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Part I

Abstract (in English and Romanian)

1 Abstract

This habilitation thesis contains two sections. In the first one, we present the main research results we have obtained, and, in the second one, we give some future directions of research.

My research field is the numerical analysis and concerns the study of the numerical methods, especially the domain decomposition methods, and their application to various problems arising from mechanics, engineer sciences or financial field. The obtained results can be included in three research directions: *a) domain decomposition methods, b) optimal control, domain embedding methods and fast algorithms,* and *c) financial problems.*

Also, some results concerning the study of mechanical problems have been mentioned in the introduction. These results are some pointwise results, they have been obtained long time ago and were not followed by other researches, even if they were cited many times in the literature. For this reason, they are mentioned only in the introduction, and are not detailed in the thesis.

Since my PhD thesis, [2], refers to the application of the Schwarz method to plasticity problems, and this field became my main research direction, I have mentioned it in the introduction together with two papers, [3] and [4], which have been published in Numerische Mathematik and SIAM Journal on Numerical Analysis, respectively, before the defence of the thesis.

1.1 Scientific and professional results

The scientific and professional results are presented in Section 3, and they will briefly described here following the three research directions we have mentioned above.

1.1.1 Domain decomposition methods

In paper *[11] (section 3.1.1)*, the convergence of an additive Schwarz method is proved for variational inequalities given by a symmetric, coercive and continuous bilinear form. The convex set is of one-obstacle type. Also, a convergence rate depending on the decomposition of the domain is obtained.

The multiplicative algorithm introduced in [4], in which the convex set is not decomposed as a sum of convex subsets, has been is resumed in *[15] (section 3.1.2)*. In this paper, the convergence rates of the one- and two-level methods associated to that algorithm are given for two-obstacle problems. The general convergence result is derived here in a more general framework than that in [4], the Gâteaux derivative of the minimized functional being considered monotonous and Lipschitz continuous. Also, in order to get the convergence rate, a stability condition is added to the assumption introduced in [4].

The results in [15] hold for a reflexive Banach space V, but the conditions imposed to the functional F are, in general, too restrictive in comparison with the generality of the space. For instance, functional $F : W^{1,\sigma}(\Omega) \to \mathbf{R}$, $F(v) = \frac{1}{\sigma}|v|^{\sigma}_{1,\sigma}$, $|\cdot|_{1,\sigma}$ being the seminorm of $W^{1,\sigma}(\Omega)$, satisfies these conditions only if $\sigma = 2$, ie. if the space is a Hilbert space. In *[16] (section 3.1.3)*, the convergence study takes also into consideration such functionals F. First, we prove the convergence of the algorithm for functionals F which are differentiable, strictly convex and coercive. The convergence rate of the algorithm is found by imposing on F a little more restrictive conditions. Finally, we prove that the assumption made to prove the general convergence result holds for convex sets in Sobolev spaces having a certain property. The convex sets of the one- or two-obstacle type have this property.

Paper *[22] (section 3.1.4)* is a continuation of [16]. We prove that the one-, two- and multi-level multiplicative Schwarz methods obtained from the general algorithm in [16] are convergent. To this end, we prove that the assumption we made in the general convergence result holds. In these cases, we can explicitly write the convergence rate as a function of the domain decomposition and mesh parameters.

In *[23] (section 3.1.5)*, the additive method in [11] is resumed in the general background introduced in [16]. Moreover, the new proof of convergence theorem is made under more general assumptions.

In *[24] (section 3.1.6)*, the convergence of the additive and multiplicative methods is studied for inequalities containing an extra term given by an operator. The framework of the general convergence result is that in [16] or [23], and it is applied to prove the convergence of the one- and two-level methods. Besides the direct use of the algorithms for the inequalities with contraction operators, we can use these results to obtain the convergence rate of the Schwarz method for other types of inequalities or nonlinear equations. In this way, we prove the convergence and estimate the error of the one- and two-level Schwarz methods for some inequalities in Hilbert spaces which are not of the variational type. Also, the general convergence result can be applied to prove the convergence of the Schwarz method for the Navier-Stokes problem. We give conditions of existence and uniqueness of the solution for all problems we consider. We point out that these conditions and the convergence conditions of the proposed algorithms are of the same type.

In *[27] (section 3.1.7)*, we present and analyze subspace correction methods for the solution of variational inequalities of the second kind and quasi-variational inequalities, and apply these theoretical results to non smooth contact problems in linear elasticity with Tresca and non-local Coulomb friction. As in [24], we introduce these methods in a reflexive Banach space, prove that they are globally convergent and give error estimates. In the context of finite element discretization, where our methods turn out to be one- and two-level Schwarz methods, we specify their convergence rate and its dependence on the discretization parameters and conclude that our methods converge optimally.

In *[28] (section 3.1.9)*, we introduce four multigrid algorithms for the constrained minimization of non-quadratic functionals. These algorithms are combinations of additive or multiplicative iterations on levels with additive or multiplicative ones over the levels. The convex set is decomposed as a sum of convex level subsets, and consequently, the algorithms have an optimal computing complexity. We estimate the global convergence rates of the proposed algorithms as functions of the number of levels, and compare them with the convergence rates of other existing multigrid methods. We prove that these algorithms optimally converge for one-obstacle problems.

