R_2)$$

*has at least one solution.*

**Corollary 12** *For any $\mu \in \mathbb{R}$ and $h \in C$ such that*

$$\int_{\mathcal{A}} h(|x|) \, dx = 0,$$

*the problem*

$$div \left( \frac{\nabla v}{\sqrt{1 - |\nabla v|^2}} \right) + \mu \sin v = h(|x|) \quad in \quad \mathcal{A}, \quad \partial_\nu v = 0 \quad on \quad \partial \mathcal{A}$$

*has at least one classical radial solution.*

*Proof.* Indeed, going to spherical coordinates, we have

$$\int_{\mathcal{A}} h(|x|) \, dx = \frac{2\pi^{n/2}}{\Gamma(n/2)} \int_{R_1}^{R_2} h(r) \, r^{N-1} dr.$$

■

**Remark 14** If $\mathcal{D}$ is a bounded domain with sufficiently smooth boundary, a necessary condition for the existence of at least one solution to the Neumann problem

$$\operatorname{div} \left( \frac{\nabla v}{\sqrt{1 - |\nabla v|^2}} \right) + \mu \sin v = h(x) \quad \text{in} \quad \mathcal{D}, \quad \partial_\nu v = 0 \quad \text{on} \quad \partial \mathcal{D} \qquad (6.11)$$

for any $\mu > 0$ is that condition

$$\int_{\mathcal{D}} h(x)\, dx = 0 \tag{6.12}$$

holds, as it is easily seen by integrating both members of (6.11) over $\mathcal{D}$ and using divergence theorem and the boundary conditions. It is an open problem to know if condition (6.12) is sufficient. A proof of the existence of a minimum for the functional

$$\mathcal{G}(u) = \int_{\mathcal{D}} \left[ -\sqrt{1 - |\nabla v(x)|^2} + \mu \cos v(x) + h(x) v(x) \right] dx$$

on the closed convex set

$$K := \{v \in W^{1,\infty}(\mathcal{D}) : |\nabla v(x)| \leq 1 \text{ a.e. on } \mathcal{D}\}$$

can be done following the lines of the proof of Proposition 6, but our way to go from the variational inequality to the differential equation seems to be specific to a one-dimensional situation, i.e. to the radial case.

# Chapter 7

# Variational methods

## 7.1 The functional framework

In what follows, we assume that $\Phi : [-a,a] \to \mathbb{R}$ satisfies the following hypothesis :

$(H_\Phi)$ $\Phi(0) = 0$, *$\Phi$ is continuous, of class $C^1$ on $(-a,a)$, with $\phi := \Phi'$ : $(-a,a) \to \mathbb{R}$ an increasing homeomorphism such that $\phi(0) = 0$.*

Clearly, $\Phi$ is strictly convex and $\Phi(x) \geq 0$ for all $x \in [-a,a]$ .

Given $0 \leq R_1 < R_2$ and $g : [R_1,R_2] \times \mathbb{R} \to \mathbb{R}$ a *continuous* function, we denote by $G : [R_1,R_2] \times \mathbb{R} \to \mathbb{R}$ the indefinite integral of $g$, i.e.,

$$G(r,x) := \int_0^x g(r,\xi)\, d\xi, \quad (r,x) \in [R_1,R_2] \times \mathbb{R}.$$

We set $C := C[R_1,R_2]$, $L^1 := L^1(R_1,R_2)$, $L^\infty := L^\infty(R_1,R_2)$ and $W^{1,\infty} := W^{1,\infty}(R_1,R_2)$. The usual norm $\|\cdot\|_\infty$ is considered on $C$ and $L^\infty$. The space $W^{1,\infty}$ is endowed with the norm

$$\|v\| = \|v\|_\infty + \|v'\|_\infty, \quad v \in W^{1,\infty}.$$

Denoting

$$L^1_{N-1} := \{v : (R_1,R_2) \to \mathbb{R} \text{ measurable } : \int_{R_1}^{R_2} r^{N-1}|v(r)|\, dr < +\infty\},$$

each $v \in L^1_{N-1}$ can be written $v(r) = \overline{v} + \tilde{v}(r)$, with

$$\overline{v} := \frac{N}{R_2^N - R_1^N} \int_{R_1}^{R_2} v(r)\, r^{N-1}\, dr, \quad \int_{R_1}^{R_2} \tilde{v}(r)\, r^{N-1}\, dr = 0.$$

If $v \in W^{1,\infty}$ then $\tilde{v}$ vanishes at some $r_0 \in (R_1,R_2)$ and

$$|\tilde{v}(r)| = |\tilde{v}(r) - \tilde{v}(r_0)| \leq \int_{R_1}^{R_2} |v'(t)|\, dt \leq (R_2 - R_1)\|v'\|_\infty,$$

so, one has that

$$||\widetilde{v}||_\infty \leq (R_2 - R_1)\|v'\|_\infty. \tag{7.1}$$

Putting

$$K := \{v \in W^{1,\infty} \ : \ \|v'\|_\infty \leq a\},$$

it is clear that $K$ is a convex subset of $W^{1,\infty}$.

Let $\Psi : C \to (-\infty, +\infty]$ be defined by

$$\Psi(v) = \begin{cases} \varphi(v), & \text{if } v \in K, \\ \\ +\infty, & \text{otherwise,} \end{cases}$$

where $\varphi : K \to \mathbb{R}$ is given by

$$\varphi(v) = \int_{R_1}^{R_2} r^{N-1}\Phi(v')\,dr, \quad v \in K.$$

Obviously, $\Psi$ is proper and convex. On the other hand, as shown in [20], we have that if $\{u_n\} \subset K$ and $u \in C$ are such that $u_n(r) \to u(r)$ for all $r \in [R_1, R_2]$, then $u \in K$ and

$$\varphi(u) \leq \liminf_{n\to\infty} \varphi(u_n). \tag{7.2}$$

This implies that $\Psi$ is lower semicontinuous on $C$. Also, note that $K$ is closed in $C$.

Next, let $\mathcal{G} : C \to \mathbb{R}$ be defined by

$$\mathcal{G}(u) = \int_{R_1}^{R_2} r^{N-1}G(r,u)\,dr, \quad u \in C.$$

A standard reasoning (also see [72, Remark 2.7]) shows that $\mathcal{G}$ is of class $C^1$ on $C$ and its derivative is given by

$$\langle \mathcal{G}'(u), v\rangle = \int_{R_1}^{R_2} r^{N-1}g(r,u)v\,dr, \quad u, v \in C.$$

The functional $I : C \to (-\infty, +\infty]$ defined by

$$I = \Psi + \mathcal{G} \tag{7.3}$$

has the structure required by Szulkin's critical point theory [112]. Accordingly, a function $u \in C$ is *a critical point* of $I$ if $u \in K$ and it satisfies the inequality

$$\Psi(v) - \Psi(u) + \langle \mathcal{G}'(u), v - u\rangle \geq 0 \quad \text{for all } v \in C,$$

or, equivalently

$$\int_{R_1}^{R_2} r^{N-1}[\Phi(v') - \Phi(u')]\,dr + \int_{R_1}^{R_2} r^{N-1} g(r,u)(v-u)\,dr \geq 0 \quad \text{for all } v \in K.$$

Now, we consider the Neumann boundary value problem (2.20) under the basic hypothesis $(H_\Phi)$. Recall that by a *solution* of (2.20) we mean a function $u \in C^1[R_1, R_2]$, such that $\|u'\|_\infty < a$, $r \mapsto r^{N-1}\phi(u')$ is differentiable and (2.20) is satisfied.

**Lemma 15** *For every $f \in C$, problem*

$$[r^{N-1}\phi(u')]' = r^{N-1}[\overline{u} + f], \quad u'(R_1) = 0 = u'(R_2) \tag{7.4}$$

*has a unique solution $u_f$, which is also the unique solution of the variational inequality*

$$\int_{R_1}^{R_2} r^{N-1}[\Phi(v') - \Phi(u') + \overline{u}(\overline{v} - \overline{u}) + f(v-u)]\,dr \geq 0 \quad \textit{for all } v \in K, \tag{7.5}$$

*and the unique minimum over $K$ of the strictly convex functional $J$ defined by*

$$J(u) = \int_{R_1}^{R_2} r^{N-1}\left[\Phi(u') + \frac{\overline{u}^2}{2} + fu\right] dr. \tag{7.6}$$

*Proof.* Problem (7.4) is equivalent to finding $u = \overline{u} + \widetilde{u}$ with $\overline{u}$ and $\widetilde{u}$ solutions of

$$\begin{cases} [r^{N-1}\phi(\widetilde{u}')]' = r^{N-1}\widetilde{f}, \quad \widetilde{u}'(R_1) = 0 = \widetilde{u}'(R_2), \\ \overline{u} = -\overline{f}, \quad \displaystyle\int_{R_1}^{R_2} r^{N-1}\widetilde{u}(r)\,dr = 0. \end{cases} \tag{7.7}$$

Now the first equation gives, using the first boundary condition,

$$\widetilde{u}'(r) = \phi^{-1}\left[r^{1-N}\int_{R_1}^{r} s^{N-1}\widetilde{f}(s)\,ds\right]. \tag{7.8}$$

From (7.8) we get

$$\|\widetilde{u}'\|_\infty < a, \quad \widetilde{u}'(R_2) = \phi^{-1}\left[R_2^{1-N}\int_{R_1}^{R_2} s^{N-1}\widetilde{f}(s)\,ds\right] = \phi^{-1}(0) = 0.$$

Then the unique solution of (7.8) is given by

$$\widetilde{u}(r) = c + \int_{R_1}^{r} \phi^{-1}\left[t^{1-N}\int_{R_1}^{t} s^{N-1}\widetilde{f}(s)\,ds\right] dt, \tag{7.9}$$

where

$$c = -\frac{N}{R_2^N - R_1^N} \int_{R_1}^{R_2} r^{N-1} \int_{R_1}^{r} \phi^{-1} \left[ t^{1-N} \int_{R_1}^{t} s^{N-1} \widetilde{f}(s)\, ds \right] dt\, dr. \tag{7.10}$$

The unique solution $u_f = \overline{u} + \widetilde{u}$ of (7.4) follows from (7.7), (7.9) and (7.10).

Now, if $u$ is a solution of (7.4), then, taking $v \in K$, multiplying each member of the differential equation by $v - u$, integrating over $[R_1, R_2]$, and using integration by parts and the boundary conditions, we get

$$\int_{R_1}^{R_2} r^{N-1} [\phi(u')(v' - u') + \overline{u}(\overline{v} - \overline{u}) + f(v - u)]\, dr = 0,$$

which gives (7.5) if we use the convexity inequality for $\Phi$

$$\Phi(v') - \Phi(u') \geq \phi(u')(v' - u').$$

The inequality $\frac{\overline{v}^2}{2} - \frac{\overline{u}^2}{2} \geq \overline{u}(\overline{v} - \overline{u})$ introduced in (7.5) implies that

$$\int_{R_1}^{R_2} r^{N-1} \left[ \Phi(v') - \Phi(u') + \frac{\overline{v}^2}{2} + fv - \frac{\overline{u}^2}{2} - fu \right] dr \geq 0 \quad \text{for all } v \in K,$$

which shows that $J$ has a minimum on $K$ at $u$. Conversely if it is the case, then, for all $\lambda \in (0, 1]$ and all $v \in K$, we get

$$\int_{R_1}^{R_2} r^{N-1} \{ \Phi[(1-\lambda)u' + \lambda v'] + \frac{[(1-\lambda)\overline{u} + \lambda \overline{v}]^2}{2} + f[(1-\lambda)u + \lambda v] \}\, dr$$
$$\geq \int_{R_1}^{R_2} r^{N-1} [\Phi(u') + \frac{\overline{u}^2}{2} + fu]\, dr,$$

which, using the convexity of $\Phi$, simplifying, dividing both members by $\lambda$ and letting $\lambda \to 0_+$, gives the variational inequality (7.5). Thus solving (7.5) is equivalent to minimizing (7.6) over $K$. Now, it is straightforward to check that $J$ is strictly convex over $K$ and therefore has a unique minimum there, which gives the required uniqueness conclusions of Lemma 15. ∎

**Proposition 8** *If $u$ is a critical point of $I$, then $u$ is a solution of problem (2.20).*

*Proof.* We set

$$f_u := g(\cdot, u) - \overline{u} \in C$$

and consider the problem

$$[r^{N-1} \phi(w')]' = r^{N-1} [\overline{w} + f_u(r)], \quad w'(R_1) = 0 = w'(R_2). \tag{7.11}$$

By virtue of Lemma 15, problem (7.11) has an unique solution $\widehat{u}$ and it is also the unique solution of

$$\int_{R_1}^{R_2} r^{N-1}[\Phi(v') - \Phi(\widehat{u}') + \overline{\widehat{u}}(\overline{v} - \overline{\widehat{u}}) + f_u(r)(v - \widehat{u})]\, dr \geq 0 \quad \text{for all } v \in K. \tag{7.12}$$

Since $u$ is a critical point of $I$, we infer that

$$\int_{R_1}^{R_2} r^{N-1}[\Phi(v') - \Phi(u') + \overline{u}(\overline{v} - \overline{u}) + f_u(r)(v - u)]\, dr \geq 0 \quad \text{for all } v \in K. \tag{7.13}$$

It follows by uniqueness that $u = \widehat{u}$. Hence, $u$ solves problem (2.20). ■

## 7.2 Ground state solutions

We begin by a lemma which is the main tool for the minimization problems in this section. With this aim, for any $\rho > 0$, set

$$\widehat{K}_\rho := \{u \in K \ : \ |\overline{u}| \leq \rho\}.$$

**Lemma 16** *Assume that there is some $\rho > 0$ such that*

$$\inf_{\widehat{K}_\rho} I = \inf_K I. \tag{7.14}$$

*Then $I$ is bounded from below on $C$ and attains its infimum at some $u \in \widehat{K}_\rho$, which solves problem (2.20).*

*Proof.* By virtue of (7.14) and $\inf_C I = \inf_K I$, it suffices to prove that there is some $u \in \widehat{K}_\rho$ such that

$$I(u) = \inf_{\widehat{K}_\rho} I. \tag{7.15}$$

Then, we get that $u$ is a minimum point of $I$ on $C$ and, on account of [112, Proposition 1.1], is a critical point of $I$. The proof will be accomplished by virtue of Proposition 8.

If $v \in \widehat{K}_\rho$ then, using (7.1) we obtain

$$|v(r)| \leq |\overline{v}| + |\tilde{v}(r)| \leq \rho + (R_2 - R_1)a.$$

This, together with $\|v'\|_\infty \leq a$ show that $\widehat{K}_\rho$ is bounded in $W^{1,\infty}$ and, by the compactness of the embedding $W^{1,\infty} \subset C$, the set $\widehat{K}_\rho$ is relatively compact in $C$. Let $\{u_n\} \subset \widehat{K}_\rho$ be a minimizing sequence for $I$. Passing to a subsequence if necessary and using [**?**, Lemma 1], we may assume that $\{u_n\}$ converges uniformly to some $u \in K$. It is easily seen that actually $u \in \widehat{K}_\rho$. From (7.2) and the continuity of $\mathcal{G}$ on $C$, we obtain

$$I(u) \leq \liminf_{n\to\infty} I(u_n) = \lim_{n\to\infty} I(u_n) = \inf_{\widehat{K}_\rho} I,$$

showing that (7.15) holds true. ■

The following result is proved in [20, Theorem 1].

**Corollary 13** *Let $f : [R_1, R_2]\times\mathbb{R} \to \mathbb{R}$ be continuous and $F : [R_1, R_2]\times\mathbb{R} \to \mathbb{R}$ be defined by*

$$F(r, x) := \int_0^x f(r, \xi)\, d\xi, \quad (r, x) \in [R_1, R_2] \times \mathbb{R}.$$

*If there is some $\omega > 0$ such that $F(r, x) = F(r, x+\omega)$ for all $(r, x) \in [R_1, R_2]\times\mathbb{R}$, then, for any $h \in C$ with $\overline{h} = 0$, the problem*

$$[r^{N-1}\phi(u')]' = r^{N-1}[f(r, u) + h(r)], \quad u'(R_1) = 0 = u'(R_2).$$

*has at least one solution $u \in \widehat{K}_\omega$ which is a minimizer of the corresponding energy functional $I$ on $C$.*

*Proof.* We have

$$G(r, x) = F(r, x) + h(r)x, \quad (r, x) \in [R_1, R_2] \times \mathbb{R}.$$

Due to the $\omega$–periodicity of $F(r, \cdot)$ and because of $\overline{h} = 0$, it holds

$$I(v + j\omega) = I(v) \quad \text{for all } v \in K \text{ and } j \in \mathbb{Z}.$$

Then, the conclusion follows from the equality

$$\{I(v) \ : \ v \in K\} = \{I(v) \ : \ v \in \widehat{K}_\omega\}$$

and Lemma 16. ■

**Theorem 22** *If $g : [R_1, R_2] \times \mathbb{R} \to \mathbb{R}$ is a continuous function such that*

$$\liminf_{|x|\to\infty} G(r, x) > 0, \qquad \textit{uniformly in } r \in [R_1, R_2], \tag{7.16}$$

*then (2.20) has at least one solution which minimizes $I$ on $C$.*

*Proof.* Using (7.1) and (7.16) it follows that there exists $\rho > 0$ such that

$$G(r, u) > 0$$

for any $u \in K$ such that $|\overline{u}| > \rho$. It follows that $I(u) > 0$ provided that $u \in K$ and $|\overline{u}| > \rho$. The proof follows from Lemma 16, as $I(0) = 0$. ■

**Remark 15** An easy adaptation of the techniques in Section 2.3 of [49] shows that *the Neumann problem for the p-Laplacian* ($p > 1$) *on a bounded domain* $\Omega \subset \mathbb{R}^N$

$$div(|\nabla v|^{p-2}\nabla v) = g(x,v) \quad in \quad \Omega, \quad \frac{\partial u}{\partial \nu} = 0 \quad on \quad \partial\Omega$$

*with* $g : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ *continuous has at least one strong solution if*

$$\liminf_{|u|\to\infty} \frac{G(x,u)}{|u|^p} > 0, \quad \textit{uniformly in } x \in \overline{\Omega},$$

a condition of the type already introduced by Hammerstein [71] for the Laplacian with Dirichlet conditions. For the radial solutions of (2.18), Theorem 22 shows that it is sufficient that such a condition holds with $p = 0$.

**Example 4** The Neumann problem

$$\operatorname{div}\left(\frac{\nabla v}{\sqrt{1-|\nabla v|^2}}\right) = \frac{v + h(|x|)}{1 + [v + h(|x|)]^2} + \cos v \quad \text{in} \quad \mathcal{A},$$
$$\frac{\partial v}{\partial \nu} = 0 \quad \text{on} \quad \partial\mathcal{A},$$

has at least one radial solution for all $h \in C$.

**Theorem 23** *Let* $g : [R_1, R_2] \times \mathbb{R} \to \mathbb{R}$ *be a continuous function and* $l \in L^1_{N-1}$ *be such that*

$$|g(r,x)| \le l(r) \tag{7.17}$$

*for a.e.* $r \in (R_1, R_2)$ *and all* $x \in \mathbb{R}$. *If*

$$\lim_{|x|\to\infty} \int_{R_1}^{R_2} r^{N-1} G(r,x)\, dr = +\infty, \tag{7.18}$$

*then (2.20) has at least one solution which minimizes* $I$ *on* $C$.

*Proof.* We shall apply Lemma 16. For arbitrary $u \in K$, using (7.1) and (7.17), we estimate $I$ as follows.

$$\begin{aligned}
I(u) &= \int_{R_1}^{R_2} r^{N-1}\Phi(u')\, dr + \int_{R_1}^{R_2} r^{N-1} G(r,u)\, dr \\
&\ge \int_{R_1}^{R_2} r^{N-1} G(r,\overline{u})\, dr + \int_{R_1}^{R_2} r^{N-1}[G(r,u) - G(r,\overline{u})]\, dr \\
&= \int_{R_1}^{R_2} r^{N-1} G(r,\overline{u})\, dr + \int_{R_1}^{R_2} r^{N-1} \int_0^1 g(r,\overline{u} + s\tilde{u})\tilde{u}\, ds\, dr \\
&\ge \int_{R_1}^{R_2} r^{N-1} G(r,\overline{u})\, dr - a(R_2 - R_1) \int_{R_1}^{R_2} r^{N-1} l(r)\, dr.
\end{aligned}$$

From (7.18) we can find $\rho > 0$ such that $I(u) > 0$ provided that $|\overline{u}| > \rho$. As by $(H_\Phi)$ we know that $\Phi(0) = 0$, one has $I(0) = 0$. Therefore, (7.14) is fulfilled and the proof is complete. ■

**Remark 16** Condition (7.18) is of the type introduced by Ahmad-Lazer-Paul [1] for the Laplacian with Dirichlet conditions. The reader will observe that the conclusion of Theorem 23 still remains true if (7.18) is replaced by the weaker but more technical condition

$$\liminf_{|x|\to\infty} \int_{R_1}^{R_2} r^{N-1} G(r,x)\,dr > a(R_2 - R_1) \int_{R_1}^{R_2} r^{N-1} l(r)\,dr.$$

**Example 5** For every $h \in C$ such that $-\frac{\pi}{2} < \overline{h} < \frac{\pi}{2}$, the Neumann problem

$$\operatorname{div}\left(\frac{\nabla v}{\sqrt{1-|\nabla v|^2}}\right) - \arctan v - \cos v = h(|x|) \quad \text{in} \quad \mathcal{A},$$
$$\frac{\partial v}{\partial \nu} = 0 \quad \text{on} \quad \partial\mathcal{A},$$

has at least one radial solution.

**Theorem 24** *Let $g : [R_1, R_2] \times \mathbb{R} \to \mathbb{R}$ be a continuous function such that $G(r,\cdot)$ is convex for all $r \in [R_1, R_2]$. Then, problem (2.20) has at least one solution if and only if there is some $c \in \mathbb{R}$ such that*

$$\int_{R_1}^{R_2} r^{N-1} g(r,c)\,dr = 0. \tag{7.19}$$

*Proof.* Define

$$\Gamma : \mathbb{R} \to \mathbb{R}, \qquad x \mapsto \int_{R_1}^{R_2} r^{N-1} G(r,x)\,dr$$

and note that

$$\Gamma'(x) = \int_{R_1}^{R_2} r^{N-1} g(r,x)\,dr \quad \text{for all } x \in \mathbb{R}.$$

Let us assume that (2.20) has a solution $u$. Clearly, we have

$$\int_{R_1}^{R_2} r^{N-1} g(r,u)\,dr = 0. \tag{7.20}$$

On account of the convexity of $G(r,\cdot)$, the function $g(r,\cdot) : \mathbb{R} \to \mathbb{R}$ is nondecreasing for any $r \in [R_1, R_2]$. Hence,

$$g(r, -||u||_\infty) \le g(r, u(r)) \le g(r, ||u||_\infty) \quad \text{for all } r \in [R_1, R_2]. \tag{7.21}$$

From (7.20) and (7.21) we infer

$$\Gamma'(-||u||_\infty) \le 0 \le \Gamma'(||u||_\infty).$$

Then, by the continuity of $\Gamma'$ there exists $c \in \mathbb{R}$ such that (7.19) holds true.

Reciprocally, assume that there exists $c \in \mathbb{R}$ such that $\Gamma'(c) = 0$. Using the fact that $\Gamma'$ is nondecreasing, we have to consider the following three cases.

(i) It holds

$$\Gamma'(x) = \Gamma'(c) = 0 \quad \text{for all } x \geq c.$$

This implies that

$$g(r, x) = g(r, c) \quad \text{for all } r \in [R_1, R_2] \text{ and } x \geq c.$$

Let $v$ be a solution of the problem

$$[r^{N-1}\phi(w')]' = r^{N-1}g(r, c), \quad w'(R_1) = 0 = w'(R_2);$$

we know that this exists by Theorem 2.3 in [**?**]. Setting $u = c + ||v||_\infty + v$, we get that $u$ solves problem (2.20).

(ii) One has that

$$\Gamma'(x) = \Gamma'(c) = 0 \quad \text{for all } x \leq c.$$

In this case the reasoning is similar to that in the case (i).

(iii) There are $x_1, x_2 \in \mathbb{R}$ with $x_1 < c < x_2$ and $\Gamma'(x_1) < 0 < \Gamma'(x_2)$. If $x \geq x_2$, then

$$\begin{aligned} \Gamma(x) &= \Gamma(x_2) + \int_{R_1}^{R_2} r^{N-1} \left( \int_{x_2}^{x} g(r, t)\, dt \right) dr \\ &\geq \Gamma(x_2) + (x - x_2)\Gamma'(x_2). \end{aligned}$$

It follows that $\Gamma(x) \to +\infty$ when $x \to +\infty$. Analogously $\Gamma(x) \to +\infty$ when $x \to -\infty$. Hence,

$$\lim_{|x| \to \infty} \Gamma(x) = +\infty. \tag{7.22}$$

On the other hand, by the convexity of $G(r, \cdot)$, we have

$$G(r, u) \geq 2G(r, \frac{\overline{u}}{2}) - G(r, -\widetilde{u}) \quad \text{for all } r \in [R_1, R_2],$$

which gives

$$I(u) \geq \int_{R_1}^{R_2} r^{N-1}\Phi(u')\, dr + 2\Gamma(\frac{\overline{u}}{2}) - \int_{R_1}^{R_2} r^{N-1} G(r, \tilde{u})\, dr \quad \text{for all } u \in K \tag{7.23}$$

The estimate (7.23) together with (7.1) and (7.22) show that we can find $\rho > 0$ such that $I(u) > 0$ provided that $u \in K$ and $|\overline{u}| > \rho$. Then, the proof follows from Lemma 16 as in the proof of Theorem 23. ∎

**Remark 17** Theorem 24 can be stated equivalently as : *Let* $g : [R_1, R_2] \times \mathbb{R} \to \mathbb{R}$ *be a continuous function such that* $G(r, \cdot)$ *is convex for all* $r \in [R_1, R_2]$. *Then, problem (2.20) has at least one solution if and only if the real convex function* $x \mapsto \int_{R_1}^{R_2} r^{N-1} G(r, x)\, dr$ *has a minimum.* Corresponding results for the Laplacian with Neumann or Dirichlet boundary conditions have been given in [88] and [87].

**Example 6** The Neumann problem with $h \in C$

$$\operatorname{div}\left(\frac{\nabla v}{\sqrt{1-|\nabla v|^2}}\right) = \arctan v - h(|x|) \quad \text{in} \quad \mathcal{A}, \quad \frac{\partial v}{\partial \nu} = 0 \quad \text{on} \quad \partial\mathcal{A},$$

has at least one radial solution if and only if $-\dfrac{\pi}{2} < \overline{h} < \dfrac{\pi}{2}$.

**Example 7** The Neumann problem with $h \in C$

$$\operatorname{div}\left(\frac{\nabla v}{\sqrt{1-|\nabla v|^2}}\right) = \arctan v^+ - h(|x|) \quad \text{in} \quad \mathcal{A}, \quad \frac{\partial v}{\partial \nu} = 0 \quad \text{on} \quad \partial\mathcal{A},$$

has at least one radial solution if and only if $0 \le \overline{h} < \dfrac{\pi}{2}$.

**Example 8** The Neumann problem with $h \in C$

$$\operatorname{div}\left(\frac{\nabla v}{\sqrt{1-|\nabla v|^2}}\right) = e^v - h(|x|) \quad \text{in} \quad \mathcal{A}, \quad \frac{\partial v}{\partial \nu} = 0 \quad \text{on} \quad \partial\mathcal{A},$$

has at least one radial solution if and only if $\overline{h} > 0$.

## 7.3 (PS)–sequences and Saddle Point solutions

Towards the application of the minimax results obtained in Szulkin [112] to the functional $I$ defined by (7.3) we have to know when $I$ satisfies the compactness *Palais-Smale* (in short, (PS)) *condition.*

Viewing our functional framework from Section 2, we say that a sequence $\{u_n\} \subset K$ is a *(PS)–sequence* if $I(u_n) \to c \in \mathbb{R}$ and

$$\int_{R_1}^{R_2} r^{N-1}[\Phi(v') - \Phi(u_n') + g(r, u_n)(v - u_n)]\, dr \ge -\varepsilon_n \|v - u_n\|_\infty \quad \text{for all } v \in K, \tag{7.24}$$

where $\varepsilon_n \to 0+$. According to [112], the functional $I$ is said to satisfy the *(PS) condition* if any (PS)–sequence has a convergent subsequence in $C$.

The lemma below provides useful properties of the *(PS)*–sequences.

**Lemma 17** *Let $\{u_n\}$ be a (PS)–sequence. Then the following hold true :*

*(i) the sequence* $\left\{\int_{R_1}^{R_2} r^{N-1}G(r,u_n)\,dr\right\}$ *is bounded;*

*(ii) if $\{\overline{u}_n\}$ is bounded, then $\{u_n\}$ has a convergent subsequence in $C$;*

*(iii) one has that*

$$-\varepsilon_n \le \int_{R_1}^{R_2} r^{N-1}g(r,u_n)\,dr \le \varepsilon_n \quad \text{for all } n \in \mathbb{N}. \tag{7.25}$$

*Proof.* *(i)* This is immediate from the fact that $\{I(u_n)\}$ and $\Phi$ are bounded.

*(ii)* From (7.1) and $u_n \in K$, the sequence $\{\widetilde{u}_n\}$ is bounded in $W^{1,\infty}$. By the compactness of the embedding $W^{1,\infty} \subset C$, we deduce that $\{\widetilde{u}_n\}$ has a convergent subsequence in $C$. Using then the boundedness of $\{\overline{u}_n\} \subset \mathbb{R}$ it follows that $\{u_n\}$ has a convergent subsequence in $C$.

*(iii)* Taking $v = u_n \pm 1$ in (7.24) one obtains (7.25). ■

**Theorem 25** *Let $g : [R_1, R_2] \times \mathbb{R} \to \mathbb{R}$ be a continuous function and $l \in L^1_{N-1}$ be such that (7.17) is satisfied for a.e. $r \in (R_1, R_2)$ and all $x \in \mathbb{R}$. If*

$$\lim_{|x|\to\infty} \int_{R_1}^{R_2} r^{N-1}G(r,x)\,dr = -\infty, \tag{7.26}$$

*then (2.20) has at least one solution.*

*Proof.* We shall apply the Saddle Point Theorem [112, Theorem 3.5].

From (7.26) the functional $I$ is not bounded from below. Indeed, if $v = c \in \mathbb{R}$ is a constant function then

$$I(c) = \int_{R_1}^{R_2} r^{N-1}G(r,c)\,dr \to -\infty \quad \text{as} \quad |c| \to \infty. \tag{7.27}$$

We split $C = \mathbb{R} \bigoplus X$, where $X = \{v \in C \;:\; \overline{v} = 0\}$. Note that

$$I(v) \ge \int_{R_1}^{R_2} r^{N-1}G(r,\widetilde{v})\,dr \quad \text{for all } v \in K \cap X,$$

which together with (7.1) imply that there is a constant $\alpha \in \mathbb{R}$ such that

$$I(v) \ge \alpha \quad \text{for all } v \in X. \tag{7.28}$$

Using (7.27) and (7.28) we can find some $R > 0$ so that

$$\sup_{S_R} I < \inf_X I,$$

where $S_R = \{c \in \mathbb{R} : |c| = R\}$.

It remains to show that $I$ satisfies the (PS) condition. Let $\{u_n\} \subset K$ be a (PS)–sequence. Since $\{I(u_n)\}$, $\{\varphi(u_n)\}$ are bounded and, by (7.17) we have

$$\begin{aligned} & \left| \int_{R_1}^{R_2} r^{N-1}[G(r,u_n) - G(r,\overline{u}_n)]\,dr \right| \\ \leq\ & \int_{R_1}^{R_2} r^{N-1} \int_0^1 |g(r, \overline{u}_n + s\tilde{u}_n)\tilde{u}_n|\,ds\,dr \\ \leq\ & a(R_2 - R_1) \int_{R_1}^{R_2} r^{N-1} l(r)\,dr, \end{aligned}$$

from

$$I(u_n) = \varphi(u_n) + \int_{R_1}^{R_2} r^{N-1} G(r,\overline{u}_n)\,dr + \int_{R_1}^{R_2} r^{N-1}[G(r,u_n) - G(r,\overline{u}_n)]\,dr$$

it follows that there exists a constant $\beta \in \mathbb{R}$ such that

$$\int_{R_1}^{R_2} r^{N-1} G(r,\overline{u}_n)\,dr \geq \beta.$$

Then by (7.26) the sequence $\{\overline{u}_n\}$ is bounded and Lemma 17 *(ii)* ensures that $\{u_n\}$ has a convergent subsequence in $C$. Consequently, $I$ satisfies the (PS) condition and the conclusion follows from [112, Theorem 3.5] and Proposition 8. ■

**Remark 18** Condition (7.26), also of the Ahmad-Lazer-Paul type [1] is, in some sense, 'dual' to condition (7.18).

**Example 9** For every $h \in C$ such that $-\frac{\pi}{2} < \overline{h} < \frac{\pi}{2}$, the Neumann problem

$$\operatorname{div}\left(\frac{\nabla v}{\sqrt{1 - |\nabla v|^2}}\right) + \arctan v + \cos v = h(|x|) \quad \text{in} \quad \mathcal{A},$$
$$\frac{\partial v}{\partial \nu} = 0 \quad \text{on} \quad \partial\mathcal{A},$$

has at least one radial solution.

**Theorem 26** *If $g : [R_1, R_2] \times \mathbb{R} \to \mathbb{R}$ is a continuous function such that*

$$\lim_{|x| \to \infty} G(r,x) = -\infty, \qquad \textit{uniformly in } r \in [R_1, R_2], \tag{7.29}$$

*then (2.20) has at least one solution.*

*Proof.* We keep the notations introduced in the proof of Theorem 25. Clearly, (7.29) implies (7.26) and from the proof of Theorem 25 it follows that $I$ has the geometry required by the Saddle Point Theorem. To show that $I$ satisfies the (PS) condition, let $\{u_n\} \subset K$ be a (PS)–sequence. If $\{|\overline{u}_n|\}$ is not bounded, we may assume going if necessary to a subsequence, that $|\overline{u}_n| \to \infty$. Using (7.1) and (7.29) we deduce that

$$G(r, u_n(r)) \to -\infty, \quad \text{uniformly in } r \in [R_1, R_2].$$

This implies

$$\int_{R_1}^{R_2} r^{N-1} G(r, u_n)\, dr \to -\infty,$$

contradicting Lemma 17 *(i)*. Hence, $\{\overline{u}_n\}$ is bounded and by Lemma 17 *(ii)*, the sequence $\{u_n\}$ has a convergent subsequence in $C$. Therefore, $I$ satisfies the (PS) condition. The proof is complete. ■

**Remark 19** No result corresponding to Theorem 26 holds for the Laplacian with Neumann (or Dirichlet) boundary conditions. Indeed, if $\lambda_k$ is a positive eigenvalue of $-\Delta$ on some bounded domain $\Omega \subset \mathbb{R}^N$ with Neumann boundary conditions, and $\varphi_k$ a corresponding eigenfunction, the problem

$$\Delta v = -\lambda_k v + \varphi_k(x) \quad in \quad \Omega, \quad \frac{\partial v}{\partial \nu} = 0 \quad on \quad \partial\Omega$$

has no solution, but $-\lambda_k \dfrac{u^2}{2} + \varphi_k(x)u \to -\infty$ uniformly in $\overline{\Omega}$ when $|u| \to \infty$.

**Example 10** The Neumann problem

$$\operatorname{div}\left(\frac{\nabla v}{\sqrt{1-|\nabla v|^2}}\right) + \frac{v + h(|x|)}{1 + [v + h(|x|)]^2} = \cos v \quad \text{in} \quad \mathcal{A},$$
$$\frac{\partial v}{\partial \nu} = 0 \quad \text{on} \quad \partial\mathcal{A},$$

has at least one radial solution for all $h \in C$.

## 7.4 Mountain Pass solutions

In this section we consider problem (2.21) with $\lambda > 0$ and $m \geq 2$ fixed real numbers, and $f : [R_1, R_2] \times \mathbb{R} \to \mathbb{R}$ a continuous function satisfying the Ambrosetti–Rabinowitz condition [7] :

$(AR)$ *There exists $\theta > m$ and $x_0 > 0$ such that*

$$0 < \theta F(r, x) \leq x f(r, x) \quad \textit{for all} \quad r \in [R_1, R_2] \quad \textit{and} \quad |x| \geq x_0.$$

Note that for problem (2.21) the function $g$ from the general functional framework in Section 2 is now defined in terms of $f$ by

$$g(r,x) = \lambda|x|^{m-2}x - f(r,x) \quad \text{for all } (r,x) \in [R_1,R_2] \times \mathbb{R}$$

and accordingly, $G$ entering in the definition of the energy functional $I$ becomes

$$G(r,x) = \lambda\frac{|x|^m}{m} - F(r,x) \quad \text{for all } (r,x) \in [R_1,R_2] \times \mathbb{R}.$$

**Lemma 18** *Let $p \geq 1$ be a real number. Then*

$$|u(r)|^p \geq |\overline{u}|^p - pa(R_2-R_1)|\overline{u}|^{p-1},\ \forall u \in K,\ \forall r \in [R_1,R_2] \tag{7.30}$$

*and there are constants $\alpha_1,\ \alpha_2 \geq 0$ such that*

$$|u(r)|^p \leq |\overline{u}|^p + \alpha_1|\overline{u}|^{p-1} + \alpha_2, \quad \forall u \in K \text{ with } |\overline{u}| \geq 1,\ \forall r \in [R_1,R_2]. \tag{7.31}$$

*Proof.* The result is trivial for $p = 1$. If $p > 1$, $u \in K$ and $r \in [R_1,R_2]$, then, using the convexity of the differentiable function $s \mapsto |s|^p$, we get

$$\begin{aligned} |u(r)|^p &= |\overline{u} + \widetilde{u}(r)|^p \geq |\overline{u}|^p + p|\overline{u}|^{p-2}\overline{u}\widetilde{u}(r) \\ &\geq |\overline{u}|^p - p|\overline{u}|^{p-1}(R_2-R_1)a. \end{aligned}$$

On the other hand, denoting by $\widetilde{p}$ the smallest integer larger or equal to $p$ and letting $M := a(R_2-R_1)$, we have, for all $r \in [R_1,R_2]$,

$$\begin{aligned} |u(r)|^p &= |\overline{u} + \widetilde{u}(r)|^p \leq (|\overline{u}| + M)^p = |\overline{u}|^p \left(1 + \frac{M}{|\overline{u}|}\right)^p \\ &\leq |\overline{u}|^p \left(1 + \frac{M}{|\overline{u}|}\right)^{\widetilde{p}} = |\overline{u}|^p \left(1 + \sum_{k=1}^{\widetilde{p}} \frac{\widetilde{p}!}{k!(\widetilde{p}-k)!}\frac{M^k}{|\overline{u}|^k}\right) \\ &= |\overline{u}|^p + \sum_{k=1}^{\widetilde{p}} \frac{\widetilde{p}!}{k!(\widetilde{p}-k)!} M^k |\overline{u}|^{p-k}, \end{aligned}$$

and (7.31) follows easily. ■

**Lemma 19** *If (AR) holds, then $I$ satisfies the (PS) condition.*

*Proof.* Let $\{u_n\} \subset K$ be a (PS)–sequence. From Lemma 17 *(i)* and (7.30) there are constants $c_1, d \in \mathbb{R}$ such that

$$\lambda\frac{R_2^N - R_1^N}{N}\frac{|\overline{u}_n|^m}{m} - c_1|\overline{u}_n|^{m-1} - \int_{R_1}^{R_2} r^{N-1}F(r,u_n)\,dr \leq d \quad \forall n \in \mathbb{N}. \tag{7.32}$$

Using Lemma 17 *(iii)* and $\varepsilon_n \to 0$, we may assume that

$$-1 \le \lambda \int_{R_1}^{R_2} r^{N-1}|u_n|^{m-2}u_n\,dr - \int_{R_1}^{R_2} r^{N-1}f(r,u_n)\,dr \le 1 \quad \forall n \in \mathbb{N}. \tag{7.33}$$

Suppose, by contradiction, that $\{|\overline{u}_n|\}$ is not bounded. Then, there is a subsequence of $\{|\overline{u}_n|\}$, still denoted by $\{|\overline{u}_n|\}$, with $|\overline{u}_n| \to \infty$. Let $n_0 \in \mathbb{N}$ be such that $|\overline{u}_n| \ge \max\{1, x_0 + a(R_2 - R_1)\}$ for all $n \ge n_0$. By virtue of (7.1) we have

$$|u_n(r)| \ge x_0 \quad \text{for all } r \in [R_1, R_2] \text{ and } n \ge n_0.$$