In *[26] (section 3.1.10)*, the coupling of the Navier-Stokes and Darcy equations is considered for modeling the interaction between surface and porous-media flows. The problem is formulated as an interface equation by means of the associated (nonlinear) Steklov-Poincaré operators, and the well-posedness is proved. Iterative methods to solve a conforming finite element approximation of the coupled problem are proposed and analyzed. Finally, numerical examples are given to illustrate the convergence of the proposed methods.

C. Neumann proposed in [72] an iterative method in which the solution of a Dirichlet problem in a domain $\Omega \subset \mathbf{R}^2$ is found by alternately solving two problems in two domains Ω_1 and Ω_2 whose intersection is the domain Ω, $\Omega = \Omega_1 \cap \Omega_2$. The two problems have the same equation as the initial one. The sum of the restrictions to Ω of the solutions in the two sequences converges to the solution of the problem in Ω. In *[17] (section 3.1.11)*, a generalization of this method, named the Schwarz-Neumann method, to more than two domains is propsed. We prove the convergence and the numerical stability of the algorithm. The results apply to both bounded and unbounded domains, and are given for the weak solution of an elliptic problem with mixed boundary conditions. Numerical results are given for both bounded and unbounded domains.

1.1.2 Optimal control, domain embedding methods and fast algorithms

In *[12] (section 3.2.1)*, a domain embedding method associated with an optimal boundary control problem, with boundary observations, to solve elliptic problems is proposed. We prove that the optimal boundary control problem has a unique solution if the controls are taken in a finite dimensional subspace of the space of the boundary conditions on the auxiliary domain. Using a controllability theorem due to J. L. Lions, we prove that the solutions of Dirichlet (or Neumann) problems can be approximated within any prescribed error, however small, by solutions of Dirichlet (or Neumann) problems in the auxiliary domain taking an appropriate subspace for such an optimal control problem. We also prove that the results obtained for the interior problems hold for the exterior problems, too. Some numerical examples are given for both the interior and the exterior Dirichlet problems.

In *[13] (section 3.2.2)*, analysis-based fast algorithms to solve inhomogeneous elliptic equations of three different types in three different two-dimensional domains are derived. Dirichlet, Neumann and mixed boundary value problems are treated in all these cases. These algorithms are derived from an exact formula for the solution of a large class of elliptic equations (the coefficients of the equation do not depend on the polar angle when we use the polar coordinates) based on Fourier series expansions and the solution of a one-dimensional ordinary differential equation. The performance of these algorithms are illustrated on several of these problems by numerical results.

In papers *[14]* and *[18] (section 3.2.3)* the domain embedding method is associated with a distributed control to solve boundary value problems. In [14], the method is based on formulating the problem as an optimal distributed control problem inside a rectangle in which the arbitrary domain is embedded. A periodic solution of the equation under consideration is constructed by making use of Fourier series. In [18], a domain embedding method is proposed to solve second order elliptic problems in arbitrary two-dimensional domains. The method is based on formulating the problem as an optimal distributed control problem inside a disc in which the arbitrary domain is embedded. The optimal distributed control problem inside the disc is solved by the fast algorithm given in [13].

1.1.3 Valuation of the American options

Papers *[9], [10]* and *[21] (section 3.3)* deal with theoretical study of existence and uniqueness as well as the numerical computation of the solution for the problem of the valuation of American options.

1.2 Future directions of research

In this section some of the future directions of research, which are presented in detail in Section 4, are briefly described.

1.2.1 Multigrid methods for nonlinear problems

As we have seen in the previous section, my main direction of research has been the study of the domain decomposition methods for variational inequalities, variational inequalities of the second kind, quasi-variational inequalities and inequalities which do not arise from a minimization problem. The multigrid methods are very efficient and robust and consequently, they deserve a particular study when they are applied for nonlinear problems. For this reason, one of my research direction in the future will be the extension of the one- and two-level methods in the previous section to multigrid methods. The results in two unpublished papers [30] and [29] are presented. They represent a first attempt in the study of the multigrid methods for nonlinear problems.

The multigrid methods presented in Section 3.1.9 (paper [28]) have been given for variational inequalities whose convex set is of one-obstacle type. In preprint *[30] (section 4.1.1)*, other four algorithms have been introduced for the variational inequalities with convex sets of two-obstacle type.

Paper *[29] (section 4.1.2)* is an attempt to introduce a multigrid method with level decomposition of the convex set for the variational inequalities with a contraction operator in Section 3.1.6 (paper [24]). This is an extension of the two-level method in [24] to more than two levels. The main difficulty is introduced by the condition in the convergence theorem. Even if this condition seems to be a natural one, it being similar with the existence and uniqueness condition of the solution, it will introduce an upper bound for the number of levels we can use in the multigrid method. Maybe another approach of the convergence proof or other conditions imposed to the operator will solve this problem, but it remains an open problem so far.

1.2.2 Domain decomposition methods for Navier-Stokes equation and for saddle point problems

The convex sets of the problems we introduced so far are of the one- and two-obstacle type, or they have a little more general property. In this section we succintly discuss the application of the Schwarz methods to problems whose convex set is not of these types, like Navier-Stokes problem or saddle point problems. It is evident that the verification of the assumptions made in the general convergence theory can not be made by using unity partitions associated to the decomposition of the domain, as in the previous sections.