The (AR) condition ensures that

$$\operatorname{sign} \overline{u}_n = \operatorname{sign} u_n(r) = \operatorname{sign} f(r, u_n(r)) \quad \text{for all } r \in [R_1, R_2] \text{ and } n \ge n_0 \tag{7.34}$$

and

$$\begin{aligned} &- \int_{R_1}^{R_2} r^{N-1}F(r,u_n)\,dr \\ &\ge -\frac{\overline{u}_n}{\theta}\int_{R_1}^{R_2} r^{N-1}f(r,u_n)\,dr - \frac{1}{\theta}\int_{R_1}^{R_2} r^{N-1}f(r,u_n)\widetilde{u}_n\,dr \\ &\text{for all } n \ge n_0. \end{aligned} \tag{7.35}$$

From (7.33) and (7.31) there are constants $c_2, c_3 \ge 0$ such that

$$\begin{aligned} &- \frac{\overline{u}_n}{\theta}\int_{R_1}^{R_2} r^{N-1}f(r,u_n)\,dr \ge -\lambda\frac{R_2^N - R_1^N}{\theta N}|\overline{u}_n|^m - c_2|\overline{u}_n|^{m-1} - c_3 \\ &\text{for all } n \ge n_0. \end{aligned} \tag{7.36}$$

Also, using (7.1), (7.31), (7.33) and (7.34) we can find constants $c_4, c_5, c_6 \ge 0$ so that

$$\begin{aligned} &- \frac{1}{\theta}\int_{R_1}^{R_2} r^{N-1}f(r,u_n)\widetilde{u}_n\,dr \ge -c_4|\overline{u}_n|^{m-1} - c_5|\overline{u}_n|^{m-2} - c_6, \\ &\text{for all } n \ge n_0. \end{aligned} \tag{7.37}$$

From (7.35), (7.36) and (7.37) we obtain

$$\begin{aligned} &- \int_{R_1}^{R_2} r^{N-1}F(r,u_n)\,dr \\ &\ge -\lambda\frac{R_2^N - R_1^N}{N}\frac{|\overline{u}_n|^m}{\theta} - (c_2+c_4)|\overline{u}_n|^{m-1} - c_5|\overline{u}_n|^{m-2} - c_3 - c_6 \\ &\text{for all } n \ge n_0. \end{aligned} \tag{7.38}$$

Then, (7.38) together with $\theta > m$ imply

$$\lambda\frac{R_2^N - R_1^N}{N}\frac{|\overline{u}_n|^m}{m} - c_1|\overline{u}_n|^{m-1} - \int_{R_1}^{R_2} r^{N-1}F(r,u_n)\,dr \to +\infty \quad \text{as } n \to \infty,$$

contradicting (7.32). Consequently, $\{\overline{u}_n\}$ is bounded and the proof follows from Lemma 17 *(ii)*. ■

**Lemma 20** *If (AR) holds and $c \in \mathbb{R}$, then $I(c) \to -\infty$ as $|c| \to \infty$.*

*Proof.* The (AR) condition implies (see [49]) that there exists $\gamma \in C$, $\gamma > 0$, such that

$$F(r,x) \geq \gamma(r)|x|^\theta \quad \text{for all } r \in [R_1, R_2] \text{ and } |x| \geq x_0. \tag{7.39}$$

From (7.39) we infer

$$\begin{aligned} I(c) &= \lambda \frac{R_2^N - R_1^N}{mN}|c|^m - \int_{R_1}^{R_2} r^{N-1} F(r,c)\, dr \\ &\leq \lambda \frac{R_2^N - R_1^N}{mN}|c|^m - |c|^\theta \int_{R_1}^{R_2} r^{N-1}\gamma(r)\, dr, \end{aligned}$$

for all $c \in \mathbb{R}$ with $|c| \geq x_0$. Then, the conclusion follows from $\theta > m$ and $\gamma > 0$. ■

**Lemma 21** *Assume that $F$ satisfies*

$$\limsup_{x \to 0} \frac{mF(r,x)}{|x|^m} < \lambda \quad \textit{uniformly in } r \in [R_1, R_2]. \tag{7.40}$$

*Then there exist $\alpha, \rho > 0$ such that*

$$\int_{R_1}^{R_2} r^{N-1}\left[\lambda \frac{|u|^m}{m} - F(r,u)\right] dr \geq \alpha \quad \textit{for all } u \in K \cap \partial B_\rho, \tag{7.41}$$

*where $\partial B_\rho := \{u \in C : ||u||_\infty = \rho\}$.*

*Proof.* Assumption (7.40) ensures that there are constants $b < \lambda$ and $\rho > 0$ such that

$$F(r,x) \leq \frac{b}{m}|x|^m \quad \text{for all } r \in [R_1, R_2] \text{ and } |x| \leq \rho. \tag{7.42}$$

We claim that:

$$\inf_{u \in K \cap \partial B_\rho} \int_{R_1}^{R_2} r^{N-1}|u|^m\, dr > 0. \tag{7.43}$$

Then, by virtue of (7.42) we have

$$\begin{aligned} &\int_{R_1}^{R_2} r^{N-1}\left[\lambda \frac{|u|^m}{m} - F(r,u)\right] dr \\ \geq\ & \frac{\lambda - b}{m}\int_{R_1}^{R_2} r^{N-1}|u|^m\, dr =: \alpha \quad \text{for all } u \in K \cap \partial B_\rho, \end{aligned}$$

and (7.43) implies (7.41). In order to prove (7.43), suppose by contradiction that there exists a sequence $\{u_n\} \subset K \cap \partial B_\rho$ such that

$$\int_{R_1}^{R_2} r^{N-1}|u_n|^m \, dr \to 0 \quad \text{as} \quad n \to \infty.$$

It is clear that $\{u_n\}$ is bounded in $W^{1,\infty}$. Passing to a subsequence if necessary, we may assume a that $\{u_n\}$ is convergent in $C$ to some $u$. This implies that $||u||_\infty = \rho$ and

$$\int_{R_1}^{R_2} r^{N-1}|u_n|^m \, dr \to \int_{R_1}^{R_2} r^{N-1}|u|^m \, dr \quad \text{as} \quad n \to \infty.$$

It follows that $u = 0$, contradiction with $||u||_\infty = \rho > 0$. Therefore, (7.43) holds true and the proof is complete. ∎

**Theorem 27** *Assume that the (AR) condition holds true. If $F$ satisfies (7.40), then problem (2.21) has at least one nontrivial solution.*

*Proof.* The proof follows immediately from Lemmas 19, 20 and 21 and the Mountain Pass Theorem [112, Theorem 3.2]) applied to the functional $I$. ∎

**Remark 20** Theorem 27 is of the type introduced by Ambrosetti and Rabinowitz [7] for nonlinear perturbations of the Laplacian with Dirichlet boundary conditions.

**Example 11** If $\theta > m \geq 2$, $\lambda > 0$ are given real numbers and $\mu \in C$ is a positive function, then the Neumann problem

$$\operatorname{div}\left(\frac{\nabla v}{\sqrt{1-|\nabla v|^2}}\right) = \lambda|v|^{m-2}v - \mu(|x|)|v|^{\theta-2}v \quad \text{in} \quad \mathcal{A}, \quad \frac{\partial v}{\partial \nu} = 0 \quad \text{on} \quad \partial\mathcal{A},$$

has at least one nontrivial radial solution.

## 7.5 The periodic case

Let $\Phi : [-a, a] \to \mathbb{R}$ and $g : [R_1, R_2] \times \mathbb{R} \to \mathbb{R}$ be as above, i.e., $\Phi$ satisfies $(H_\Phi)$ and $g$ is continuous. The periodic problem (2.22) can be treated quite similarly to problem (2.20) with the following modifications. Taking $N = 1$, one works with

$$K_P := \{v \in W^{1,\infty} \ : \ \|v'\|_\infty \leq a, \ v(R_1) = v(R_2)\}$$

instead of $K$, and $\Psi_P : C \to (-\infty, +\infty]$ given by

$$\Psi_P(v) = \begin{cases} \displaystyle\int_{R_1}^{R_2} \Phi(v'), & \text{if } v \in K_P, \\ +\infty, & \text{otherwise,} \end{cases}$$

instead of $\Psi$. With $\mathcal{G}_P : C \to \mathbb{R}$ defined by

$$\mathcal{G}_P(u) = \int_{R_1}^{R_2} G(r,u)\,dr, \quad u \in C,$$

the energy functional $I_P : C \to (-\infty, +\infty]$ will be now $I_P = \Psi_P + \mathcal{G}_P$.

The references from [20] are replaced by the similar ones from [29].

We only state the following existence results which are obtained as the corresponding ones for problems (2.20) and (2.21) by no longer than "mutatis mutandis" arguments.

**Proposition 9** *If $u \in K_P$ is a critical point of $I_P$, then $u$ is a solution of problem (2.22).*

Denoting

$$\widehat{K}_{P,\rho} := \{u \in K_P \ : \ |\overline{u}| \leq \rho\},$$

we have the following

**Lemma 22** *Assume that there is some $\rho > 0$ such that*

$$\inf_{\widehat{K}_{P,\rho}} I_P = \inf_{K_P} I_P.$$

*Then $I_P$ is bounded from below and attains its infimum at some $u \in \widehat{K}_{P,\rho}$, which solves problem (2.22).*

By means of Lemma 22 we can easily reformulate Corollary 13, Theorem 22 and Theorem 26 for the periodic problem (2.22). Also we note the following versions of the other theorems.

**Theorem 28** *Assume that there exists $l \in L^1$ such that*

$$|g(r,x)| \leq l(r)$$

*for a.e. $r \in (R_1, R_2)$ and all $x \in \mathbb{R}$. If either*

$$\liminf_{|x|\to\infty} \int_{R_1}^{R_2} G(r,x)\,dr > (R_2 - R_1)\left(a \int_{R_1}^{R_2} l(r)\,dr\right) \tag{7.44}$$

*or*

$$\lim_{|x|\to\infty} \int_{R_1}^{R_2} G(r,x)\,dr = -\infty,$$

*then problem (2.22) has at least one solution $u$. Moreover, if (7.44) holds true then $u$ minimizes $I_P$ on $C$.*

**Theorem 29** *Let $g : [R_1, R_2] \times \mathbb{R} \to \mathbb{R}$ be a continuous function such that $G(r, \cdot)$ is convex for all $r \in [R_1, R_2]$. Then, problem (2.22) has at least one solution if and only if there is some $c \in \mathbb{R}$ such that*

$$\int_{R_1}^{R_2} g(r, c)\, dr = 0.$$

**Theorem 30** *Let $f : [R_1, R_2] \times \mathbb{R} \to \mathbb{R}$ be a continuous function such that the (AR) condition is fulfilled. If $F$ satisfies (7.40), then the problem*

$$[\phi(u')]' = \lambda |u|^{m-2} u - f(r, u), \quad u(R_1) - u(R_2) = 0 = u'(R_1) - u'(R_2),$$

*has at least one nontrivial solution for any $\lambda > 0$ and $m \geq 2$.*

# Chapter 8

# One parameter Neumann problems

## 8.1 Preliminaries

Let $(X, \|\cdot\|)$ be a real Banach space and $I$ be a functional of the type

$$I = \mathcal{F} + \psi,$$

where $\psi : X \to (-\infty, +\infty]$ is proper (i.e., $D(\psi) := \{v \in X; : \psi(v) < +\infty\} \neq \emptyset$), convex, lower semicontinuous (in short, l.s.c.) and $\mathcal{F} \in C^1(X; \mathbb{R})$.

According to Szulkin [112], a point $u \in X$ is said to be *a critical point* of $I$ if it satisfies the inequality

$$\langle \mathcal{F}'(u), v - u \rangle + \psi(v) - \psi(u) \geq 0, \quad \forall v \in X.$$

A number $c \in \mathbb{R}$ such that $I^{-1}(c)$ contains a critical point is called *a critical value* of $I$.

The functional $I$ is said *to satisfy the Palais-Smale* (in short, *(PS)*) *condition* if every sequence $\{u_n\} \subset X$ for which $I(u_n) \to c \in \mathbb{R}$ and

$$\langle \mathcal{F}'(u_n), v - u_n \rangle + \psi(v) - \psi(u_n) \geq -\varepsilon_n \|v - u_n\|, \quad \forall v \in X,$$

where $\varepsilon_n \to 0$ (called *(PS)-sequence*), possesses a convergent subsequence.

**Proposition 10** *Suppose that $I$ satisfies the (PS) condition and there exists an open set $U$ such that*

$$-\infty < \inf_{\overline{U}} I < \inf_{\partial U} I. \tag{8.1}$$

*Then $I$ has at least one critical point $u \in U$ such that $I(u) = \inf_{U} I$.*

*Proof.* Let

$$c_0 = \inf_{\overline{U}} I \tag{8.2}$$

and $\{\varepsilon_n\}$ be a sequence with $\varepsilon_n \to 0$ and

$$0 < \varepsilon_n < \inf_{\partial U} I - c_0 \quad \text{for all } n \in I\!N. \tag{8.3}$$

Using Ekeland's variational principle, applied to $I|_{\overline{U}}$, for each $n \in I\!N$, we can find $v_n \in \overline{U}$ such that

$$I(v_n) \leq c_0 + \varepsilon_n \tag{8.4}$$

and

$$I(v) \geq I(v_n) - \varepsilon_n \|v - v_n\| \quad \text{for all } v \in \overline{U}. \tag{8.5}$$

From (8.3) and (8.4) it follows $I(v_n) < \inf_{\partial U} I$, which ensures that $v_n \in U$, for all $n \in I\!N$. Let $v \in X$, $n \in I\!N$ be arbitrarily chosen and $t_0 := t_0(v, n) \in (0, 1)$ be so that $v_n + t(v - v_n) \in U$, for all $t \in (0, t_0)$. Using (8.5) and the convexity of $\psi$, we get

$$\frac{\mathcal{F}(v_n + t(v - v_n)) - \mathcal{F}(v_n)}{t} + \psi(v) - \psi(v_n) \geq -\varepsilon_n \|v - v_n\|$$

and, letting $t \to 0+$, one obtains

$$\langle \mathcal{F}'(v_n), v - v_n \rangle + \psi(v) - \psi(v_n) \geq -\varepsilon_n \|v - v_n\| \quad \text{for all } v \in X. \tag{8.6}$$

On the other hand, from (8.4) it is clear that

$$I(v_n) \to c_0. \tag{8.7}$$

Since $I$ satisfies the (PS) condition, (8.6) and (8.7) ensure that $\{v_n\}$ contains a subsequence, still denoted by $\{v_n\}$, convergent to some $u \in \overline{U}$.

By the lower semicontinuity of $\psi$ it holds

$$\psi(u) \leq \liminf_{n \to \infty} \psi(v_n) \tag{8.8}$$

and, on account of $\mathcal{F} \in C^1(X; I\!R)$, one obtains

$$\lim_{n \to \infty} \langle \mathcal{F}'(v_n), v - v_n \rangle = \langle \mathcal{F}'(u), v - u \rangle \quad \text{for all } v \in X. \tag{8.9}$$

From (8.6), (8.8) and (8.9) we deduce

$$\langle \mathcal{F}'(u), v - u \rangle + \psi(v) - \psi(u) \geq 0 \quad \text{for all } v \in X. \tag{8.10}$$

Also, from (8.2), (8.7) and (8.8) we have

$$c_0 \leq I(u) \leq \lim_{n \to \infty} \mathcal{F}(v_n) + \liminf_{n \to \infty} \psi(v_n) = \liminf_{n \to \infty} I(v_n) = c_0,$$

hence $I(u) = c_0$ and from (8.1), $u \in U$. This together with (8.10) shows that $c_0$ is a critical value of $I$. ■

For $\sigma > 0$, we shall denote $B_\sigma = \{v \in X \ : \ \|v\| < \sigma\}$ and by $\overline{B}_\sigma$ its closure.

**Proposition 11** *Suppose that $I$ satisfies the (PS) condition together with*

*(i) $I(0) = 0$ and there exists $\rho > 0$ such that*

$$-\infty < \inf_{\overline{B}_\rho} I < 0 < \inf_{\partial B_\rho} I; \tag{8.11}$$

*(ii) $I(e) \leq 0$ for some $e \in X \setminus \overline{B}_\rho$.*

*Then $I$ has at least two nontrivial critical points.*

*Proof.* From the Mountain Pass Theorem [112, Theorem 3.2] there exists a first nontrivial critical point $u_0 \in X$ with $I(u_0) > 0$. On the other hand, using Proposition 10 with $U = B_\rho$ and (8.11), it follows that $\inf_{B_\rho} I$ is a critical value of $I$. This implies the existence of a second critical point $u_1$ with $I(u_1) < 0$. We have that $u_1$ is nontrivial and different from $u_0$ because $I(0) = 0$ and $I(u_0) > 0$. ■

**Remark 21** (i) It is a simple matter to check that if, in addition $\psi$ and $\mathcal{F}$ are even, then $I$ has at least four nontrivial critical points.

(ii) If the operator $\mathcal{F}' : X \to X^*$ maps bounded sets into bounded sets, then condition $-\infty < \inf_{\overline{B}_\rho} I$ in (8.11) is automatically satisfied. Indeed, in this case, by the mean value theorem one has

$$|\mathcal{F}(u) - \mathcal{F}(0)| \leq \rho \sup_{v \in \overline{B}_\rho} \|\mathcal{F}'(v)\| \quad \text{for all } u \in \overline{B}_\rho,$$

showing that $\mathcal{F}$ is bounded on $\overline{B}_\rho$. On the other hand, we know that the proper, convex and l.s.c. function $\psi$ is bounded from below by a continuous affine function.

(iii) Proposition 11 is implicitly employed in [77] to derive the existence of at least two nontrivial solutions for a variational inequality on the half line.

## 8.2 Hypotheses and the functional framework

Throughout this paper we assume that *hypotheses $(H_f)$ and $(H_\Phi)$ from Section 1 hold true.* Clearly, from $(H_\Phi)$ we have that $\Phi$ is strictly convex and $\Phi(x) \geq 0$ for all $x \in [-a, a]$. Also, it is worth noticing that choosing $\Phi(y) = 1 - \sqrt{1 - y^2}$, $\forall y \in [-1, 1]$, one has $\phi(y) = \dfrac{y}{\sqrt{1 - y^2}}$, $\forall y \in (-1, 1)$, as it is particularly involved when dealing with problems (2.23)–(2.26).

The approaches for problems (2.27) and (2.29) (resp. (2.28) and (2.30)) are based on the Szulkin's critical point theory and are quite similar. That is why we shall treat in detail problem (2.27) (resp. (2.28)) and we restrict ourselves to only point out the corresponding adaptations for the treatment of problem (2.29) (resp. (2.30)).

We set $C := C[R_1, R_2]$, $L^1 := L^1(R_1, R_2)$, $L^\infty := L^\infty(R_1, R_2)$ and $W^{1,\infty} := W^{1,\infty}(R_1, R_2)$. The usual norm $\|\cdot\|_\infty$ is considered on $C$ and $L^\infty$. The space $W^{1,\infty}$ is endowed with the norm

$$\|v\| = \|v\|_\infty + \|v'\|_\infty, \quad v \in W^{1,\infty}.$$

Denoting

$$L^1_{N-1} := \{v : (R_1, R_2) \to \mathbb{R} \text{ measurable} \; : \; \int_{R_1}^{R_2} r^{N-1} |v(r)|\, dr < +\infty\},$$

each $v \in L^1_{N-1}$ can be written $v(r) = \overline{v} + \tilde{v}(r)$, with

$$\overline{v} := \frac{N}{R_2^N - R_1^N} \int_{R_1}^{R_2} v(r)\, r^{N-1}\, dr, \qquad \int_{R_1}^{R_2} \tilde{v}(r)\, r^{N-1}\, dr = 0.$$

If $v \in W^{1,\infty}$ then $\tilde{v}$ vanishes at some $r_0 \in (R_1, R_2)$ and

$$|\tilde{v}(r)| = |\tilde{v}(r) - \tilde{v}(r_0)| \le \int_{R_1}^{R_2} |v'(t)|\, dt \le (R_2 - R_1)\|v'\|_\infty,$$

so, one has that

$$||\widetilde{v}||_\infty \le (R_2 - R_1)\|v'\|_\infty. \tag{8.12}$$

Putting

$$K := \{v \in W^{1,\infty} \; : \; \|v'\|_\infty \le a\},$$

it is clear that $K$ is a convex subset of $W^{1,\infty}$.