In *Section 4.2*, we first discuss the Schwarz method for the Navier-Stokes equation whose convergence has been proved in Section 3.1.6. At the end, we introduce a saddle point formulation of the plasticity problem with hardening. In [2], the iterative Uzawa's method (which decouples the stresses and the hardening parameter from the displacements) associated with the Schwarz method have been used to solve this problem. We hope to provide in the future more direct domain decomposition methods.

2 Rezumat

Această teză de abilitare conţine două secţiuni. In prima secţiune, prezentăm principalele rezultate de cercetare pe care le-am obţinut, iar, în secţiunea a doua, descriem unele direcţii de cercetare viitoare.

Domeniul meu de cercetare este analiza numerică şi priveşte studiul metodelor numerice, în special metodele de descompunerea domeniilor, şi aplicarea lor la diverse probleme provenind din mecanică, ştiinţele inginereşti şi domeniul financiar. Rezultatele obţinute pot fi incluse în trei direcţii de cercetare: *a) metode de descompunerea domeniilor, b) control optimal, metode de scufundarea domeniilor şi algorotmi rapizi* şi *c) probleme financiare*.

De asemenea, unele rezultate privind studiul unor probleme din mecanică au fost menţionate în introducerea tezei. Aceste rezultate sunt punctuale, au fost obţinute cu mult timp în urmă şi nu au fost urmate de alte cercetări, chiar dacă au fost citate de multe ori în literatură. Din acest motiv, ele sunt menţionate numai în introducere şi nu sunt detaliate în teză.

Deoarece teza mea de doctorat, [2], se referă la aplicarea metodei lui Schwarz la probleme de plasticitate, iar acest domeniu a devenit principala mea direcţie de cercetare, am menţionat-o în introducere împreună cu două lucrari, [3] şi [4], care au fost publicate în Numerische Mathematik şi SIAM Journal on Numerical Analysis înainte de susţinerea tezei.

2.1 Rezultate ştiinţifice şi profesionale

Rezultatele ştiinţifice şi profesionale sunt prezentate în Secţiunea 3, iar în acest rezumat vor fi descrise pe scurt conform cu cele trei direcţii de cercetare menţionate mai sus.

2.1.1 Metode de descompunerea domeniilor

In lucrarea *[11] (secţiunea 3.1.1)*, se demonstrează convergenţa unei metode Schwarz aditive pentru inegaliaţi variaţionale date de o formă biliniară, simetrică, coercivă şi continuă. Mulţimea convexă este de tip un-obstacol. De asemenea, se obţine o rată de convergenţă depinzând de descompunerea domeniului.

Algoritmul multiplicativ introdus în [4], în care mulţimea convexă nu este descompusă ca o sumă de submulţimi convexe, a fost reluat în *[15] (secţiunea 3.1.2)*. In această lucrare, ratele de convergenţă ale metodelor cu unul sau două nivele de discretizare asociate algoritmului sunt date pentru probleme cu două obstacole. Rezultatul de convergenţă general este demonstrat în această lucrare într-un cadru mai general decât acel din [4], derivata Gâteaux a funcţionalei de minimizat este considerată ca fiind monotonă şi Lipschitz continuă. De asemenea, pentru a obţine rata de convergenţă, o condiţie de stabilitate este adugată ipotezei introdusă în [4].

Rezultatele din [15] sunt demonstrate pentru un spaţiu Banach reflexiv V, dar condiţiile impuse funcţionalei F sunt în general prea restrictive în comparaţie cu generalitatea spaţiului. De exemplu, funcţionala $F : W^{1,\sigma}(\Omega) \to \mathbf{R}$, $F(v) = \frac{1}{\sigma}|v|^{\sigma}_{1,\sigma}$, $|\cdot|_{1,\sigma}$ fiind seminorma din $W^{1,\sigma}(\Omega)$, satisface aceste condiţii numai dacă $\sigma = 2$, adică, dacă spaţiul este Hilbert. In *[16] (secţiunea 3.1.3)*, studiul convergenţei ia în considerare şi astfel de funcţionale F. Mai întâi, se demonstrează convergenţa algoritmului pentru funcţionale F care sunt diferenţiabile, strict convexe şi coercive. Rata de convergenţă a algoritmului este gasită impunând condiţii puţin mai restrictive asupra lui F. In sfârşit, se demonstrează că ipoteza facută pentru a demonstra rezultatul de convergenţă general este satisfacută pentru mulţimi convexe din spaţii Sobolev având o anumită proprietate. Mulţimile convexe de tipul unul- sau două-obstacole au această proprietate.

Lucrarea *[22] (secţiunea 3.1.4)* este o continuare a lucrării [16]. In această lucrare se demonstrează că metodele Schwarz multiplicative cu unul sau două nivele de discretizare obţinute din algoritmul general din [16] sunt convergente. Pentru aceasta, demonstrăm că ipoteza facută în rezultatul de convergenţă general este satisfacută. In aceste cazuri, putem scrie explicit rata de convergenţă în funcţie de parametrii de descompunere a domeniului şi cei ai reţelelor de discretizare.