Let $\Psi : C \to (-\infty, +\infty]$ be defined by

$$\Psi(v) = \begin{cases} \displaystyle\int_{R_1}^{R_2} r^{N-1} \Phi(v')\, dr, & \text{if } v \in K, \\ +\infty, & \text{otherwise.} \end{cases}$$

Obviously, $\Psi$ is proper and convex. On the other hand, as shown in [20] (also, see [18]), $K \subset C$ is closed and $\Psi$ is lower semicontinuous on $C$.

Next, denoting by $F : [R_1, R_2] \times I\!R \to I\!R$ the primitive of $f$, i.e.,

$$F(r, x) := \int_0^x f(r, \xi)\, d\xi, \quad (r, x) \in [R_1, R_2] \times I\!R,$$

we define $\mathcal{F}_\lambda : C \to I\!R$ by

$$\mathcal{F}_\lambda(u) = \int_{R_1}^{R_2} r^{N-1} \left[ \frac{\alpha}{p}|u|^p - F(r, u) - \frac{\lambda}{q} b(r)|u|^q \right] dr, \quad u \in C$$

and $\widehat{\mathcal{F}}_\lambda : C \to I\!R$ by

$$\widehat{\mathcal{F}}_\lambda(u) = \int_{R_1}^{R_2} r^{N-1} \left[\frac{\lambda}{m}|u|^m - F(r,u) - h(r)u\right] dr, \quad u \in C.$$

A standard reasoning (also see [72, Remark 2.7]) shows that $\mathcal{F}_\lambda$ and $\widehat{\mathcal{F}}_\lambda$ are of class $C^1$ on $C$ and

$$\langle \mathcal{F}_\lambda'(u), v\rangle = \int_{R_1}^{R_2} r^{N-1} \left[\alpha|u|^{p-2}u - f(r,u) - \lambda b(r)|u|^{q-2}u\right] v\, dr, \quad u, v \in C,$$

$$\langle \widehat{\mathcal{F}}_\lambda'(u), v\rangle = \int_{R_1}^{R_2} r^{N-1} \left[\lambda|u|^{m-2}u - f(r,u) - h(r)\right] v\, dr, \quad u, v \in C,$$

Then it is clear that $I_\lambda, \widehat{I}_\lambda : C \to (-\infty, +\infty]$ defined by

$$I_\lambda = \mathcal{F}_\lambda + \Psi, \qquad \widehat{I}_\lambda = \widehat{\mathcal{F}}_\lambda + \Psi$$

have the structure required by Szulkin's critical point theory. At this stage, the search of solutions of problem (2.27) (resp. (2.28)) reduces to finding critical points of the energy functional $I_\lambda$ (resp. $\widehat{I}_\lambda$) by the following Proposition which is proved in [18, Proposition 1].

**Proposition 12** *If $u \in C$ is a critical point of $I_\lambda$ (resp. $\widehat{I}_\lambda$), then $u$ is a solution (2.27) (resp. (2.28)).*

In the case of the periodic problems (2.29) and (2.30), taking $N = 1$, one works with

$$K_P := \{v \in W^{1,\infty} \ : \ \|v'\|_\infty \le a,\ v(R_1) = v(R_2)\}$$

instead of $K$, and $\Psi_P : C \to (-\infty, +\infty]$ given by

$$\Psi_P(v) = \begin{cases} \displaystyle\int_{R_1}^{R_2} \Phi(v'), & \text{if } v \in K_P, \\ +\infty, & \text{otherwise} \end{cases}$$

instead of $\Psi$. With $\mathcal{F}_{P,\lambda}, \widehat{\mathcal{F}}_{P,\lambda} : C \to I\!R$ defined by

$$\mathcal{F}_{P,\lambda}(u) = \int_{R_1}^{R_2} \left[\frac{\alpha}{p}|u|^p - F(r,u) - \frac{\lambda}{q} b(r)|u|^q\right] dr, \quad u \in C,$$

$$\widehat{\mathcal{F}}_{P,\lambda}(u) = \int_{R_1}^{R_2} \left[\frac{\lambda}{m}|u|^m - F(r,u) - h(r)u\right] dr, \quad u \in C,$$

the energy functionals $I_{P,\lambda}, \widehat{I}_{P,\lambda} : C \to (-\infty, +\infty]$ will be now $I_{P,\lambda} = \Psi_P + \mathcal{F}_{P,\lambda}$ and $\widehat{I}_{P,\lambda} = \Psi_P + \widehat{\mathcal{F}}_{P,\lambda}$. We have (see [18, Proposition 2]) the following

**Proposition 13** *If $u \in C$ is a critical point of $I_{P,\lambda}$ (resp. $\widehat{I}_{P,\lambda}$), then $u$ is a solution of (2.29) (resp. (2.30)).*

## 8.3 Nontrivial solutions

**The Neumann problem (2.27).** Towards the application of Proposition 11, we have to know that the energy functional satisfies the (PS) condition. In this respect, we need the following inequalities which are proved in [18, Lemma 4].

**Lemma 23** *Let $s \geq 1$ be a real number. Then*

$$|u(r)|^s \geq |\overline{u}|^s - sa(R_2 - R_1)|\overline{u}|^{s-1},\ \forall u \in K,\ \forall r \in [R_1, R_2] \tag{8.13}$$

*and there are constants $k_1,\ k_2 \geq 0$ such that*

$$|u(r)|^s \leq |\overline{u}|^s + k_1|\overline{u}|^{s-1} + k_2, \quad \forall u \in K,\ \forall r \in [R_1, R_2]. \tag{8.14}$$

The following Lemma states that under the hypothesis (AR) the functional $I_\lambda$ satisfies the (PS) condition and is anticoercive on the subpace of constant functions.

**Lemma 24** *If (2.32) holds, then $I_\lambda$ satisfies the (PS) condition and*

$$I_\lambda(c) \to -\infty \quad as \quad |c| \to \infty\,,\ c \in I\!R, \tag{8.15}$$

*for any $\lambda > 0$.*

*Proof.* We shall denote by $c_i$ a generic constant, which may depend on $\lambda$. Also, we shall invoke the positive constant

$$A = \frac{\alpha(R_2^N - R_1^N)}{pN}. \tag{8.16}$$

Let $\{u_n\} \subset K$ be a sequence for which $I_\lambda(u_n) \to c \in \mathbb{R}$ and

$$\langle \mathcal{F}_\lambda'(u_n), v - u_n\rangle + \Psi(v) - \Psi(u_n) \geq -\varepsilon_n\|v - u_n\|_\infty,\ \ \forall v \in C, \tag{8.17}$$

where $\varepsilon_n \to 0$.

We *claim* that $\{\overline{u}_n\}$ is bounded.

To see this, let $j \in (\max\{p-1, q\}, p)$ be fixed. From (8.13), (8.14) we infer

$$\int_{R_1}^{R_2} r^{N-1}\left[\frac{\alpha}{p}|u_n|^p - \frac{\lambda}{q}b(r)|u_n|^q\right] dr \geq A|\overline{u}_n|^p - c_1|\overline{u}_n|^j - c_2$$

and, since $\{I_\lambda(u_n)\}$ and $\Phi$ are bounded, it follows

$$A|\overline{u}_n|^p - c_1|\overline{u}_n|^j - \int_{R_1}^{R_2} r^{N-1}F(r, u_n) \leq c_3 \quad \text{for all } n \in I\!N. \tag{8.18}$$

Letting $v = u_n \pm 1$ in (8.17), as $\varepsilon \to 0$, we may assume that

$$-1 \leq \int_{R_1}^{R_2} r^{N-1}\left[\alpha|u_n|^{p-2}u_n - f(r, u_n) - \lambda b(r)|u_n|^{q-2}u_n\right] dr \leq 1,$$

for all $n \in I\!N$ hence, setting

$$\beta(u_n) := \int_{R_1}^{R_2} r^{N-1} \left[\alpha |u_n|^{p-2} u_n - \lambda b(r) |u_n|^{q-2} u_n\right] dr,$$

we have

$$-1 - \beta(u_n) \leq - \int_{R_1}^{R_2} r^{N-1} f(r, u_n)\, dr \leq 1 - \beta(u_n) \quad \text{for all } n \in I\!N. \tag{8.19}$$

Using (8.14) and taking into account that $j - 1 \in (\max\{p-2, q-1\}, p-1)$ we obtain the estimate

$$|\beta(u_n)| \leq pA|\overline{u}_n|^{p-1} + c_4 |\overline{u}_n|^{j-1} + c_5,$$

which, by virtue of (8.19), gives

$$\left| \int_{R_1}^{R_2} r^{N-1} f(r, u_n)\, dr \right| \leq pA|\overline{u}_n|^{p-1} + c_4 |\overline{u}_n|^{j-1} + c_6 \quad \text{for all } n \in I\!N. \tag{8.20}$$

Clearly, we have

$$\left| \frac{1}{\theta} \int_{R_1}^{R_2} r^{N-1} f(r, u_n) \overline{u}_n\, dr \right| \leq \frac{pA}{\theta} |\overline{u}_n|^p + c_7 |\overline{u}_n|^j + c_8 |\overline{u}_n| \quad \text{for all } n \in I\!N. \tag{8.21}$$

Now, suppose, by contradiction, that $\{|\overline{u}_n|\}$ is not bounded. Then, there is a subsequence of $\{|\overline{u}_n|\}$, still denoted by $\{|\overline{u}_n|\}$, with $|\overline{u}_n| \to \infty$. Let $n_0 \in \mathbb{N}$ be such that $|\overline{u}_n| \geq x_0 + a(R_2 - R_1)$ for all $n \geq n_0$. Condition (2.32) ensures that

$$\text{sign}\, \overline{u}_n = \text{sign}\, u_n(r) = \text{sign}\, f(r, u_n(r)) \quad \text{for all } r \in [R_1, R_2], \ \ n \geq n_0.$$

As $\{u_n\} \subset K$, using (8.12) and (8.20) we obtain

$$\left| \frac{1}{\theta} \int_{R_1}^{R_2} r^{N-1} f(r, u_n) \tilde{u}_n\, dr \right| \leq c_9 |\overline{u}_n|^{p-1} + c_{10} |\overline{u}_n|^{j-1} + c_{11} \quad \text{for all } n \geq n_0. \tag{8.22}$$

From (2.32) it holds

$$\begin{aligned} - & \int_{R_1}^{R_2} r^{N-1} F(r, u_n)\, dr \\ \geq & -\frac{1}{\theta} \int_{R_1}^{R_2} r^{N-1} f(r, u_n) \overline{u}_n\, dr - \frac{1}{\theta} \int_{R_1}^{R_2} r^{N-1} f(r, u_n) \widetilde{u}_n\, dr, \end{aligned}$$

for all $n \geq n_0$. Then, on account of (8.21) and (8.22), we get

$$- \int_{R_1}^{R_2} r^{N-1} F(r, u_n)\, dr \geq -\frac{pA}{\theta} |\overline{u}_n|^p - \gamma(|\overline{u}_n|) \quad \text{for all } n \geq n_0,$$

where

$$\gamma(|\overline{u}_n|) := c_9|\overline{u}_n|^{p-1} + c_7|\overline{u}_n|^j + c_{10}|\overline{u}_n|^{j-1} + c_8|\overline{u}_n| + c_{11}.$$

This together with $\theta > p$ imply

$$A|\overline{u}_n|^p - c_1|\overline{u}_n|^j - \int_{R_1}^{R_2} r^{N-1}F(r,u_n)$$

$$\geq A\frac{\theta - p}{p}|\overline{u}_n|^p - c_1|\overline{u}_n|^j - \gamma(|\overline{u}_n|) \to +\infty \quad \text{as } n \to \infty,$$

contradicting (8.18). Consequently, $\{\overline{u}_n\}$ is bounded, as claimed.

Since $\{u_n\} \subset K$, the sequence $\{u_n\}$ is bounded in $W^{1,\infty}$. By the compactness of the embedding $W^{1,\infty} \subset C$, we deduce that $\{u_n\}$ has a convergent subsequence in $C$. Therefore, $I_\lambda$ satisfies the (PS) condition.

Condition (2.32) implies (see [49]) that there exists $\gamma \in C$, $\gamma > 0$, such that

$$F(r,x) \geq \gamma(r)|x|^\theta \quad \text{for all } r \in [R_1, R_2] \text{ and } |x| \geq x_0.$$

We infer (see (8.16)):

$$\begin{aligned} I_\lambda(c) &= A|c|^p - \int_{R_1}^{R_2} r^{N-1}F(r,c)\,dr - \frac{\lambda}{q}|c|^q \int_{R_1}^{R_2} r^{N-1}b(r)\,dr \\ &\leq A|c|^p - |c|^\theta \int_{R_1}^{R_2} r^{N-1}\gamma(r) - \frac{\lambda}{q}|c|^q \int_{R_1}^{R_2} r^{N-1}b(r)\,dr\ , \end{aligned}$$

for all $c \in I\!R$ with $|c| \geq x_0$. Then, (8.15) follows from $\theta > p > q$ and $\gamma > 0$. ■

**Lemma 25** *If $\overline{b} > 0$ and either condition (2.33) or condition (2.34) holds true, then*

$$\inf_{B_\eta} I_\lambda < 0, \tag{8.23}$$

*for all $\eta$, $\lambda > 0$.*

*Proof.* Let us suppose that (2.33) holds true. A similar argument works under assumption (2.34). Condition (2.33) means that

$$\lim_{\varepsilon\to 0+} \inf_{x\in(-\varepsilon,0)} \frac{F(r,x)}{|x|^p} = h(r), \text{ uniformly in } r \in [R_1,R_2] \text{ and } h \geq 0 \text{ on } [R_1,R_2].$$

This yields the existence of some $\varepsilon_1 > 0$ so that

$$F(r,x) \geq -|x|^p \quad \text{for all } r \in [R_1,R_2],\ \ x \in (-\varepsilon_1, 0]. \tag{8.24}$$

Clearly, we may assume that $\eta < \varepsilon_1$. For $c \in (-\eta, 0) \subset (-\varepsilon_1, 0]$, using (8.24) and $\overline{b} > 0$, we estimate $I_\lambda(c)$ as follows (see (8.16)) :

$$\begin{aligned}
I_\lambda(c) &= A|c|^p - \int_{R_1}^{R_2} r^{N-1} F(r,c) - \frac{\lambda}{q}\left(\int_{R_1}^{R_2} r^{N-1} b(r)\, dr\right)|c|^q \\
&\leq A\left(1+\frac{p}{\alpha}\right)|c|^p - \frac{\lambda}{q}\left(\int_{R_1}^{R_2} r^{N-1} b(r)\, dr\right)|c|^q \\
&= |c|^q\left[A\left(1+\frac{p}{\alpha}\right)|c|^{p-q} - \frac{\lambda}{q}\left(\int_{R_1}^{R_2} r^{N-1} b(r)\, dr\right)\right] < 0,
\end{aligned}$$

provided that $|c| > 0$ is small enough. Obviously, this implies (8.23) and the proof is complete. ■

**Lemma 26** *If $\overline{b} > 0$ and (2.31) holds true, then there exist $\rho$, $\lambda_0 > 0$ such that*

$$\inf_{\partial B_\rho} I_\lambda > 0, \tag{8.25}$$

*for all $\lambda \in (0, \lambda_0)$.*

*Proof.* Assumption (2.31) ensures that there are constants $\varepsilon$, $\rho > 0$ such that

$$F(r,x) \leq \frac{\alpha - \varepsilon}{p}|x|^p \quad \text{for all} r \in [R_1, R_2] \text{ and } |x| \leq \rho. \tag{8.26}$$

We know (see the proof of Lemma 7 in [18]) that

$$\beta_0 := \inf_{u \in K \cap \partial B_\rho} \int_{R_1}^{R_2} r^{N-1} |u|^p\, dr > 0.$$

Also, from $\overline{b} > 0$ it follows

$$\beta_1 := \int_{R_1}^{R_2} r^{N-1} b^+(r)\, dr > 0.$$

We set

$$\lambda_0 := \frac{\varepsilon p^{-1} \beta_0}{\rho^q q^{-1} \beta_1} \ \ (> 0).$$

Using (8.26), for arbitrary $\lambda \in (0, \lambda_0)$ and $u \in K \cap \partial B_\rho$ one obtains

$$\begin{aligned}
I_\lambda(u) &\geq \frac{\alpha}{p}\int_{R_1}^{R_2} r^{N-1}|u|^p dr - \int_{R_1}^{R_2} r^{N-1} F(r,u) dr - \frac{\lambda}{q}\int_{R_1}^{R_2} r^{N-1} b^+(r)|u|^q dr \\
&\geq \frac{\varepsilon}{p}\int_{R_1}^{R_2} r^{N-1}|u|^p\, dr - \lambda\frac{\rho^q}{q}\int_{R_1}^{R_2} r^{N-1} b^+(r)\, dr \\
&\geq \frac{\varepsilon}{p}\beta_0 - \lambda\frac{\rho^q}{q}\beta_1 = \frac{\rho^q}{q}\beta_1(\lambda_0 - \lambda) =: c_\lambda > 0\,.
\end{aligned}$$

Then (8.25) follows from

$$\inf_{\partial B_\rho} I_\lambda = \inf_{u\in K\cap\partial B_\rho} I_\lambda(u) \geq c_\lambda.$$

■

**Theorem 31** *Assume (2.32), (2.31) and that* $\overline{b} > 0$. *If either (2.33) or (2.34) holds true, then there exists* $\lambda_0 > 0$ *such that problem (2.27) has at least two nontrivial solutions for any* $\lambda \in (0, \lambda_0)$.

*Proof.* It is clear that $I_\lambda$ is bounded from below on bounded subsets of $C$. Then, the conclusion follows from Proposition 11, Lemmas 24, 25, 26 and Proposition 12. ■

**Remark 22** On account of Remark 21 (i) it is easy to see that under the hypotheses of Theorem 31, if, in addition, $\Phi$ is even and $f(r, \cdot)$ is odd for all $r \in [R_1, R_2]$, then (2.27) has at least four nontrivial solutions for any $\lambda \in (0, \lambda_0)$.

**Corollary 14** *Assume (2.32) and that* $\overline{b} > 0$. *If*

$$0 \leq \lim_{x\to 0} \frac{F(r,x)}{|x|^p} < \frac{\alpha}{p} \quad \textit{uniformly in } r \in [R_1, R_2], \tag{8.27}$$

*then there exists* $\lambda_0 > 0$ *such that problem (2.23) has at least two nontrivial solutions for any* $\lambda \in (0, \lambda_0)$. *If, in addition,* $f(r, \cdot)$ *is odd for all* $r \in [R_1, R_2]$, *then (2.23) has at least four nontrivial radial solutions for any* $\lambda \in (0, \lambda_0)$.

**Example 12** If $\alpha > 0$, $\theta > p > q \geq 2$ are constants and $\gamma, b \in C$, $\gamma > 0$, $\overline{b} > 0$, then there exists $\lambda_0 > 0$ such that the Neumannproblem

$$-\mathrm{div}\left(\frac{\nabla v}{\sqrt{1-|\nabla v|^2}}\right) + \alpha|v|^{p-2}v = \gamma(|x|)|v|^{\theta-2}v + \lambda b(|x|)|v|^{q-2}v \quad \text{in } \mathcal{A},$$
$$\frac{\partial v}{\partial \nu} = 0 \quad \text{on} \quad \partial\mathcal{A}$$

has at least four nontrivial radial solutions for any $\lambda \in (0, \lambda_0)$.

**The periodic problem (2.29).** It is easy to check that Lemma 24 remains valid with $I_{P,\lambda}$ instead of $I_\lambda$. Also, if condition "$\overline{b} > 0$" is replaced by

$$\int_{R_1}^{R_2} b(r)\, dr > 0 \tag{8.28}$$

then Lemmas 25 and 26 remain true with $I_{P,\lambda}$ instead of $I_\lambda$. Thus, we obtain the following.

**Theorem 32** *Assume (2.32), (2.31) and (8.28). If either (2.33) or (2.34) holds true, then there exists* $\lambda_0 > 0$ *such that problem (2.29) has at least two nontrivial solutions for any* $\lambda \in (0, \lambda_0)$.