In *[23] (secţiunea 3.1.5)*, se reia metoda aditivă din [11] în cadrul general introdus în [16]. In plus, demonstraţia teoremei de convergenţă este facută în ipoteze mai generale.

In *[24] (secţiunea 3.1.6)*, se studiază convergenţa metodelor multiplicative şi aditive pentru inegalităţi conţinând un termen dat de un operator. Cadrul de demonstraţie al rezultatului de convergenţă general este cel din [16] sau [23], iar acest rezultat este aplicat pentru demonstrarea convergenţei metodelor cu unul sau două nivele de discretizare. Pe lângă folosirea directă a algoritmilor pentru inegalităţi cu operatori de contracţie, aceste rezultate se pot utiliza pentru obţinerea ratelor de convergenţă ale metodei Schwarz pentru alte tipuri de inegalităţi sau ecuaţii neliniare. In acest fel, se demonstrează convergenţa şi se estimează eroarea metodelor Schwarz cu unul sau două nivele de discretizare pentru unele inegalităţi dintr-un spaţiu Hilbert care nu sunt de tip variaţional. De asemenea, rezultatul general de convergenţă poate fi aplicat pentru a demonstra convergenţa metodei Schwarz pentru problema Navier-Stokes. In sfârşit, în lucrare se dau condiţii de existenţă şi unicitate a soluţiei pentru problemele considerate. Subliniem faptul că aceste condiţii şi condiţiile de convergenţă ale algoritmilor propuşi sunt de acelaş tip.

In *[27] (secţiunea 3.1.7)*, sunt prezentate şi analizate metode de corecţii pe subspaţii pentru soluţia inegalităţilor variaţionale de speţa a doua şi a inegalităţilor quasi-variaţionale. Aceste rezultate teoretice sunt aplicate la probleme de contact, cu frecare Tresca şi Coulomb ne locală, din elasticitatea liniară. Ca şi în [24], aceste metode sunt introduse într-un spaţiu Banach reflexiv, se demonstrează că ele sunt global convergente şi se da o estimare a erorii. In contextul discretizării prin elemente finite, unde metodele introduse devin metode Schwarz cu unul sau două nivele de discretizare, se calculează rata de convergenţă, dependenţa ei de parametrii de dicretizare şi se ajunge la concluzia că metodele converg optimal.

In *[28] (secţiunea 3.1.9)*, sunt introduşi patru algoritmi multigrid pentru minimizarea cu restricţii a funcţionalelor ne pătratice. Aceşti algoritmi sunt combinaţii de iterări aditive sau multiplicative pe nivele de discretizare cu iterări aditive sau multiplicative între nivele. Mulţimea convexă este descompusă ca o sumă de submulţimi convexe de nivel, şi în consecinţă, algoritmii au o complexitate de calcul optimală. Ratele de convergenţă globală ale algoritmilor propuşi sunt estimate în funcţie de numarul de nivele de discretizare şi sunt comparate cu ratele de convergenţă ale altor metode multigrid existente. Se demonstrează că aceşti algoritmi converg optimal pentru probleme cu un singur obstacol.

In *[26] (secţiunea 3.1.10)*, modelarea interacţiunii dintre curgerea de suprafaţă şi cea dintr-un mediu poros a unui fluid este facută prin cuplarea ecuaţiei Navier-Stokes cu cea a lui Darcy. Problema este formulată ca o ecuaţie de interfaţă cu ajutorul operatorilor Steklov-Poincaré neliniari şi se demonstrează că problema este bine pusă. Se propun şi se analizează metode iterative pentru rezolvarea aproximării prin elemente finite conforme a problemei cuplate. In sfârşit, sunt date exemple numerice pentru a ilustra convergenţa metodelor propuse.

C. Neumann a propus în [72] o metodă iterativă în care soluţia unei probleme Dirichlet într-un domeniu $\Omega \subset \mathbf{R^2}$ este găsită prin rezolvarea alternativă a două probleme din două domenii Ω_1 şi Ω_2 a căror intersecţie este domeniul Ω, $\Omega = \Omega_1 \cap \Omega_2$. Cele două probleme au aceiaşi ecuaţie ca şi cea iniţială. Suma restricţiilor la Ω a soluţiilor din cele două şiruri obţinute converge la soluţia problemei din Ω. In *[17] (secţiunea 3.1.11)*, se propune o generalizare la mai mult de două domenii a acestei metode, numită metoda Schwarz-Neumann. Se demonstrează convergenţa şi stabilitatea numerică a acestui algoritm. Rezultatele obţinute se pot aplica atât la domenii mărginite cât şi la cele nemărginite şi sunt date pentru soluţia slabă a unei probleme eliptice cu condiţii la limtă mixte. Rezultate numerice sunt date pentru ambele tipuri de domenii, mărginite şi nemărginite.