**Corollary 15** *Assume (2.32) and (8.28). If (8.27) holds true, then there exists* $\lambda_0 > 0$ *such that problem (2.25) has at least two nontrivial solutions for any* $\lambda \in (0, \lambda_0)$. *If, in addition,* $f(r, \cdot)$ *is odd for all* $r \in [R_1, R_2]$, *then (2.25) has at least four nontrivial solutions for any* $\lambda \in (0, \lambda_0)$.

**Example 13** Let $\alpha > 0$, $\theta > p > q \geq 2$ be constants and $\gamma, b \in C$, $\gamma > 0$ and $b$ satisfying (8.28). Then there exists $\lambda_0 > 0$ such that the periodic problem

$$-\left(\frac{u'}{\sqrt{1-|u'|^2}}\right)' + \alpha|u|^{p-2}v = \gamma(r)|u|^{\theta-2}u + \lambda b(r)|u|^{q-2}v \quad \text{in } [R_1, R_2],$$
$$u(R_1) - u(R_2) = 0 = u'(R_1) - u'(R_2)$$

has at least four nontrivial solutions for any $\lambda \in (0, \lambda_0)$.

## 8.4 Multiple solutions

**The Neumann problem (2.28).** The following existence result, inspired from [108, 83] provides a useful tool in obtaining multiple solutions.

**Lemma 27** *We assume* $\overline{h} = 0$ *and there exists* $k_1, k_2 > 0$ *and* $0 < \sigma < m$ *such that*

$$-l(r) \leq F(r, x) \leq k_1|x|^\sigma + k_2, \quad \text{for all} \quad (r, x) \in [R_1, R_2] \times \mathbb{R}_+, \tag{8.29}$$

*with some* $l \in L^1_{N-1}$, $l \geq 0$, *together with either*

$$\lim_{x \to +\infty} \int_{R_1}^{R_2} r^{N-1} F(r, x) dr = +\infty, \tag{8.30}$$

*or the limit* $F_+(r) = \lim_{x \to +\infty} F(r, x)$ *exists for all* $r \in [R_1, R_2]$ *and*

$$F(r, x) < F_+(r), \quad \forall r \in [R_1, R_2],\ x \geq 0. \tag{8.31}$$

*Then there exists* $\lambda_+ > 0$ *such that problem (2.28) has at least one solution* $u_\lambda > 0$ *for any* $0 < \lambda < \lambda_+$ *which minimize* $\widehat{I}_\lambda$ *on* $C^+ = \{v \in C : v \geq 0\}$. *Moreover,* $u_\lambda$ *is a local minimum for* $\widehat{I}_\lambda$.

*Proof.* First, notice that from (8.12) it holds

$$||\widetilde{u}||_\infty \leq a(R_2 - R_1) \quad \text{for all} \quad u \in K. \tag{8.32}$$

This implies that

$$\overline{u} - a(R_2 - R_1) \leq u(r) \leq \overline{u} + a(R_2 - R_1) \quad \text{for all} \quad u \in K, \tag{8.33}$$

hence

$$\overline{u} \to +\infty \quad \text{as} \quad ||u||_\infty \to \infty,\ u \in C^+ \cap K. \tag{8.34}$$

Also, it is clear that

$$|u(r)| \leq |\overline{u}| + a(R_2 - R_1) \quad \text{for all} \quad u \in K,\ r \in [R_1, R_2]. \tag{8.35}$$

From (8.29) it follows that

$$\widehat{I}_\lambda(u) \geq \int_{R_1}^{R_2} r^{N-1} \left[ \frac{\lambda}{m}|u|^m - k_1|u|^\sigma - k_2 - ||h||_\infty |u| \right] dr,$$

for all $u \in C^+$. Hence, using (8.13), (8.35), (8.34) and $\sigma < m$, we deduce immediately that

$$\widehat{I}_\lambda(u) \to +\infty \quad \text{whenever} \quad ||u||_\infty \to \infty,\ u \in C^+, \tag{8.36}$$

that is $\widehat{I}_\lambda$ is coercive on $C^+$. This immediately implies that $\widehat{I}_\lambda$ is bounded from below on $C^+$. Now, let $\{u_n\} \subset C^+ \cap K$ be a minimizing sequence, $\widehat{I}_\lambda(u_n) \to \inf_{C^+} \widehat{I}_\lambda$ as $n \to \infty$. Then, from (8.36) it follows that $\{u_n\}$ is bounded in $C$, and using that $\{u_n\} \subset K$, we infer that $\{u_n\}$ is bounded in $W^{1,\infty}$. But $W^{1,\infty}$ is compactly embedded in $C$, hence $\{u_n\}$ has a convergent subsequence in $C$ to some $u_\lambda \in C^+ \cap K$. By the lower semicontinuity of $\widehat{I}_\lambda$ it follows

$$\widehat{I}_\lambda(u_\lambda) = \inf_{C^+} \widehat{I}_\lambda.$$

We *claim* that

$$\overline{u}_\lambda \to +\infty \quad \text{as} \quad \lambda \to 0. \tag{8.37}$$

Assuming this for the moment, it follows from (8.33) and (8.37) that there exists $\lambda_+ > 0$ such that $u_\lambda > 0$ for any $0 < \lambda < \lambda_+$, implying that $u_\lambda$ is a local minimum for $\widehat{I}_\lambda$. Consequently, from Proposition 1.1 in [112], $u_\lambda$ is a critical point of $\widehat{I}_\lambda$, and hence a solution of (2.28) (by Proposition 12) for any $0 < \lambda < \lambda_+$.

In order to prove the claim, assume first that (8.30) holds true. Then, consider $M > 0$ and $x_M > 0$ such that

$$\int_{R_1}^{R_2} r^{N-1} F(r, x_M) dr > 2M. \tag{8.38}$$

On the other hand, as $\overline{h} = 0$, one has that for all $\lambda > 0$,

$$\widehat{I}_\lambda(x) = \frac{\lambda(R_2^N - R_1^N)}{Nm}|x|^m - \int_{R_1}^{R_2} r^{N-1} F(r, x) dr \quad (x \in \mathbb{R}). \tag{8.39}$$

So, choosing $\lambda_M > 0$ such that

$$\frac{\lambda_M(R_2^N - R_1^N)}{Nm} x_M^m < M,$$

and using (8.38), (8.39), it follows that

$$\widehat{I}_\lambda(x_M) < -M \quad \text{for all} \quad 0 < \lambda < \lambda_M.$$

Consequently, one has that

$$\inf_{C^+} \widehat{I}_\lambda \to -\infty \quad \text{as} \quad \lambda \to 0,$$

which, together with (8.33) imply (8.37), as claimed.

Now, let (8.31) holds true, and assume also by contradiction that there exists $\lambda_n \to 0$ such that $\{\overline{u}_{\lambda_n}\}$ is bounded. On account of (8.33) and of the compactness of the embedding in $W^{1,\infty} \subset C$, one can assume, passing if necessary to a subsequence, that $\{u_{\lambda_n}\}$ is convergent in $C$ to some $u \in C^+$. Using (8.31) and Fatou's lemma it follows that

$$\int_{R_1}^{R_2} r^{N-1} F(r,u) dr < \int_{R_1}^{R_2} r^{-1} F_+(r) dr \leq \liminf_{s\to\infty} \int_{R_1}^{R_2} r^{N-1} F(r, s + \widetilde{u}) dr,$$

which imply that there exists $s_0 > 0$ sufficiently large, with $s_0 + \widetilde{v} \in C^+$ for all $v \in K$, and $\rho > 0$ such that

$$\int_{R_1}^{R_2} r^{N-1} [F(r,u) - F(r, s_0 + \widetilde{u})] dr < -\rho.$$

So, for $n$ sufficiently large, we have

$$\int_{R_1}^{R_2} r^{N-1} [F(r, u_{\lambda_n}) - F(r, s_0 + \widetilde{u}_{\lambda_n})] dr < -\rho. \tag{8.40}$$

On the other hand, using (8.32) and (8.33) it follows

$$\int_{R_1}^{R_2} r^{-1} \frac{\lambda_n}{m} [|s_0 + \widetilde{u}_{\lambda_n}|^m - |u_{\lambda_n}|^m] dr \to 0 \quad \text{as} \quad n \to \infty. \tag{8.41}$$

Notice that, as $\overline{h} = 0$, for all $\lambda > 0$ and $s \in \mathbb{R}$, one has

$$\begin{aligned}\widehat{I}_\lambda(s + \widetilde{u}_\lambda) = &\int_{R_1}^{R_2} r^{N-1} \Phi(u'_\lambda) dr + \int_{R_1}^{R_2} r^{N-1} \frac{\lambda}{m} |s + \widetilde{u}_\lambda|^m dr \\ &- \int_{R_1}^{R_2} r^{N-1} F(r, s + \widetilde{u}_\lambda) dr - \int_{R_1}^{R_2} r^{N-1} h(r) \widetilde{u}_\lambda dr.\end{aligned}$$

Then, by (8.40) and (8.41) we obtain

$$\widehat{I}_{\lambda_n}(s_0 + \widetilde{u}_{\lambda_n}) < \widehat{I}_{\lambda_n}(u_{\lambda_n}),$$

for $n$ sufficiently large, contradicting the definition of $u_{\lambda_n}$. This proves the claim and the proof is complete. ■

**Theorem 33** *Assume that conditions $\overline{h} = 0$, (2.35) and either (2.36) or (2.37) hold true. Then there exists $\lambda_0 > 0$ such that problem (2.28) has at least three solutions for any $\lambda \in (0, \lambda_0)$.*

*Proof.* From Lemma 27, it follows that there exists $\lambda_+ > 0$ such that $\widehat{I}_\lambda$ has a local minimum at some $u_{\lambda,1} > 0$ for any $0 < \lambda < \lambda_+$. Using exactly the same strategy, we can find $\lambda_- > 0$ such that $\widehat{I}_\lambda$ has a local minimum at some $u_{\lambda,2} < 0$ for any $0 < \lambda < \lambda_-$. Taking $\lambda_0 = \min\{\lambda_-, \lambda_+\}$ it follows that $\widehat{I}_\lambda$ has two local minima for any $\lambda \in (0, \lambda_0)$. On the other hand, from the proof of Lemma 27, it is easy to see that $\widehat{I}_\lambda$ is coercive on $C$, implying that $\widehat{I}_\lambda$ satisfies the (PS) condition for any $\lambda > 0$. Hence, from Corollary 3.3 in [112] we infer that $\widehat{I}_\lambda$ has at least three critical points for all $\lambda \in (0, \lambda_0)$ which are solutions of (2.28) (Proposition 12). ■

**Corollary 16** *Under the assumptions of Theorem 33, there exists $\lambda_0 > 0$ such that problem (2.24) has at least three radial solutions for any $\lambda \in (0, \lambda_0)$.*

**Remark 23** (i) When $f$ is bounded, it is well known [1] that the Ahmad-Lazer-Paul condition (2.36) generalizes the Landesman-Lazer condition

$$\int_{R_1}^{R_2} r^{N-1} f^-(r)dr < 0 < \int_{R_1}^{R_2} r^{N-1} f_+(r)dr,$$

where $f^-(r) = \limsup_{x\to-\infty} f(r,x)$ and $f_+(r) = \liminf_{x\to+\infty} f(r,x)$.

(ii) Condition (2.37) holds true whenever one has the sign condition

$$xf(r,x) > 0 \quad \text{for all} \quad r \in [R_1, R_2] \quad \text{and} \quad x \neq 0.$$

(iii) The condition :

*there exists* $0 < \theta < m$ *such that*

$$xf(r,x) - \theta F(r,x) \to -\infty \quad \text{as} \quad |x| \to \infty, \text{ uniformly in } r \in [R_1, R_2],$$

introduced in [109, 84], together with the sign condition

$$xf(r,x) > 0 \text{ for all } r \in [R_1, R_2] \text{ and } |x| \geq x_0$$

for some $x_0 > 0$, imply (2.35) and (2.36).

**Example 14** Let $m \in \mathbb{N}$ be even and $h \in C$ be with $\overline{h} = 0$. Then, using Corollary 16 and Remark 23 (iii), it follows that there exists $\lambda_0 > 0$ such that the Neumann problem

$$-\mathrm{div}\left(\frac{\nabla v}{\sqrt{1-|\nabla v|^2}}\right) + \lambda |v|^{m-2}v = \frac{v^{m-1}}{1+v^m} + h(|x|) \quad \text{in } \mathcal{A},$$
$$\frac{\partial v}{\partial \nu} = 0 \quad \text{on} \quad \partial\mathcal{A},$$

has at least three radial solutions for all $\lambda \in (0, \lambda_0)$.

**The periodic problem (2.30).** Using exactly the same strategy as above, but with $\widehat{I}_{P,\lambda}$ instead of $\widehat{I}_{\lambda}$, we have the following

**Theorem 34** *Assume that conditions*

$$\int_{R1}^{R_2} h(r)\,dr = 0, \tag{8.42}$$

*(2.35), and either (2.36) or (2.37) hold true for* $N = 1$. *Then there exists* $\lambda_0 > 0$ *such that problem (2.30) has at least three solutions for any* $\lambda \in (0, \lambda_0)$.

**Corollary 17** *Under the assumptions of Theorem 34, there exists* $\lambda_0 > 0$ *such that problem (2.26) has at least three solutions for any* $\lambda \in (0, \lambda_0)$.

**Example 15** Let $m \in \mathbb{N}$ be even and $h \in C$ satisfying (8.42). Then there exists $\lambda_0 > 0$ such that the periodic problem

$$-\left(\frac{u'}{\sqrt{1-|u'|^2}}\right)' + \lambda|u|^{m-2}u = \frac{u^{m-1}}{1+u^m} + h(r),$$
$$u(R_1) - u(R_2) = 0 = u'(R_1) - u'(R_2)$$

has at least three solutions for all $\lambda \in (0, \lambda_0)$.

# Chapter 9

# Multiple critical orbits

## 9.1 A nonsmooth variational approach

Consider the periodic boundary value problem (2.41) under the hypotheses $(H_\phi)$, $(H_F)$ and $(H_h)$. The following variational setting is taken from [18] when $N = 1$ and [97] in the general case.

We set $C = C([0,T], \mathbb{R}^N)$ and $W^{1,\infty} = W^{1,\infty}([0,T], \mathbb{R}^N)$. The usual norm $||\cdot||_\infty$ is considered on $C$ and $L^\infty$. Setting

$$\widetilde{C} := \{u \in C : \int_0^T u(t)dt = 0\},$$

we can split

$$C = \mathbb{R}^N \oplus \widetilde{C}$$

and each $v \in C$ can be uniquely written as

$$u = \overline{u} + \widetilde{u}, \quad \text{ with } \overline{u} \in \mathbb{R}^N,\ \widetilde{u} \in \widetilde{C}.$$

Also, note that setting

$$G_p := \left\{ \sum_{k=1}^{N} k_i \omega_i e_i \ : \ k_i \in \mathbb{Z},\ 1 \le i \le N \right\},$$

one has that $G_p \simeq \mathbb{Z}^N$ and span $G_p = \mathbb{R}^N$. Putting

$$\widehat{K} = \{v \in W^{1,\infty} : \ ||v'||_\infty \le a,\ v(0) = v(T)\},$$

we have that $\widehat{K}$ is a convex and closed set in $C$.

Let $\Psi_p : C \to (-\infty, +\infty]$ be defined by

$$\Psi_p(v) = \int_0^T \Phi(v') \text{ if } v \in \widehat{K}, \qquad \Psi_p(v) = +\infty \text{ if } v \in C \setminus \widehat{K}.$$

and $\mathcal{G}_p : C \to \mathbb{R}$ be defined by

$$\mathcal{G}_p(v) = \int_0^T F(t,u)dt + \int_0^T h(t)u \qquad (u \in C).$$

The following hold true.

$(p_1)$ $\mathcal{G}_p \in C^1(C,\mathbb{R})$ and $\mathcal{G}_p'$ takes bounded sets into bounded sets; $\Psi_p$ is convex, lower semicontinuous and $D(\Psi_p) = \{u \in C : \ \Psi_p(u) < +\infty\} = \widehat{K}$ is a closed set in $C$. Note also that

$$\mathcal{G}_p(u+g) = \mathcal{G}_p(u) \ \text{ and } \ \Psi_p(u+g) = \Psi_p(u) \quad \forall u \in C, \ g \in G_p.$$

$(p_2)$ One has that $\Psi_p(0) = 0$ and

$$\Psi_p(u) = \Psi_p(\widetilde{u}) \quad \text{for all} \quad u \in C.$$

$(p_3)$ There exists $\rho > 0$ such that

$$||\widetilde{u}||_\infty \leq \rho, \quad |\Psi(u)| \leq \rho \quad \text{for all} \quad u \in \widehat{K}.$$

$(p_4)$ Any sequence $\{u_n\} \subset \widehat{K}$ with $\{\overline{u}_n\}$ bounded, has a convergent subsequence.

With $\Psi_p$ and $\mathcal{G}_p$ as above, we define $I_p := \Psi_p + \mathcal{G}_p$.

**Proposition 14** *If $u \in C$ is a critical point of $I_p$, i.e.,*

$$\langle \mathcal{G}_p'(u), v-u\rangle + \Psi_p(v) - \Psi_p(u) \geq 0, \qquad \forall v \in C,$$

*then $u$ is a solution of problem (2.41).*

## 9.2 Notations and hypotheses

The space $\mathbb{R}^N$ $(N \geq 1)$ will be endowed with the norm

$$|u| = \max_{i=1}^{N} |u_i| \quad \text{for all} \quad u = (u_1, ..., u_N) \in \mathbb{R}^N.$$

Let $(X, ||\cdot||_X)$ be a real Banach space with the dual denoted by $X^*$ and $G$ be a discrete subgroup of $X$. We denote by $\pi : X \to X/G$ the canonical projection. The following definitions are taken from [101] and are classical. A set $A \subset X$ is said to be *$G$-invariant* if

$$A = \pi^{-1}(\pi(A)).$$

Notice that a set $A$ is $G$-invariant if and only if $u + g \in A$ for all $u \in A$ and $g \in G$. If $M$ is an arbitrary set and $f : X \to M$ is a function, then $f$ is called *$G$-invariant* if

$$f(u+g) = f(u) \quad \text{for all} \quad u \in X, g \in G.$$

For any $G$-invariant functional $\mathcal{G} \in C^1(X, \mathbb{R})$, one has that $\mathcal{G}' : X \to X^*$ is $G$-invariant. In what follows we assume that

$$\dim\, (\text{span } G) = N.$$

Then, we have

$$G \simeq \mathbb{Z}^N, \qquad X \simeq \mathbb{R}^N \oplus Y,$$

where $Y$ is a closed subspace of $X$. So, any $u \in X$ can be uniquely decomposed as

$$u = \overline{u} + \widetilde{u}, \quad \text{ with } \overline{u} \in \mathbb{R}^N,\ \widetilde{u} \in Y,$$

and the mappings $u \mapsto \overline{u}$, $u \mapsto \widetilde{u}$ are bounded linear projections. We will consider on $X$ the equivalent norm

$$||u|| = |\overline{u}| + ||\widetilde{u}||_X \quad (u \in X).$$

In the sequel we assume the following hypotheses.

$(H_1)$ The functional $\mathcal{G} \in C^1(X, \mathbb{R})$ is $G$-invariant and $\mathcal{G}'$ takes bounded sets into bounded sets. One the other hand, $\Psi : X \to (-\infty, +\infty]$ is $G$-invariant, convex, lower semicontinuous and $D(\Psi) = \{u \in X :\ \Psi(u) < +\infty\}$ is a closed nonempty set.

$(H_2)$ One has that $\Psi(0) = 0$ and

$$\Psi(u) = \Psi(\widetilde{u}) \quad \text{for all} \quad u \in X.$$

$(H_3)$ There exists $\rho > 0$ such that

$$||\widetilde{u}|| \leq \rho, \quad |\Psi(u)| \leq \rho \quad \text{for all} \quad u \in D(\Psi).$$

$(H_4)$ Any sequence $\{u_n\} \subset D(\Psi)$ with $\{\overline{u}_n\}$ bounded, has a convergent subsequence.

Note that from $(H_2)$ it follows that $\Psi$ is $G$-invariant and

$$\Psi(\overline{u}) = 0 \quad \text{for all} \quad \overline{u} \in \mathbb{R}^N.$$

With $\Psi$ and $\mathcal{G}$ as above, we shall consider the functional

$$I = \Psi + \mathcal{G}. \tag{9.1}$$

According to Szulkin [112], a point $u \in X$ is said to be *a critical point* of $I$ if $u \in D(\Psi)$ and it holds

$$\langle \mathcal{G}'(u), v - u \rangle + \Psi(v) - \Psi(u) \geq 0 \quad \text{for all} \quad v \in X. \tag{9.2}$$

For any $c \in \mathbb{R}$, we shall use the notations:

$$K = \{u \in X :\ u \text{ is a critical point}\}, \quad K_c = \{u \in K :\ I(u) = c\}.$$

Since $\mathcal{G}'$ and $\Psi$ are $G$-invariant, it follows immediately that if $u \in K$, then $\pi^{-1}(\pi(u)) \subset K$. In this case the set $\pi^{-1}(\pi(u))$ is called *a critical orbit* of $I$. Moreover, using that $I$ is $G$-invariant, it follows that if $u \in K_c$, then $\pi^{-1}(\pi(u)) \subset K_c$.

If $\mathcal{N}$ is an open neighborhood of $K_c$ and $\epsilon > 0$, we denote

$$\mathcal{N}_\epsilon = \{u \in X \setminus \mathcal{N} : \ |\overline{u}| \le 2, \ I(u) \le c + \epsilon\}.$$

If $K_c = \emptyset$, then we will consider $\mathcal{N} = \emptyset$. Notice that $\mathcal{N}_\epsilon$ is a compact set. Indeed, using that $I$ is lower semicontinuous, it follows that $\mathcal{N}_\epsilon$ is closed. If $\{u_n\}$ is a sequence in $\mathcal{N}_\epsilon$, then $\{u_n\} \subset D(\Psi)$ and $\{\overline{u}_n\}$ is bounded. Hence from $(H_4)$ it follows that $\{u_n\}$ has a convergent subsequence. So, if $\mathcal{N}_\epsilon \neq \emptyset$, we can define

$$\alpha = \max_{u \in \mathcal{N}_\epsilon} |\langle \mathcal{G}'(u), \overline{u} \rangle|. \tag{9.3}$$

## 9.3 Some auxiliary results

Below, all the neighborhoods will be assumed to be open.

**Lemma 28** *Let $c \in \mathbb{R}$ and $\mathcal{N}$ be a $G$-invariant neighborhood of $K_c$. Then, for each $\overline{\epsilon} > 0$, there exists $\epsilon \in (0, \overline{\epsilon}]$ such that for any $u_0 \in X \setminus \mathcal{N}$ with $c - \epsilon \le I(u_0) \le c + \epsilon$, there exists $v_0 \in X$ satisfying*

$$\langle \mathcal{G}'(u_0), v_0 - u_0 \rangle + \Psi(v_0) - \Psi(u_0) < -3\epsilon.$$

*Proof.* By contradiction, assume that for any positive integer $n$ there exists $u_n \in X \setminus \mathcal{N}$ with

$$c - 1/n \le I(u_n) \le c + 1/n,$$

and

$$\langle \mathcal{G}'(u_n), v - u_n \rangle + \Psi(v) - \Psi(u_n) \ge -3/n, \quad \forall v \in X. \tag{9.4}$$

Clearly, one has that $\{u_n\} \subset D(\Psi)$. On the other hand, using that $\mathcal{G}'$, $\Psi$ and $\mathcal{N}$ are $G$-invariant, we may assume that $\{\overline{u}_n\} \subset [0, 1)^N$. So, using $(H_4)$, passing if necessary to a subsequence, it follows that $\{u_n\}$ converges to some $u \in D(\Psi)$. We deduce that

$$\mathcal{G}(u_n) \to \mathcal{G}(u) \quad \text{and} \quad \Psi(u_n) \to c - \mathcal{G}(u).$$

As $\Psi$ is lower semicontinuous, it follows that

$$c - \mathcal{G}(u) = \liminf_{n \to \infty} \Psi(u_n) \ge \Psi(u).$$

On the other hand, taking in (9.4) $v = u$ we obtain

$$\limsup_{n \to \infty} \Psi(u_n) \le \Psi(u).$$

Hence,

$$\Psi(u_n) \to \Psi(u)$$

and using (9.4), we infer that $u \in K$. But, $I(u_n) \to I(u)$ and $I(u_n) \to c$, hence $I(u) = c$ and $u \in K_c$. This is in contradiction with $u_n \to u$, $\{u_n\} \subset X \setminus \mathcal{N}$ and $\mathcal{N}$ is a neighborhood of $K_c$. ∎

**Lemma 29** *Let $c \in \mathbb{R}$ and $\mathcal{N}$ be a $G$-invariant neighborhood of $K_c$. Then, for each $\overline{\epsilon} > 0$, there exists $\epsilon \in (0, \overline{\epsilon}]$ such that for any $u_0 \in X \setminus \mathcal{N}$ with $I(u_0) \leq c + \epsilon$, there are $\epsilon_0 \in (0, \epsilon]$, $v_0 \in X$ and $U_0$ a neighborhood of $u_0$, satisfying*

$$\langle \mathcal{G}'(u), v_0 - u \rangle + \Psi(v_0) - \Psi(u) \leq 1, \quad \forall u \in U_0, \tag{9.5}$$

*and*

$$\langle \mathcal{G}'(u), v_0 - u \rangle + \Psi(v_0) - \Psi(u) \leq -2\epsilon_0, \quad \forall u \in U_0 \text{ with } I(u) \geq c - \epsilon. \tag{9.6}$$

*Proof.* Let $\overline{\epsilon} > 0$ and the corresponding $\epsilon \in (0, \overline{\epsilon}]$ be given in Lemma 28. We have to consider the following three cases.

*Case 1*: $\underline{u_0 \in K.}$ In this case, we shall prove the assertions with $v_0 = u_0$. We have

$$\langle \mathcal{G}'(u_0), u - u_0 \rangle + \Psi(u) - \Psi(u_0) \geq 0 \quad \text{for all} \quad u \in X.$$

Then, from the continuity of $\mathcal{G}'$, we infer that

$$\begin{aligned} \langle \mathcal{G}'(u), u_0 - u \rangle + \Psi(u_0) - \Psi(u) &\leq \langle \mathcal{G}'(u) - \mathcal{G}'(u_0), u_0 - u \rangle \\ &\leq ||\mathcal{G}'(u) - \mathcal{G}'(u_0)||\, ||u - u_0|| \\ &\leq 1, \end{aligned}$$

for all $u \in U_1$, where $U_1$ is a sufficiently small neighborhood of $u_0$. On the other hand, using Lemma 28, it follows

$$[u \in K,\ c - \epsilon \leq I(u) \leq c + \epsilon] \Rightarrow u \in \mathcal{N},$$

which ensures that

$$I(u_0) < c - \epsilon.$$

Next, we prove that there exists $U_2$ a neighborhood of $u_0$ and $\epsilon_0 \in (0, \epsilon]$ such that

$$\Psi(u) - \Psi(u_0) > 3\epsilon_0, \qquad \forall u \in U_2,\ I(u) \geq c - \epsilon.$$

Assume by contradiction that there exists a sequence $\{u_n\}$ converging to $u_0$, with

$$I(u_n) \geq c - \epsilon, \qquad \Psi(u_n) - \Psi(u_0) \leq 1/n$$

for all $n \geq 1$. This, together with the lower semicontinuity of $\Psi$ imply that $\Psi(u_n) \to \Psi(u_0)$, hence $I(u_n) \to I(u_0)$. But $I(u_0) < c - \epsilon$ and $I(u_n) \geq c - \epsilon$, which give a contradiction. Note that, as $\mathcal{G}'$ takes bounded sets into bounded sets, we may assume

$$||\mathcal{G}'(u)||\ ||u-u_0|| \le \epsilon_0, \qquad \forall u \in U_2.$$

It follows that

$$\langle \mathcal{G}'(u), u_0-u\rangle + \Psi(u_0) - \Psi(u) \le ||\mathcal{G}'(u)||\ ||u-u_0|| - 3\epsilon_0 \le -2\epsilon_0,$$

for all $u \in U_2$ with $I(u) \ge c-\epsilon$. So, in this case we take $U_0 = U_1 \cap U_2$.

*Case 2*: $\underline{u_0 \notin K,\ I(u_0) < c-\epsilon.}$ Let $v_0 \in D(\Psi)$ be with the property

$$\langle \mathcal{G}'(u_0), v_0-u_0\rangle + \Psi(v_0) - \Psi(u_0) < 0. \tag{9.7}$$

We may assume that $v_0$ is arbitrarily close to $u_0$. Indeed, consider $t \in (0,1)$ and $w_0 = tv_0 + (1-t)u_0$. Then, from (9.7) and the convexity of $\Psi$, it follows that

$$\langle \mathcal{G}'(u_0), w_0-u_0\rangle + \Psi(w_0) - \Psi(u_0) < 0.$$

The assertion follows by taking $t \to 0_+$.

First, we deal with (9.6). Using that $I(u_0) < c-\epsilon$ and arguing exactly as in the previous case, there exists $U_3$ a neighborhood of $u_0$ and $\epsilon_0 \in (0,\epsilon]$ such that

$$\Psi(u) - \Psi(u_0) > 4\epsilon_0, \qquad \forall u \in U_3,\ I(u) \ge c-\epsilon.$$

Using that $\mathcal{G}'$ takes bounded sets into bounded sets, it follows that there exists $M_0 > \epsilon_0/2$ with

$$||\mathcal{G}'(u)|| < M_0, \qquad \forall u \in X,\ ||u-u_0|| \le 1.$$

Now, let us consider $v_0$ satisfying (9.7) and $||v_0-u_0|| < \epsilon_0/(2M_0)$. From the choice of $M_0$, it follows

$$||\mathcal{G}'(u)||\ ||v_0-u_0|| \le \frac{\epsilon_0}{2} \quad \text{and} \quad ||\mathcal{G}'(u)||\ ||u-u_0|| \le \frac{\epsilon_0}{2},$$

for all $u \in X$ with $||u-u_0|| \le \epsilon_0/(2M_0)$. Set $U_4 = U_3 \cap B(u_0, \epsilon_0/(2M_0))$. One has the following estimates:

$$\begin{aligned} \Psi(v_0) - \Psi(u) &= (\Psi(v_0) - \Psi(u_0)) + (\Psi(u_0) - \Psi(u)) \\ &< \langle \mathcal{G}'(u_0), u_0-v_0\rangle + (\Psi(u_0) - \Psi(u)) \\ &\le ||\mathcal{G}'(u_0)||\ ||v_0-u_0|| + (\Psi(u_0) - \Psi(u)) \\ &\le \epsilon_0/2 - 4\epsilon_0 < -3\epsilon_0, \end{aligned}$$

for all $u \in U_4$ with $I(u) \ge c-\epsilon$. We infer

$$\begin{aligned} \langle \mathcal{G}'(u), v_0-u\rangle + \Psi(v_0) - \Psi(u) &\le ||\mathcal{G}'(u)||\ (||v_0-u_0|| + ||u_0-u||) - 3\epsilon_0 \\ &\le \epsilon_0/2 + \epsilon_0/2 - 3\epsilon_0 = -2\epsilon_0, \end{aligned}$$

for all $u \in U_4$, $I(u) \geq c - \epsilon$, and (9.6) is proved.

Next, we have in view (9.5). Let $\delta_0 > 0$ be such that

$$\langle \mathcal{G}'(u_0), v_0 - u_0 \rangle + \Psi(v_0) - \Psi(u_0) = -2\delta_0.$$

Using the continuity of $\mathcal{G}'$, it follows that there exists a neighborhood of $u_0$ denoted by $U_5$ such that

$$||\mathcal{G}'(u) - \mathcal{G}'(u_0)||\ ||v_0 - u|| < \delta_0/4 \quad \text{and} \quad ||\mathcal{G}'(u_0)||\ ||u - u_0|| < \delta_0/4,$$

for all $u \in U_5$. We get

$$\begin{aligned} \langle \mathcal{G}'(u), v_0 - u \rangle &= \langle \mathcal{G}'(u) - \mathcal{G}'(u_0), v_0 - u \rangle \\ &+ \langle \mathcal{G}'(u_0), v_0 - u_0 \rangle + \langle \mathcal{G}'(u_0), u_0 - u \rangle \\ &\leq ||\mathcal{G}'(u) - \mathcal{G}'(u_0)||\ ||v_0 - u|| \\ &+ \langle \mathcal{G}'(u_0), v_0 - u_0 \rangle + ||\mathcal{G}'(u_0)||\ ||u - u_0|| \\ &\leq \delta_0/2 + \langle \mathcal{G}'(u_0), v_0 - u_0 \rangle, \end{aligned}$$

for all $u \in U_5$. On the other hand, by the lower semicontinuity of $\Psi$, there exists $U_6$ a neighborhood of $u_0$ such that

$$\Psi(u_0) - \Psi(u) \leq \delta_0/2, \qquad \forall u \in U_6.$$

Consequently, taking $U_7 = U_5 \cap U_6$, one has

$$\begin{aligned} \langle \mathcal{G}'(u), v_0 - u \rangle + \Psi(v_0) - \Psi(u) \leq \delta_0/2 + \langle \mathcal{G}'(u_0), v_0 - u_0 \rangle + \Psi(v_0) - \Psi(u_0) \\ + \Psi(u_0) - \Psi(u) \leq -\delta_0, \end{aligned}$$

for all $u \in U_7$, and (9.5) is proved. Therefore, in this case $U_0$ will be $U_4 \cap U_7$.

*Case 3*: $\underline{u_0 \notin K,\ I(u_0) \geq c - \epsilon}$. From Lemma 28, there exists $v_0 \in X$ satisfying

$$\langle \mathcal{G}'(u_0), v_0 - u_0 \rangle + \Psi(v_0) - \Psi(u_0) < -3\epsilon.$$

Now, arguing exactly as in the proof of (9.5) in Case 2, it follows that there exists $U_0$ a neighborhood of $u_0$ such that

$$\langle \mathcal{G}'(u), v_0 - u \rangle \leq \epsilon/2 + \langle \mathcal{G}'(u_0), v_0 - u_0 \rangle \quad \text{and} \quad \Psi(u_0) - \Psi(u) \leq \epsilon/2,$$

for all $u \in U_0$. Also, an argument similar to that used in the proof of (9.5) in Case 2 yields

$$\langle \mathcal{G}'(u), v_0 - u \rangle + \Psi(v_0) - \Psi(u) \leq -2\epsilon \qquad \forall u \in U_0.$$

■

**Lemma 30** *Let $c \in \mathbb{R}$ and $\mathcal{N}$ be a $G$-invariant neighborhood of $K_c$. Then, for each $\overline{\epsilon} > 0$, there exist $\epsilon \in (0, \overline{\epsilon}]$, $M_\epsilon > 0$, $\epsilon' \in (0, \epsilon]$ such that: $\forall u_0 \in \mathcal{N}_\epsilon$, $\exists v_0 \in X$ with $||v_0|| \leq M_\epsilon$, $\exists U_0$ a neighborhood of $u_0$ satisfying (9.5) and*

$$\langle \mathcal{G}'(u), v_0 - u \rangle + \Psi(v_0) - \Psi(u) \leq -2\epsilon', \quad \forall u \in U_0 \text{ with } I(u) \geq c - \epsilon. \tag{9.8}$$

*Proof.* Let $\overline{\epsilon} > 0$ and the corresponding $\epsilon \in (0, \overline{\epsilon}]$ be given by Lemma 29. For each $u_0 \in \mathcal{N}_\epsilon$, let $\epsilon_0$, $v_0$ and $U_0$ constructed in Lemma 29. The sets $U_0$ cover $\mathcal{N}_\epsilon$. Using that $\mathcal{N}_\epsilon$ is compact, it follows that there exists $(U_j)_{j=1}^l$ a finite subcovering. Let $u_j, \epsilon_j, v_j$ be related to $U_j$ in the same way as $u_0, \epsilon_0, v_0$ are related to $U_0$. We set

$$M_\epsilon = \max_{j=1}^{l} ||v_j|| \quad \text{ and } \quad \epsilon' = \min_{j=1}^{l} \epsilon_j.$$

Then, for $u_0 \in \mathcal{N}_\epsilon$, there exists $U_{j_0}$ such that $u_0 \in U_{j_0}$. We take $v_0 = v_{j_0}$ and $U_0 = U_{j_0}$. The proof follows now from Lemma 29. ∎

**Lemma 31** *Let $u_0 \in D(\Psi)$ be such that*

$$\mathcal{G}'(u_0)|_{\mathbb{R}^N} \neq 0. \tag{9.9}$$

*Then, for any $r > 0$, there exists $v_r \in X$ and $U_0$ a neighborhood of $u_0$ such that*

$$\langle \mathcal{G}'(u+g), v_r - (u+g) \rangle + \Psi(v_r) - \Psi(u+g) \leq -r, \tag{9.10}$$

*for all $g \in G$ with $|g| \leq 6$ and $u \in U_0$.*

*Proof.* Since $\mathcal{G}'$ is bounded on bounded subsets of $X$, we can fix some $\rho_0 > 0$ such that

$$|\langle \mathcal{G}'(u), g \rangle| \leq \rho_0, \quad \forall u \in B(u_0, 1) \text{ and } g \in G \text{ with } |g| \leq 6.$$

On the other hand, one has that there exists some $e_j = (0, ..., 1, ..., 0) \in \mathbb{R}^N$ with

$$\langle \mathcal{G}'(u_0), e_j \rangle \neq 0.$$

We may assume that

$$\langle \mathcal{G}'(u_0), e_j \rangle \ > 0.$$

Let $r > 0$ and consider $v_r = u_0 + \overline{v}_r \in D(\Psi)$, where $\overline{v}_r = (0, ..., w_r, ..., 0)$, ($w_r \in \mathbb{R}$). We have

$$\langle \mathcal{G}'(u_0), v_r - u_0 \rangle = \langle \mathcal{G}'(u_0), \overline{v}_r \rangle = w_r \langle \mathcal{G}'(u_0), e_j \rangle.$$

It follows that there is some $w_r < 0$ such that

$$\langle \mathcal{G}'(u_0), v_r - u_0 \rangle \ < -r - 2(\rho_0 + 2\rho),$$

with $\rho$ entering in $(H_3)$. Then, for $u \in X$, we write

$$\begin{aligned} \langle \mathcal{G}'(u), v_r - u \rangle &\leq ||\mathcal{G}'(u) - \mathcal{G}'(u_0)|| \, ||v_r - u|| + ||\mathcal{G}'(u_0)|| \, ||u_0 - u|| \\ &+ \langle \mathcal{G}'(u_0), v_r - u_0 \rangle. \end{aligned}$$

Using the continuity of $\mathcal{G}'$, it follows that there exists $U_r \subset B(u_0, 1)$ a neighborhood of $u_0$ such that

$$||\mathcal{G}'(u) - \mathcal{G}'(u_0)|| \leq \frac{\rho_0 + 2\rho}{2(|w_r| + 1)}, \quad \forall u \in U_r.$$

Also, we may assume that

$$||\mathcal{G}'(u_0)||\ ||u_0 - u|| \leq \frac{\rho_0 + 2\rho}{2}, \quad \forall u \in U_r.$$

Then, from

$$||v_r - u|| \leq |w_r| + 1$$

we obtain

$$||\mathcal{G}'(u) - \mathcal{G}'(u_0)||\ ||v_r - u|| \leq \frac{\rho_0 + 2\rho}{2}, \quad \forall u \in U_r.$$

Hence,

$$\langle \mathcal{G}'(u), v_r - u\rangle \leq -r - (\rho_0 + 2\rho), \quad \forall u \in U_r.$$

Now, the result follows immediately from the $G$-invariance of $\mathcal{G}'$ and $\Psi$ and from $(H_3)$. ∎

**Lemma 32** *Let $c \in \mathbb{R}$ and $\mathcal{N}$ be a neighborhood of $K_c$. Then, for any $\epsilon, r > 0$, there exists $M_{\epsilon,r} > 0$ such that $\forall u_0 \in \mathcal{N}_\epsilon$, satisfying (9.9), $\exists v_0 \in X$ with $||v_0|| \leq M_{\epsilon,r}$, $\exists U_0$ a neighborhood of $u_0$ such that (9.10) holds true, for all $g \in G$ with $|g| \leq 6$ and $u \in U_0$.*

*Proof.* Using that the set $\mathcal{N}_\epsilon \subset D(\Psi)$ is compact, the argument is similar to that employed in the proof of Lemma 30, but with Lemma 31 instead of Lemma 29. ∎

**Remark 24** Let $U$ be an open subset of $X$ $u_0 \in U$. Using that $G$ is discrete, there exists $\mu_0 \in (0, 1]$ such that the square

$$D(u_0, \mu_0) = \{u \in X: \ |\overline{u} - \overline{u}_0| < \mu_0, \ ||\widetilde{u} - \widetilde{u}_0|| < \mu_0\}$$

satisfies $D(u_0, \mu_0) \subset U$ and

$$u \in D(u_0, \mu_0) \Rightarrow u + g \notin D(u_0, \mu_0) \quad \forall g \in G \setminus \{0\}. \tag{9.11}$$

It follows that $U_0$ in the above Lemmas 29 - 32 can be supposed to be such a square.