2.1.2 Control optimal, metode de scufundarea domeniului si algoritmi rapizi

In *[12] (secţiunea 3.2.1)*, pentru rezolvarea problemelor eliptice, se propune asocierea unei metode de scufundarea domeniilor cu o metodă de control frontieră optimal având observaţii pe frontiera domeniului problemei date. Se demonstrează că problema de control optimal frontieră are o soluţie unică dacă controlul este luat într-un subspaţiu finit dimensional al spaţiului condiţiilor la limită de pe domeniul auxiliar. Folosind o teoremă de controlabilitate datorată lui J. L. Lions, se demonstrează că soluţiile problemelor Dirichlet (sau Neumann) pot fi aproximate cu orice eroare, oricât de mică, cu soluţii ale problemelor Dirichlet (sau Neumann) din domeniul auxiliar folosind un subspaţiu adecvat pentru o astfel de problemă de control optimal. De asemenea, se demonstrează că rezultatele obţinute pentru probleme interioare sunt valabile şi pentru probleme exterioare. In lucrare, se dau exemple numerice pentru rezolvarea unor probleme Dirichlet, atât interioare cât şi exterioare.

In *[13] (secţiunea 3.2.2)*, sunt introduşi algoritmi rapizi pentru rezolvarea unor ecuaţii eliptice neomogene, de trei tipuri diferite şi în trei variante de domenii bidimensionale. In toate cazurile, problemele pot avea condiţii la limită de tip Dirichlet, Neumann şi mixt. Aceşti algoritmi au la bază soluţia exactă a unei clase largi de ecuaţii eliptice (coeficienţii ecuaţiei nu depind de unghiul polar când se folosesc coordonatele polare) bazată pe dezvoltări în serii Fourier şi pe soluţii ale unor ecuaţii difernţiale ordinare. Performanţa acestor algoritmi este ilustrată prin rezultate numerice pentru mai multe astfel de probleme.

In lucrările *[14]* şi *[18] (secţiunea 3.2.3)*, metoda scufundării domeniilor este asociată cu un control distribuit pentru a rezolva probleme la limită. In [14], metoda este bazată pe formularea problemei ca o problemă de control optimal distribuit în interiorul unui dreptunghi în care domeniul problemei este scufundat. Pe domeniul dreptunghiular, se consideră soluţii periodice folosind seriile Fourier. In [18], se propune o metodă de scufundarea domeniilor pentru rezolvarea problemelor eliptice de ordinul doi în domenii bidimensionale arbitrare. Metoda se bazează pe formularea problemei ca o problemă de control optimal distribuit în interiorul unui disc în care domeniul arbitrar al problemei este scufundat. Problema de control optimal de pe disc este rezolvată prin algoritmul rapid introdus în [13].

2.1.3 Evaluarea opţiunilor de tip american

Lucrările *[9], [10]* şi *[21] (secţiunea 3.3)* se ocupa cu studiul teoretic privind existenţa şi unicitatea precum şi calculul numeric al soluţiei problemei evaluării opţiunilor de tip american.

2.2 Direcţii de cercetare viitoare

In această secţiune vor fi descrise pe scurt câteva din direcţiile de cercetare viitoare care sunt detaliate în Secţiunea 4.

2.2.1 Metode multigrid pentru probleme neliniare

Aşa cum s-a văzut în secţiunea anterioară, principala mea direcţie de cercetare a fost studiul metodelor de descompunerea domeniilor pentru inegalităţi variaţionale, inegalităţi variaţionale de speţa a doua, inegalităţi quasivariaţionale şi inegalităţi care nu provin dintr-o problemă de minimizare. Considerăm că metodele multigrid sunt foarte eficiente şi robuste şi de aceea credem că merită un studiu deosebit pentru aplicarea lor la probleme neliniare. Din acest motiv, una din direcţiile de cercetare în viitor va fi extinderea metodelor cu unul sau două nivele de disretizare din secţiunea precedentă la mnetode multigrid. In continuare, vom prezenta rezultatele din două lucrări nepublicate, [30] şi [29]. Ele reprezintă o primă încercare de studiu al metodelor multigrid pentru probleme neliniare.

Metodele multigrid prezentate în Secţiunea 3.1.9 (lucrarea [28]) au fost introduse pentru inegalitaţi variaţionale a căror mulţime convexă este de tipul un-obstacol. In preprintul *[30] (secţiunea 4.1.1)*, au fost introduşi alţi patru algoritmi pentru inegalitaţi variaţionale având mulţimea convexă de tipul două-obstacole.

Lucrarea *[29] (secţiunea 4.1.2)* este o încercare de a introduce o metodă multigrid cu descopunere pe nivele a mulţimii convexe pentru inegalităţile variaţionale cu un operator de contracţie din Secţiunea 3.1.6 (lucrarea [24]). Aceasta este o extindere a metodei cu două nivele de discretizare din [24] la o metoda cu mai mult de două nivele. Principala dificultate este introdusă de condiţia din teorema de convergenţă. Chiar dacă această condiţie pare să fie naturală, fiind similară cu condiţia de existenţă şi unicitate a soluţiei, ea va introduce o limită superioară pentru numărul de nivele de discretizare pe care le putem utiliza în metoda multigrid. Poate o altă abordare a demonstraţiei convergenţei sau alte condiţii impuse operatorului vor rezolva această problemă, dar ea este o problema deschisă în prezent.

2.2.2 Metode de descopunerea domeniilor pentru ecuaţia Navier-Stokes şi pentru probleme de punct şa

Mulţimile convexe ale problemelor pe care le-am introdus până acum sunt de tipul unul- sau două-obstacole, sau au o proprietate puţin mai generală. In această secţiune disctăm pe scurt aplicarea metodelor Schwarz la probleme a căror multime convexă nu este de aceste tipuri, cum ar fi problema Navier-

Stokes sau problemele de punct şa. Este evident că verificarea ipotezelor făcute în teoria generală de convergenţă nu poate fi realizată folosind partiţii ale unităţii asociate descompunerii domeniului, ca în secţiunile anterioare.

In *Secţiunea 4.2*, discutăm mai întâi metoda Schwarz pentru ecuaţia Navier-Stokes a cărei convergeţă a fost demonstrată în Secţiunea 3.1.6. La sfârşit, introducem o formulare de punct şa pentru problema de plasticitate cu ecruisare. In [2], pentru rezolvarea acestei probleme s-a folosit metoda interativă a lui Uzawa (care decuplează eforturile şi parametrul de ecruisaj de deplasări) asociată cu metoda lui Schwarz. Sperăm să putem propune metode de descopunerea domeniilor mai directe decât aceasta.

Part II

Scientific and Professional Results and Future Directions of Research

Mainly, my research is of numerical analysis and concerns the study of the numerical methods and their application to various problems arising from mechanics, engineer sciences or financial field. In this thesis, the obtained research results are summarized in three directions:

1. Domain decomposition methods
2. Optimal control, domain embedding methods and fast algorithms
3. Financial problems

It would exist a fourth direction given by the study of some mechanical problems. The most important results are obtained in [6], [8], [7] and [19].

Papers *[6]* and *[8]* deal with the dynamical cavitation phenomenon in viscoplastic materials. We consider a hollow sphere submitted to a symmetric traction loading in the dynamical case. The initial void radius is considered to be infinitesimal, and consequently, we shall take this radius as vanishing. We find an expression for the critical load, at which the void grows rapidly without bound ("cavitation instability"), and we give some theoretical results concerning the dependence of the solution on this critical load and various initial conditions.

In *[7]*, we give a three dimensional generalization of the one dimensional elasto-plastic theory, which was elaborated by A. Chrysochoos, O. Maisonneuve and their collaborators, concerning the evolution of the stored energy ratio. The theoretical results in the paper shows that the stored energy ratio is an increasing function of the time during a torsion test. The same behavior of the stored energy ratio was been obtained in the one dimensional theory concerning the axial deformation in a traction test.

Paper *[19]* aims to investigate the distribution of the first and second moments of the mechanical fields in polycrystals. The material is viewed as a composite with a large number of anisotropic phases. An efficient semianalytical procedure is proposed for the computation of the intra-phase fields variance.

As we see, these results are some pointwise results, they have been obtained long time ago and were not followed by other researches, even if they were cited many times in the literature. For this reason, they are mentioned only here, and we will not give details on them in the following.

Now, I would mention in this introduction my PhD thesis *[2]* and the papers *[3]* and *[4]*, because they are related to the Schwarz method and this topic has became my main research direction. The PhD thesis had as a topic the application of the Schwarz method to the elasto-plastic problem, and the board of examiners has been composed by: G. Duvaut (Chairman), P. Le Tallec (Referee), Q. S. Nguyen (Referee), P. Ladevèze (Examiner), P. L. Lions (Examiner), J. F. Maitre (Examiner) and M. Predeleanu (Examiner). To verify the effectiveness of the proposed method, my colleague from LMT Pierre Gilormini solved the numerical examples in thesis with ABAQUS (the parallelizable version). The comparison of the results led to the conclusion that the proposed algorithm based on Schwarz method has a lower CPU time when using more than four processors. These results have also been published in [5]. The above mentioned two papers have been published before the thesis defence, and, since then, they have been cited in the literature many times: [3] is cited in 13 papers (5 citations in journals with an influence relative score greater than 0.5) and [4] is cited in 36 papers (14 citations in journals with an influence relative score greater than 0.5). Seeing the title of [3], "A generalization of the Schwarz alternating method to an arbitrary number of subdomains", the today's reader might have serious doubts concerning the novelty of the paper, but the paper has been submitted to Numerische Mathematik in 1987, the same year in which The First International Conference on Domain Decomposition Methods has been held in Paris, and where P. L. Lions presented his seminal paper [64]. We would point out that the proof techniques of the convergence are different in the two papers. In paper [4] the convergence of the Schwarz method for variational inequalities is proved, in both variants, with and without the decomposition of the convex set of the problem. The proof uses a maximum principle, and, in the case of a general convex set a certain assumption is made. This assumption is very useful for the convergence proof (for the estimation of the convergence rate, too) of the algorithms based on the Schwarz method and will be resumed, in an easily modified and strengthened form, in all subsequent papers.

3 Scientific and professional results

In this section, only the main and significant results I have obtained are presented according with the above mentioned directions of research. (see List of Publications for a complete list of papers).

3.1 Domain decomposition methods

The study of the overlapping domain decomposition methods for nonlinear problems has been one of my constant research concerns.These methods have originated in the Schwarz method published in [74] (see also [75]). In [66], a Schwarz method without overlap has been introduced, and then, many variants and improvements of this method have been obtained (see [68], for instance).