**Remark 25** (i) Let $u_0 \in X$ be such that

$$\mathcal{G}'(u_0)|_{\mathbb{R}^N} = 0. \tag{9.12}$$

From the continuity of $\mathcal{G}'$ in $u_0$, we infer that for any $\eta > 0$, there exists $\delta_\eta > 0$ so that

$$|\langle \mathcal{G}'(u), v\rangle| \leq \eta |v|, \quad \forall v \in \mathbb{R}^N, \ \forall u \in X \text{ with } ||u - u_0|| \leq \delta_\eta.$$

(ii) Let $U_0 = D(u_0, \mu_0)$ be as in Lemma 30 (see also Remark 24) with $\overline{\epsilon} \leq 1$. Assume that $u_0$ is such that (9.12) holds true. Let $\eta = \epsilon'/12$ and $\delta_\eta >$

0 be the corresponding number associated to $\eta$ by (i). Consider also $\nu_0 \in (0, \min\{\mu_0, \delta_\eta/2\})$, and note that $D(u_0, \nu_0) \subset B(u_0, \delta_\eta) \cap D(u_0, \mu_0)$. It is clear that

$$|\langle \mathcal{G}'(u+g), v\rangle| \leq (\epsilon'/12)|v|, \tag{9.13}$$

for all $g \in G$, $v \in \mathbb{R}^N$ and $u \in D(u_0, \nu_0)$. Then, for all $g \in G$ with $|g| \leq 6$ and $u \in D(u_0, \nu_0)$, one has

$$|\langle \mathcal{G}'(u+g), -g\rangle| \leq \epsilon'/2.$$

This, together with Lemma 30 and the $G$-invariance of $\mathcal{G}'$ and $\Psi$, imply that

$$\langle \mathcal{G}'(u+g), v_0 - (u+g)\rangle + \Psi(v_0) - \Psi(u+g) \leq 2, \tag{9.14}$$

for all $g \in G$ with $|g| \leq 6$ and $u \in D(u_0, \nu_0)$. If, moreover, $I(u) \geq c - \epsilon$, then

$$\langle \mathcal{G}'(u+g), v_0 - (u+g)\rangle + \Psi(v_0) - \Psi(u+g) \leq -\epsilon'. \tag{9.15}$$

**Remark 26** Let $\alpha$ be defined in (9.3) and $\epsilon > 0$. Then, taking in Lemma 32, $r = \epsilon + \alpha$, we obtain that there exists $M'_\epsilon := M_{\epsilon,\epsilon+\alpha} > 0$ such that for any $u_0 \in \mathcal{N}_\epsilon$ satisfying (9.9), $\exists v_0 \in X$ with $||v_0|| \leq M'_\epsilon$, $\exists D(u_0, \mu_0)$, such that

$$\langle \mathcal{G}'(u+g), v_0 - (u+g)\rangle + \Psi(v_0) - \Psi(u+g) \leq -(\epsilon + \alpha), \tag{9.16}$$

for all $g \in G$ with $|g| \leq 6$ and $u \in D(u_0, \mu_0)$.

The main result of this Section is the following

**Proposition 15** *Let $c \in \mathbb{R}$ and $\mathcal{N}$ be a $G$-invariant neighborhood of $K_c$. Then, for each $\bar{\epsilon} \in (0, 1]$ there exist $\epsilon \in (0, \bar{\epsilon}]$, $m_\epsilon > 0$ and $\epsilon' \in (0, \epsilon]$ with the following properties.*

$1^0$ *For any $u_0 \in \mathcal{N}_\epsilon$ with $\mathcal{G}'(u_0)|_{\mathbb{R}^N} = 0$, $\exists v_0 \in X$ with $||v_0|| \leq m_\epsilon$, $\exists \mu_0 > 0$, such that*

*(i) (9.13) holds true for all $g \in G$, $v \in \mathbb{R}^N$ and $u \in D(u_0, \mu_0)$;*

*(ii) (9.14) holds true for all $g \in G$ with $|g| \leq 6$ and $u \in D(u_0, \mu_0)$;*

*(iii) (9.15) holds true for all $g \in G$ with $|g| \leq 6$ and $u \in D(u_0, \mu_0)$ with $I(u) \geq c - \epsilon$.*

$2^0$ *For any $u_0 \in \mathcal{N}_\epsilon$ with $\mathcal{G}'(u_0)|_{\mathbb{R}^N} \neq 0$, $\exists v_0 \in X$ with $||v_0|| \leq m_\epsilon$, $\exists \mu_0 > 0$, such that (9.16) holds true for all $g \in G$ with $|g| \leq 6$ and $u \in D(u_0, \mu_0)$.*

*Note that $\mu_0$ above is taken such that (9.11) holds true.*

*Proof.* For $1^0$ one applies Lemma 30 and Remark 25, while $2^0$ follows from Lemma 32 and Remark 26; one takes $m_\epsilon = \max\{M_\epsilon, M'_\epsilon\}$. ■

## 9.4 A deformation lemma

**Lemma 33** *Let $c \in \mathbb{R}$ and $\mathcal{N}$ be a $G$-invariant neighborhood of $K_c$. Then, for each $\overline{\epsilon} \in (0,1]$ there exist $\epsilon \in (0,\overline{\epsilon}]$, $d_\epsilon > 0$, $\epsilon' \in (0,\epsilon]$ and $\eta : [0,\overline{t}] \times \mathcal{N}_\epsilon \to X$ a continuous function, with the following properties.*

*(i)* $\eta(0,\cdot) = id_{\mathcal{N}_\epsilon}$.

*(ii)* $\eta(t,u+g) = \eta(t,u)+g, \qquad \forall (t,u) \in [0,\overline{t}] \times \mathcal{N}_\epsilon,\ \forall g \in G$ *with* $u+g \in \mathcal{N}_\epsilon$.

*(iii)* $||\eta(t,u) - u|| \leq d_\epsilon t, \qquad \forall (t,u) \in [0,\overline{t}] \times \mathcal{N}_\epsilon$.

*(iv)* $I(\eta(t,u)) - I(u) \leq d_\epsilon t, \qquad \forall (t,u) \in [0,\overline{t}] \times \mathcal{N}_\epsilon$.

*(v)* $I(\eta(t,u)) - I(u) \leq -\epsilon' t/2, \quad \forall (t,u) \in [0,\overline{t}] \times \mathcal{N}_\epsilon$ *with* $I(u) \geq c - \epsilon$.

*(vi) If $A$ is a closed subset of $\mathcal{N}_\epsilon$ with $c \leq \sup_A I$, then*

$$\sup_{u\in A} I(\eta(t,u)) - \sup_{u\in A} I(u) \leq -\epsilon' t/2, \qquad \forall t \in [0,\overline{t}].$$

*Proof.* *Covering.* Let $\overline{\epsilon} \in (0,1]$ and the corresponding $\epsilon \in (0,\overline{\epsilon}]$, $m_\epsilon > 0$ and $\epsilon' \in (0,\epsilon]$ be given in Proposition 15. Also, for each $u_0 \in \mathcal{N}_\epsilon$, let $v_0$, $\mu_0$ and $D(u_0,\mu_0)$ be as in Proposition 15. Since the sets $D(u_0,\mu_0)$ cover the compact set $\mathcal{N}_\epsilon$, it follows that there exists $(D_j)_{j=1}^l$ a finite subcovering. Below, $u_j, v_j, \mu_j$ will be related to $D_j$ in the same way as $u_0, v_0, \mu_0$ are related to $D(u_0,\mu_0)$.

*Partition of unity.* Let $\rho_i^1 : X \to [0,\infty)$ be a continuous function (we can take the distance function $d(\cdot, X \setminus D_i)$) such that

$$\rho_i^1(u) > 0, \quad \forall u \in D_i \quad \text{and} \quad \rho_i^1(u) = 0, \quad \forall u \in X \setminus D_i.$$

Consider the $G$-invariant set

$$V_i = \bigcup_{g\in G} (D_i + g).$$

Note that, from the choice of the squares $D_i$ (see (9.11)), one has that the sets $D_i + g$ $(g \in G)$ are mutually disjoint. It follows that, the function $\rho_i^2 : X \to [0,\infty)$ given by $\rho_i^2(u+g) = \rho_i^1(u)$ for all $u \in D_i$, $g \in G$, and $\rho_i^2(u) = 0$ for all $u \in X \setminus V_i$ is correctly defined, continuous and $G$-invariant.

Now, let us define

$$D = \bigcup_{i=1}^{l} D_i$$

and

$$\sigma_i : D \to [0,1], \quad \sigma_i = \frac{\rho_i^2}{\sum_{j=1}^l \rho_j^2}.$$

One has that $\sigma_i$ is correctly defined, continuous and $G$-invariant in the sense that

$$\sigma_i(u+g) = \sigma_i(u), \quad \forall u \in D,\ g \in G \text{ with } u+g \in D.$$

Also, we have

$$\sum_{i=1}^{l} \sigma_i = 1$$

and

$$\sigma_i(w) \neq 0 \ \Leftrightarrow\ w = w_i + g_i \text{ with some } w_i \in D_i,\ g_i \in G. \tag{9.17}$$

*Deformation.* Consider the function $\eta : [0,1] \times \mathcal{N}_\epsilon \to X$ given by

$$\eta(t,u) = (1-t)u + t\sum_{i=1}^{l} \sigma_i(u) v_i + t\overline{u} \qquad ((t,u) \in [0,1] \times \mathcal{N}_\epsilon).$$

It is clear that $\eta$ is continuous and $\eta(0,\cdot) = id_{\mathcal{N}_\epsilon}$.

To prove (ii), let $(t,u) \in [0,1] \times \mathcal{N}_\epsilon$ and $g \in G$ be with $u+g \in \mathcal{N}_\epsilon$. Then, we have

$$\begin{aligned}
\eta(t,u+g) &= (1-t)(u+g) + t\sum_{i=1}^{l} \sigma_i(u+g) v_i + t\overline{[u+g]} \\
&= (1-t)u + (1-t)g + t\sum_{i=1}^{l} \sigma_i(u) v_i + t\overline{u} + tg \\
&= \eta(t,u) + g.
\end{aligned}$$

In order to prove (iii), let us consider $(t,u) \in [0,1] \times \mathcal{N}_\epsilon$. Using $(H_3)$ and denoting $d_\epsilon^1 = m_\epsilon + \rho$, one has:

$$\begin{aligned}
||\eta(t,u) - u|| &= t||\sum_{i=1}^{l} \sigma_i(u) v_i - \widetilde{u}|| \\
&\leq t\left[\sum_{i=1}^{l} \sigma_i(u)||v_i|| + ||\widetilde{u}||\right] \\
&\leq t\left[m_\epsilon \sum_{i=1}^{l} \sigma_i(u) + \rho\right] = td_\epsilon^1.
\end{aligned}$$

*Estimations.* Let us consider $(t,u) \in [0,1] \times \mathcal{N}_\epsilon$. Setting

$$w := \sum_{i=1}^{l} \sigma_i(u) v_i - \widetilde{u},$$

we have $\|w\| \leq d_\epsilon^1$ and

$$\eta(t,u) = u + tw.$$

By the mean value theorem, we can write

$$\mathcal{G}(u+tw)-\mathcal{G}(u)=t\langle \mathcal{G}'(u+\theta tw),w\rangle,$$

with some $\theta\in(0,1)$. Hence,

$$I(\eta(t,u))=\mathcal{G}(u)+t\langle \mathcal{G}'(u+\theta tw),w\rangle+\Psi(u+tw). \tag{9.18}$$

On the other hand, from $(H_2)$ and the convexity of $\Psi$ we get

$$\begin{aligned}\Psi(u+tw) &= \Psi\left((1-t)u+t\sum_{i=1}^{l}\sigma_i(u)v_i+t\overline{u}\right)\\ &= \Psi\left((1-t)u+t\sum_{i=1}^{l}\sigma_i(u)v_i\right)\\ &\le (1-t)\Psi(u)+t\sum_{i=1}^{l}\sigma_i(u)\Psi(v_i).\end{aligned}$$

Then, using (9.18), it follows

$$\begin{aligned}I(\eta(t,u))-I(u) &\le t\sum_{i=1}^{l}\sigma_i(u)\left[\Psi(v_i)-\Psi(u)\right]+t\langle \mathcal{G}'(u+\theta tw),w\rangle\\ &= t\sum_{i=1}^{l}\sigma_i(u)\left[\langle \mathcal{G}'(u),v_i-u\rangle+\Psi(v_i)-\Psi(u)\right]\\ &+ t\left[\langle \mathcal{G}'(u+\theta tw),w\rangle-\langle \mathcal{G}'(u),w\rangle+\langle \mathcal{G}'(u),\overline{u}\rangle\right]\\ &\le t\sum_{i=1}^{l}\sigma_i(u)\left[\langle \mathcal{G}'(u),v_i-u\rangle+\Psi(v_i)-\Psi(u)\right]\\ &+ t\left[(\|\mathcal{G}'(u+\theta tw)-\mathcal{G}'(u)\|)d_\epsilon^1+\langle \mathcal{G}'(u),\overline{u}\rangle\right]\end{aligned}$$

Next, as $\mathcal{G}'$ is continuous and $\mathcal{N}_\epsilon$ is compact, there exists $\delta=\delta(\epsilon,\epsilon')>0$ such that

$$||\mathcal{G}'(v)-\mathcal{G}'(u)||\le \epsilon'/(4d_\epsilon^1),\quad \forall u\in\mathcal{N}_\epsilon, v\in X \text{ with } ||v-u||\le\delta.$$

Then, denoting

$$\bar{t}_1:=\delta/d_\epsilon^1,$$

it follows

$$||\mathcal{G}'(u+\theta tw)-\mathcal{G}'(u)||\le\epsilon'/(4d_\epsilon^1),\quad \forall t\in[0,\bar{t}_1],\quad \forall u\in\mathcal{N}_\epsilon.$$

So, we obtain

$$\begin{aligned}I(\eta(t,u))-I(u) &\le t\sum_{i=1}^{l}\sigma_i(u)\left[\langle \mathcal{G}'(u),v_i-u\rangle+\Psi(v_i)-\Psi(u)\right]\\ &+ t\left[\epsilon'/4+\langle \mathcal{G}'(u),\overline{u}\rangle\right]\end{aligned} \tag{9.19}$$

for all $t \in [0, \bar{t}_1]$ and $u \in \mathcal{N}_\epsilon$.

Let us prove (iv). Consider $(t, u) \in [0, \bar{t}_1] \times \mathcal{N}_\epsilon$. From (9.17), if $\sigma_i(u) \neq 0$ then $u = u_i' + g_i$, with $u_i' \in D_i$ and $g_i \in G$. In this situation we have

$$|g_i| \leq |\overline{u}| + |\overline{u_i'}| \leq |\overline{u}| + |\overline{u_i'} - \overline{u_i}| + |\overline{u_i}| \leq 2 + \mu_i + 2 \leq 6$$

and, from Proposition 15 (ii) it follows

$$\langle \mathcal{G}'(u), v_i - u \rangle + \Psi(v_i) - \Psi(u) \leq 2.$$

This, together with (9.3) and (9.19) yield

$$I(\eta(t, u)) - I(u) \leq t(\alpha + 3).$$

To prove (v), let $(t, u) \in [0, \bar{t}_1] \times \mathcal{N}_\epsilon$ be such that $I(u) \geq c - \epsilon$. We rewrite (9.19) as follows

$$I(\eta(t, u)) - I(u) \leq \tag{9.20}$$
$$t \sum_{i=1}^{l} \sigma_i(u) \left[ \langle \mathcal{G}'(u), v_i - u \rangle + \Psi(v_i) - \Psi(u) + \langle \mathcal{G}'(u), \overline{u} \rangle \right] + t\epsilon'/4.$$

As above, if $\sigma_i(u) \neq 0$ then $u = u_i' + g_i$, with $u_i' \in D_i$, $g_i \in G$ and $|g_i| \leq 6$. From Proposition 15, if $\mathcal{G}'(u_i)|_{\mathbb{R}^N} = 0$, then

$$\langle \mathcal{G}'(u), v_i - u \rangle + \Psi(v_i) - \Psi(u) \leq -\epsilon',$$

and

$$|\langle \mathcal{G}'(u), \overline{u} \rangle| \leq \epsilon'/6,$$

while, if $\mathcal{G}'(u_i)|_{\mathbb{R}^N} \neq 0$, then

$$\langle \mathcal{G}'(u), v_i - u \rangle + \Psi(v_i) - \Psi(u) \leq -\epsilon - \alpha.$$

In both cases, one has that

$$\langle \mathcal{G}'(u), v_i - u \rangle + \Psi(v_i) - \Psi(u) + \langle \mathcal{G}'(u), \overline{u} \rangle \leq -\epsilon' + (\epsilon'/6).$$

This, together with (9.20) give

$$I(\eta(t, u)) - I(u) \leq t \sum_{i=1}^{l} \sigma_i(u)[-\epsilon' + (\epsilon'/6)] + t\epsilon'/4 < -\epsilon' t/2.$$

In order to prove (vi), we set $\bar{t} := \min\{\bar{t}_1, 1/2, \frac{\epsilon}{2(\alpha+3)}\}$ and let $A \subset \mathcal{N}_\epsilon$ be closed such that $c \leq \sup_A I$. For $t \in [0, \bar{t}]$, we have two cases.

If

$$\sup_{u \in A} I(\eta(t, u)) \leq c - (\epsilon/2),$$

then, using that $t \leq 1/2$, it follows

$$\sup_{u \in A} I(\eta(t,u)) - \sup_{u \in A} I(u) \leq -\epsilon t \leq -\epsilon' t.$$

If

$$\sup_{u \in A} I(\eta(t,u)) > c - (\epsilon/2),$$

then, putting

$$B := \{u \in A : \ I(u) \geq c - \epsilon\},$$

it follows

$$I(\eta(t,u)) \leq I(u) + (\alpha+3)t < c - \epsilon + (\alpha+3)t \leq c - (\epsilon/2),$$

for all $u \in A \setminus B$. We infer that

$$\sup_{u \in A} I(\eta(t,u)) = \sup_{u \in B} I(\eta(t,u)).$$

Consequently,

$$\begin{aligned} \sup_{u \in A} I(\eta(t,u)) - \sup_{u \in A} I(u) &\leq \sup_{u \in B} I(\eta(t,u)) - \sup_{u \in B} I(u) \\ &\leq \sup_{u \in B} [I(\eta(t,u)) - I(u)] \\ &\leq -\epsilon' t/2. \end{aligned}$$

Now, to finish the proof it suffices to take $d_\epsilon := \max\{d_\epsilon^1, \alpha+3\}$. ■

The main result of this Section is the following

**Proposition 16** *Let $c \in \mathbb{R}$ and $\mathcal{N}$ be a $G$-invariant neighborhood of $K_c$. Then, for each $\overline{\epsilon} > 0$ there exist $d > 0$, $\epsilon'' \in (0, \overline{\epsilon}]$ with $2d\epsilon'' < \overline{\epsilon}$, and $\eta : [0,\overline{t}] \times \mathcal{N}_{\epsilon''} \to X$ a continuous function, with the following properties.*

*(i)* $\eta(0,\cdot) = id_{\mathcal{N}_{\epsilon''}}$.

*(ii)* $\eta(t,u+g) = \eta(t,u)+g, \qquad \forall (t,u) \in [0,\overline{t}] \times \mathcal{N}_{\epsilon''}, \ \forall g \in G$ *with* $u+g \in \mathcal{N}_{\epsilon''}$.

*(iii)* $||\eta(t,u) - u|| \leq dt, \qquad \forall (t,u) \in [0,\overline{t}] \times \mathcal{N}_{\epsilon''}$.

*(iv) If $A$ is a closed subset of $\mathcal{N}_{\epsilon''}$ with $c \leq \sup_A I$, then*

$$\sup_{u \in A} I(\eta(t,u)) - \sup_{u \in A} I(u) \leq -\epsilon'' t, \qquad \forall t \in [0,\overline{t}].$$

*Proof.* The result follows immediately from Lemma 33 taking

$$0 < \epsilon'' < \min\{\epsilon'/2, \overline{\epsilon}/2d_\epsilon\}.$$

Note that $\mathcal{N}_{\epsilon''} \subset \mathcal{N}_\epsilon$. ■

**Lemma 34** *Let $\eta$ be as in Proposition 16. Then $\widehat{\eta}: \ [0,\overline{t}] \times \pi(\mathcal{N}_{\epsilon''}) \to \pi(X)$ defined by*

$$\widehat{\eta}(t,\Gamma) = \pi(\eta(t,v)), \quad \text{for } v \in \mathcal{N}_{\epsilon''} \text{ with } \pi(v) = \Gamma \quad (t \in [0,\overline{t}])$$

*is well defined and continuous.*

*Proof.* Let $(t,\Gamma) \in [0,\overline{t}] \times \pi(\mathcal{N}_{\epsilon''})$. It follows that there exists $u \in \mathcal{N}_{\epsilon''}$ such that $\pi(u) = \Gamma$. Assume that $u_1, u_2 \in \mathcal{N}_{\epsilon''}$ are such that $\pi(u_1) = \Gamma = \pi(u_2)$. It follows that $u_2 = u_1 + g$, for some $g \in G$. Then, using Proposition 16 (ii), we get

$$\eta(t,u_2) = \eta(t,u_1+g) = \eta(t,u_1) + g,$$

which means that $\pi(\eta(t,u_1)) = \pi(\eta(t,u_2))$ and $\widehat{\eta}$ is well defined.

For the continuity of $\widehat{\eta}$, consider a sequence $\{(t_k,\Gamma_k)\} \subset [0,\overline{t}] \times \pi(\mathcal{N}_{\epsilon''})$ converging to some $(t,\Gamma) \in [0,\overline{t}] \times \pi(\mathcal{N}_{\epsilon''})$. It follows that there exists $\{u_k\} \subset X$ with $\pi(u_k) = \Gamma_k$ such that $u_k \to u \in X$ and $\pi(u) = \Gamma$. Note that $\widetilde{u}_k \to \widetilde{u}$ and $\overline{u}_k \to \overline{u}$. On the other hand, $u_k = v_k + g_k$ with some $v_k \in \mathcal{N}_{\epsilon''}$ and $g_k \in G$. So, using that $I$ is $G$-invariant, we deduce $I(u_k) \le c + \epsilon''$. Similarly, $u = v + g$ with $v \in \mathcal{N}_{\epsilon''}$, $g \in G$ and $I(u) \le c + \epsilon''$. Consider $g' \in G$ with $\overline{u} + g' \in [0,1)^N$. Then, we may assume that $|\overline{u}_k + g'| \le 2$ for all $k \in \mathbb{N}$. Using that $\mathcal{N}$ and $I$ are $G$-invariant, it follows that $w_k := u_k + g' \in \mathcal{N}_{\epsilon''}$ and $w := u + g' \in \mathcal{N}_{\epsilon''}$. By the continuity of $\eta$ and $\pi$, we have

$$\widehat{\eta}(t_k,\Gamma_k) = \pi(\eta(t_k,w_k)) \to \pi(\eta(t,w)) = \widehat{\eta}(t,\Gamma)$$

and the proof is complete. ■

**Remark 27** If $A \subset [0,1)^N + Y$ is compact, $b \in X$ and $\inf_{a\in A} ||b-a|| \le 1$, then $|\overline{b}| \le 2$. Indeed, using the compactness of $A$, it follows that there exists $a_0 \in A$ such that $||b - a_0|| = \inf_{a\in A} ||b-a||$. As $||b-a_0|| = |\overline{b} - \overline{a}_0| + ||\widetilde{b} - \widetilde{a}_0||_X$, one has that $|\overline{b} - \overline{a}_0| \le 1$. It follows that $|\overline{b}| \le |\overline{b} - \overline{a}_0| + |\overline{a}_0| \le 2$.

## 9.5 Main tools

The results in this section are proved in [101].

**1. Lusternik-Schnirelman category.** Recall, a subset $C$ of a topological spaces $E$ is called *contractible* in $E$ if there exists a continuous function $h: [0,1] \times C \to E$ and $e \in E$ such that

$$h(0,\cdot) = id_C, \qquad h(1,\cdot) = e.$$

A subset $A$ of a topological space $E$ is said to has *category* $k$ in $E$ if $k$ is the least integer such that $A$ can be covered by $k$ closed sets contractible in $E$. The category of $A$ in $E$ is denoted by $\mathrm{cat}_E(A)$.

The main properties of the Lusternik-Schnirelman category are given in the following

**Lemma 35** *Let $E$ be a topological space and let $A, B \subset E$.*

*(i) If $A \subset B$, then $cat_E(A) \leq cat_E(B)$.*

*(ii) $cat_E(A \cup B) \leq cat_E(A) + cat_E(B)$.*

*(iii) If $A$ is closed and $B = \eta(\bar{t}, A)$, where $\eta : [0, \bar{t}] \times A \to E$ is a continuous function such that $\eta(0, \cdot) = id_A$, then $cat_E(A) \leq cat_E(B)$.*

**Remark 28** In the functional framework from the previous section, if $A = [0,1]^N + \{0\}$ $(\subset X = \mathbb{R}^N \oplus Y)$, then $\mathrm{cat}_{\pi(X)}(\pi(A)) = N + 1$.

**2. Ekeland variational principle.** *Let $(E, d)$ be a complete metric space and $\gamma : E \to (-\infty, +\infty]$ a proper, lower semi-continuous function bounded from below. Given $\delta, \lambda > 0$, and $x \in E$ with*

$$\gamma(x) \leq \inf_E \gamma + \delta,$$

*there exists $y \in E$ such that*

$$\begin{aligned} \gamma(y) &\leq \gamma(x), \\ d(x, y) &\leq 1/\lambda, \\ \gamma(z) - \gamma(y) &\geq -\delta\lambda d(y, z), \qquad \forall z \in E. \end{aligned}$$

**3. Hausdorff distance and a complete metric space.** On account of Remark 28, it follows that, for $1 \leq j \leq N + 1$, the set

$$\mathcal{A}_j = \{A \subset X : \ A \text{ is compact and } \mathrm{cat}_{\pi(X)}(\pi(A)) \geq j\}$$

is nonempty. In order to apply Ekeland's variational principle, we need the following

**Lemma 36** *Let $1 \leq j \leq N + 1$ be fixed.*

*(i) The space $\mathcal{A}_j$ with the Hausdorff distance*

$$\delta(A, B) = \max\{\sup_{a \in A} dist(a, B), \sup_{b \in B} dist(b, A)\}$$

*is a complete metric space.*

*(ii) If $I : X \to (-\infty, +\infty]$ is lower semicontinuous, then the function $\gamma : \mathcal{A}_j \to (-\infty, +\infty]$, defined by*

$$\gamma(A) = \sup_A I \quad (A \in \mathcal{A}_j) \tag{9.21}$$

*is lower semicontinuous.*

## 9.6 Main result

The main abstract result of the paper is the following

**Theorem 35** *Under the assumptions $(H_1) - (H_4)$, the functional $I$ defined in (9.1) is bounded from below and has at least $N + 1$ critical orbits.*

*Proof.* First, let us note that, $D(\Psi)$ closed and $(H_4)$ imply that $\{u \in D(\Psi) : |\overline{u}| \leq 1\}$ is a compact set. This, together with the $G$-invariance and the continuity of $\mathcal{G}$, imply that $\mathcal{G}$ is bounded on $D(\Psi)$. So, from $(H_3)$, we deduce that $I$ is bounded from below on $X$.

For $1 \leq j \leq N+1$, let $\gamma : \mathcal{A}_j \to (-\infty, +\infty]$ be defined by (9.21) and

$$c_j := \inf_{A \in \mathcal{A}_j} \gamma(A).$$

Using also that $\mathcal{A}_{j+1} \subset \mathcal{A}_j$, one has that

$$-\infty < \inf_X I \leq c_1 \leq ... \leq c_{N+1}.$$

Moreover, from Remark 28 one has $A = [0,1]^N + \{0\} \in \mathcal{A}_{N+1}$, and using $(H_2)$, we have that $I(u) = \mathcal{G}(u)$ for all $u \in A$. This together with the continuity of $\mathcal{G}$ and the compactness of $A$ imply that

$$c_{N+1} < \infty.$$

We will show that $K_{c_j} \neq \emptyset$. By contradiction, assume that $K_{c_j} = \emptyset$. Then, let $d, \epsilon'', \eta$ be given by Proposition 16 with $\mathcal{N} = \emptyset$ and $\overline{\epsilon} = 1/2$. Consider $B \in \mathcal{A}_j$ with

$$\gamma(B) \leq c_j + \epsilon''^2.$$

Using the $G$-invariance of $I$, we may assume that $B \subset [0,1)^N + Y$. Using Ekeland's variational principle (see Lemma 36) with $\delta = \epsilon''^2$ and $\lambda = 1/2\epsilon'' d$, it follows that there exists $C_B \in \mathcal{A}_j$ such that

$$\gamma(C_B) \leq \gamma(B) \leq c_j + \epsilon''^2, \tag{9.22}$$

$$\delta(B, C_B) \leq 2\epsilon'' d < 1/2,$$

$$\gamma(D) - \gamma(C_B) \geq -\frac{\epsilon''}{2d}\delta(C_B, D), \qquad \forall D \in \mathcal{A}_j. \tag{9.23}$$

In particular, one has that $\delta(B, C_B) < 1$ and $\gamma(C_B) \leq c_j + \epsilon''$, which together with $B \subset [0,1)^N + Y$ and Remark 27 imply $C_B \subset \mathcal{N}_{\epsilon''}$. So, we can consider the compact set $D_B := \eta(\overline{t}, C_B)$. Then, with $\widehat{\eta}$ introduced in Lemma 34, we have

$$\pi(D_B) = \widehat{\eta}(\overline{t}, \pi(C_B))$$

and from Lemma 35 (iii) it follows

$$\operatorname{cat}_{\pi(X)}(\pi(D_B)) \geq \operatorname{cat}_{\pi(X)}(\pi(C_B)) \geq j,$$

showing that $D_B \in \mathcal{A}_j$. So, $\gamma(D_B) \geq c_j$. On the other hand, from Proposition 16, one has

$$\delta(C_B, D_B) \leq d\bar{t}, \qquad \gamma(D_B) - \gamma(C_B) \leq -\epsilon''\bar{t}.$$

Consequently,

$$-\epsilon''\bar{t} \geq \gamma(D_B) - \gamma(C_B) \geq -\frac{\epsilon''}{2d}\delta(C_B, D_B) \geq -\frac{\epsilon''}{2d}d\bar{t},$$

giving $1 \leq 1/2$, a contradiction.

It suffices to prove that, if $c_k = c_j = c$ for some $1 \leq j < k \leq N+1$, then $K_c$ contains at least $k - j + 1$ critical orbits. By contradiction, assume that $K_c$ contains at most $n \leq k-j$ critical orbits denoted by $\pi^{-1}(\pi(u_1)), ..., \pi^{-1}(\pi(u_n))$. Note that, from the above step it follows that $n \geq 1$. Let $\rho \in (0,1)$ be such that $\pi$ restricted to $\overline{B(u_m, \rho)}$ is injective. We introduce the $G$-invariant set

$$\mathcal{M}_\rho := \bigcup_{m=1}^{n} \bigcup_{g \in G} B(u_m + g, \rho),$$

which, clearly is an open neighborhood of $K_c$.

Let $d, \epsilon''$, and $\eta$ be given by Proposition 16 with $\mathcal{N} = \mathcal{M}_{\rho/2}$ and $\bar{\epsilon} = \rho/2$. Pick $A \in \mathcal{A}_k$ such that

$$\gamma(A) \leq c + \epsilon''^2.$$

Using the $G$-invariance of $I$, we may assume that $A \subset [0,1)^N + Y$. Setting $B = A \setminus \mathcal{M}_\rho$ and using Lemma 35, we have

$$\begin{aligned} k &\leq \operatorname{cat}_{\pi(X)}(\pi(A)) \\ &\leq \operatorname{cat}_{\pi(X)}(\pi(B) \cup \pi(\mathcal{M}_\rho)) \\ &\leq \operatorname{cat}_{\pi(X)}(\pi(B)) + \operatorname{cat}_{\pi(X)}(\pi(\mathcal{M}_\rho)). \end{aligned}$$

Since from the injectivity of $\pi$ on $\overline{B(u_m, \rho)}$ and Lemma 35 (ii), one has that $\operatorname{cat}_{\pi(X)}(\pi(\mathcal{M}_\rho)) \leq n$, it follows

$$k \leq \operatorname{cat}_{\pi(X)}(\pi(B)) + n \leq \operatorname{cat}_{\pi(X)}(\pi(B)) + k - j,$$

hence $B \in \mathcal{A}_j$. It is clear that

$$\gamma(B) \leq \gamma(A) \leq c + \epsilon''^2.$$

By Ekeland's variational principle with $\delta = \epsilon''^2$ and $\lambda = 1/2\epsilon'' d$, there exists $C_B \in \mathcal{A}_j$ such that (9.22), (9.23) hold true and

$$\delta(B, C_B) \leq 2\epsilon'' d < \rho/2.$$

Note that $B \cap \mathcal{M}_\rho = \emptyset$ and $\delta(B, C_B) < \rho/2$ imply $C_B \cap \mathcal{M}_{\rho/2} = \emptyset$. Then $C_B \subset \mathcal{N}_{\epsilon''}$, and reasoning as above we arrive at the same contradiction $(1 \leq 1/2)$, and the proof is completed. ∎

**Corollary 18** *Under the hypothesis* $(H_\phi)$, $(H_F)$ *and* $(H_h)$, *the differential system* (2.41) *has at least* $N+1$ *geometrically distinct solutions.*

*Proof.* It follows immediately from the Theorem 35 and the results of Section 2. ∎

# Chapter 10

# Further developments

## 10.1 Positive radial solutions

**The Laplacian.** It is proved in the seminal paper [52] that the second order boundary value problem

$$u'' + a(t)f(u) = 0,$$
$$\alpha u(0) - \beta u'(0) = 0, \quad \gamma u(1) + \delta u'(1) = 0,$$

has at least one positive solution if the following assumptions hold true: the nonlinearity $f : [0, \infty) \to [0, \infty)$ is continuous, $f(0) = 0$ and superlinear, that is

$$\lim_{u \to 0} \frac{f(u)}{u} = 0, \qquad \lim_{u \to \infty} \frac{f(u)}{u} = \infty,$$

the weight function $a : [0, 1] \to [0, \infty)$ is continuous and non-identically zero on any subinterval of $[0, 1]$, and $\alpha, \beta, \gamma, \delta \geq 0$ are such that $\gamma\beta + \alpha\gamma + \alpha\delta > 0$. The proof of this result is based on an application of Krasnoselskii's fixed point theorem on compression-expansion of conical shells on a Banach space.

On the other hand, consider the Dirichlet boundary value problem

$$\Delta u + a(|x|)f(u) = 0 \quad \text{in } \mathcal{A}, \qquad u = 0 \quad \text{on } \partial\mathcal{A}, \tag{10.1}$$

where $0 < R_1 < R_2$, $\mathcal{A} = \{x \in \mathbb{R}^N : R_1 < |x| < R_2\}$ is an annular domain, $a, f$ satisfy the above assumptions and $f > 0$ on $(0, \infty)$. Then, it is proved independently in [8, 39, 78] using the shooting method combined with the Sturm comparison theorem and phase-plane method and in [119] using Krasnoselskii fixed point theorem, that (10.1) has at least one positive radial solution.

Analogous results can be proved if the superlinearity condition on $f$ is replaced by the fact that $f$ is sublinear, that is

$$\lim_{u \to 0} \frac{f(u)}{u} = \infty, \qquad \lim_{u \to \infty} \frac{f(u)}{u} = 0.$$

Typical examples are $f(u) = u^q$ with $0 < q < 1$ in the sublinear case and $q > 1$ in the superlinear case.

**The $\phi$-Laplacian.** In [42] it is proved that the Dirichlet boundary value problem

$$\operatorname{div}(a(|\nabla u|)\nabla u) + f(u) = 0 \quad \text{in} \quad \mathcal{A}, \qquad u = 0 \quad \text{on} \quad \partial\mathcal{A},$$

where $\mathcal{A}$ is an annular domain, the coefficient function $a$ satisfies some appropriate condition (for example $a(s) = |s|^{p-2}$ with $p > 1$) and $f$ is superlinear. To prove that the above problem has at least one positive radial solution, the authors use the properties of the Leray-Schauder degree. The main idea is to show that the degree around zero is one and the degree in a large ball is zero. For related results see [60, 64].

**Prescribed mean curvature problems.** Consider the prescribed mean curvature problem

$$\operatorname{div}\left(\frac{\nabla v}{\sqrt{1+|\nabla v|^2}}\right) + \lambda\, v^q = 0 \quad \text{in} \quad \mathcal{B}(R), \quad v = 0 \quad \text{on} \quad \partial\mathcal{B}(R), \tag{10.2}$$

with $1 < q < \frac{N+2}{N-2}$. This assumption is natural because, from [105] it follows that (10.2) has no nontrivial solutions in case $q \geq \frac{N+2}{N-2}$. Notice also that from [65] it follows that all positive solutions of (10.2) have radial symmetry. By using critical point theory, it is proved in [40] that (10.2) has at least one positive radial solution provided $\lambda$ is sufficiently large. One the other hand, if $\lambda = 1$, it is shown in [38], by mainly using a generalization of a Liouville type theorem concerning ground states due to Ni and Serrin, that there exists a non-negative number $R^*$ such that (10.2) has at least one positive radial solution for every $R > R^*$. Notice that it has been proved in [110] that there exists $R_* > 0$ such that (10.2) has no positive radial solution when $R < R_*$. The case $q = 1$ is considered in [104] for $\lambda$ in a left neighborhood of the principal eigenvalue of $-\Delta$ in $H_0^1$, and the case $q < 1$ is considered in [67] for $\lambda$ small. Finally, in dimension one, in [68] it is given a complete description of the exact number of positive solutions of (10.2).

**Open problems.** Existence of classical positive radial solutions for Dirichlet problems of type

$$\operatorname{div}\left(\frac{\nabla v}{\sqrt{1-|\nabla v|^2}}\right) + f(|x|, v) = 0 \quad \text{in} \quad \mathcal{B}(R), \quad v = 0 \quad \text{on} \quad \partial\mathcal{B}(R),$$

where $f : [0, R] \times [0, \infty) \to \mathbb{R}$ is a continuous function, which is positive on $(0, R] \times (0, \infty)$ and satisfies some appropriate conditions inspired from the above results.

## 10.2 Generalized Robertson - Walker spacetimes

Let $I \subset \mathbb{R}$ be an open interval, $f : I \to \mathbb{R}$ be a positive smooth function and $(F, g)$ be an $n$-dimensional Riemannian manifold. The product $I \times F$ is endowed with the Lorentzian metric

$$\langle \cdot, \cdot \rangle = -\tau_I^*(dt^2) + f(\tau_I)^2 \tau_F^*(g),$$

where $\tau_I$ and $\tau_F$ denote the projections onto $I$ and $F$, respectively. One has that $(I \times F, \langle \cdot, \cdot \rangle)$ is called, following [3], a generalized Robertson - Walker spacetime.

Let $u \in C^\infty(F)$ be a function such that $u(F) \subset I$. The graph of $u$ is spacelike and has the constant mean curvature $H$ if and only if $|\nabla u| < f(u)$ and

$$\operatorname{div}\left(\frac{\nabla u}{f(u)\sqrt{f(u)^2 - |\nabla u|^2}}\right) = -nH - \frac{f'(u)}{\sqrt{f(u)^2 - |\nabla u|^2}}\left(n + \frac{|\nabla u|^2}{f(u)^2}\right),$$

see [31]. The case $f = 1$ corresponds to the Mimkowski space.

**Research line.** Prove in this context results analogous with those contained in the present work.

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