Literature on the domain decomposition methods is very large, and it is motivated by their capability in providing efficient and parallelizable algorithms for large scale problems. We can see, for instance, the papers in the proceedings of the annual conferences on domain decomposition methods starting in 1987 with [41] or those cited in the books [55], [73], [76] and [81]. Naturally, most of the papers dealing with these methods are dedicated to the linear elliptic problems. For the variational inequalities, the convergence proofs refer in general to the inequalities coming from the minimization of quadratic functionals. Also, most of papers consider the convex set decomposed according to the space decomposition as a sum of convex subsets. To our knowledge very few papers deal with the application of these methods to nonlinear problems. We can cite in this direction the papers written by Tarvainen [80], for one-obstacle problems, Boglaev [32], Dryja and Hackbusch [36], Lui [57], [58] and [59], Tai and Espedal [78], and Tai and Xu [79], for nonlinear equations, Hoffmann and Zhou [47], Zeng and Zhou [85], for inequalities having nonlinear source terms.

The multilevel or multigrid methods can be viewed as domain decomposition methods. For the constrained minimization of functionals, these methods have been studied almost exclusively for the complementarity problems. Such a method has been proposed by Mandel in [69], [70] and [39]. Related methods have been introduced by Brandt and Cryer in [33] and Hackbush and Mittelmann in [45]. The method has been studied later by Kornhuber in [53] and extended to variational inequalities of the second kind in [54] and [55]. A variant of this method using truncated nodal basis functions has been introduced by Hoppe and Kornhuber in [48] and analyzed by Kornhuber and Yserentant in [56]. Also, versions of this method have been applied to Signorini's problem in elasticity by Kornhuber and Krause in [52] and Wohlmuth and Krause in [82]. Evidently, the above list of citations is not exhaustive and, for further information, we recommend the review article [44] written by Gräser and Kornhuber. A global convergence rate has been also estimated by Tai in [77] for a subset decomposition method.

We start this section with the convergence study of the domain decomposition methods with overlapping for variational inequalities, variational inequalities of the second kind, quasi-variational inequalities, and other inequalities related to them. We first prove the convergence and estimate the error of subspace correction algorithms in a general (usually, reflexive Banach) space, provided that the convex set verifies a certain assumption. In the case of the Sobolev spaces, our algorithms are domain decomposition methods, and we prove that the introduced assumption holds. In the case of the one- two- and multilevel methods the convergence rates are written in terms of the mesh and overlapping parameters. Some applications of these methods are given for contact problems with friction, and in particular, for geophysical problems concerning the initiation phase of rupture in a earthquake. Also, the convergence of the Schwarz method for the Navier-Stokes equation is derived from a general theory.

Next, we present a result concerning the existence and uniqueness of the solution for the Navier-Stokes/Darcy coupled problem. Also, the convergence of several methods (fixed-point, Newton and preconditioned Richardson methods) for this problem is proved and illustrated by numerical experiments.

Finally, a generalization to more than two domains of an almost unknown method, the Schwarz-Neumann method, is presented at the end of the section. In this method, the domain of the problem is the intersection of some larger domains, and, like in the Schwarz method, the values of the solution on a domain are obtained in function of the solution on the other domains. This method has been proposed by C. Neumann in [72], and, to our knowledge, the last reference, except [17], on this method was in the book [52].

3.1.1 Additive Schwarz method (paper [11])

In paper [11], the convergence of an additive Schwarz method is proved for variational inequalities given by a symmetric, coercive and continuous form. The convex set is of one-obstacle type. Also, a convergence rate depending on the decomposition of the domain is obtained.

First, a general framework is considered. In a Hilbert space V we consider the problem

$$u \in K : a(u, v-u) \geq f(v-u) \text{ for any } v \in K \tag{1}$$

where $a : V \times V \to \mathbf{R}$ is a bilinear, symmetric, continuous and coercive form, and $f \in V'$, V' being the dual of V. We consider $V_1, \ldots, V_m$ are some closed subspaces of V, and to solve this problem, the following subspace correction algorithm is proposed

Algorithm 3.1 *We start the algorithm with an arbitrary $u^0 \in K$. At iteration $n+1$, having $u^n \in K$, $n \geq 0$, we compute $w_i^{n+1} \in V_i$, $u^n + w_i^{n+1} \in K$, the solution of the inequality*

$$a(u^n + w_i^{n+1}, v_i - w_i^{n+1}) \geq f(v_i - w_i^{n+1}), \text{ for any } v_i \in V_i,\ u^n + v_i \in K \tag{2}$$

for $i = 1, \cdots, m$, and then we update $u^{n+1} = u^n + \varrho \sum_{i=1}^{m} w_i^{n+1}$, where parameter ϱ is selected such that $u^{n+1} = u^n + \varrho \sum_{i=1}^{m} w_i^{n+1} \in K$. In particular, we can take $0 < \varrho \leq \frac{1}{m}$ for any $n \geq 0$.

To prove the convergence of this general algorithm we need some assumptions. The first one can be written as

Assumption 3.1 *There exists a constant $C_0 > 0$ such that for any $w, v \in K$ there exist $v_i \in V_i$, $i = 1, \cdots, m$, which satisfy*

$$v - w = \sum_{i=1}^{m} v_i, \quad w + v_i \in K, \quad \sum_{i=1}^{m} ||v_i||^2 \leq C_0 ||v - w||^2.$$

The last condition in the above assumption is essential in finding the convergence rate, and, in the case of the linear problems, is well-known as the stability condition of the decomposition of the space as a sum of subspaces. The convergence proof uses the techniques proposed by P. L. Lions, where the projection operator defined by the bilinear form a plays an important role. For this reason, another assumption has been introduced,

Assumption 3.2 *Problem (2) is equivalent to finding $w_i^{n+1} \in V_i$, $u^n + w_i^{n+1} \in K$, the solution of the inequality*

$$a(u^n + w_i^{n+1}, v_i - w_i^{n+1}) \geq a(u, v_i - w_i^{n+1}), \text{ for any } v_i \in V_i,\ u^n + v_i \in K$$

where u is the solution of problem (1).

We shall see in the presentation of the next papers that this second assumption can be avoided.

Now, for $v \in K$ and $i = 1, \ldots .m$, let $P_i(u - v)$ be the projection of $u - v$ in $(K - v) \cap V_i$, where u is the solution of (1). Also, let $\tau_{ij} \in [0, 1]$, $i, j = 1, \ldots, m$, be some constants such that

$$(P_i(u - v), P_i(u - w)) \leq \tau_{ij} ||P_i(u - v)|| ||P_i(u - w)|| \text{ for any } v, w \in K$$

Denoting by $|\tau|$ the l^2 norm of the matrix $(\tau_{ij})_{i,j=1}^{m}$, we have

Theorem 3.1 *Let u^n, $n \geq 0$, be the approximation sequence obtained from Algorithm 3.1, and let u be the solution of problem (1). Assume that Assumptions 3.1 and 3.2 are satisfied. Then, there exists a fixed $\tau_0 \in (0,1)$ such that*

$$||u^{n+1} - u||^2 \leq \tau_0 ||u^n - u||^2 \text{ for any } n \geq 0$$

provided that the parameter ϱ is sufficiently small.

In the above theorem,

$$\tau_0 = 1 - 2\rho(2 + C_0)^{-1} + \rho^2 |\tau|^2.$$

This general convergence result is used to prove that Algorithm 3.1 is geometrically convergent for one-obstacle problems. Let Ω be an open bounded domain in $\mathbf{R}^n$, $n \in \mathbf{N}$, with Lipschitz continuous boundary $\Gamma = \partial\Omega$. Assume that $\partial\Omega = \bar{\Gamma}_1 \cup \bar{\Gamma}_2$, $\Gamma_1 \cap \Gamma_2 = \emptyset$, is a partition of the boundary such that $\text{meas}(\Gamma_1) > 0$. We consider problem (1) in the the Sobolev space

$$V = \{v \in H^1(\Omega) : v = 0 \text{ on } \Gamma_1\} \tag{3}$$

with the convex set

$$K = \{v \in V : v \geq 0\}, \tag{4}$$

and

$$a(u, v) = \int_\Omega \nabla u \cdot \nabla v. \tag{5}$$

We associate to a decomposition of the domain Ω,

$$\Omega = \cup_{i=1}^m \Omega_i \tag{6}$$

the subspaces $V_1, \ldots, V_m$,

$$V_i = \{v_i \in V : v = 0 \text{ in } \Omega \backslash \Omega_i\} \tag{7}$$

The parameter ρ is not necessarily a constant in the Schwarz method. For example, one can use any smooth positive function $\rho = \rho(x)$ in Algorithm 3.1. The following lemma provides a useful criterion for the selection of ρ.

Lemma 3.1 *For any $x \in \Omega$, let $N(x)$ be the number of subdomains containing x. If ρ is selected as a smooth positive function such that*

$$\rho(x) N(x) \leq 1 \text{ for any } x \in \Omega$$

then the approximation u^{n+1} in Algorithm 3.1 is a function in the convex set K. Also, the convergence proved in Theorem 3.1 holds in this case if

$$1 - 2\rho_1(2 + C_0)^{-1} + \rho_2^2|\tau|^2 = \tau_0 \leq 1$$

where $\rho_1 = \min_x \rho(x)$ and $\rho_2 = \max_x \rho(x)$.

Also, it is proved in [11] that Assumption 3.2 holds for the above one-obstacle problem. To prove Assumption 3.1, we can take $v_i = \theta_i(v - w)$ where $\theta_i \in C^1(\bar{\Omega}$ and

$$0 \leq \theta_i \leq 1, i = 1, \ldots, m, \text{ and } \sum_{i=1}^{m} \theta_i = 1 \text{ in } \Omega \tag{8}$$

Now, let $\mathcal{T}_h$ be a simplicial regular mesh partition of mesh size h over this domain. We assume that $\mathcal{T}_h$ supplies a mesh partition for each subdomain Ω_i, $i = 1, \ldots, m$. In this case, we can consider the linear finite element spaces V_h, V_h^i, $i = 1, \ldots, m$, and the convex set K_h corresponding to spaces in (3), (7) and (4), respectively, and prove that Assumptions 3.2 and 3.1 hold. For Assumption 3.1, we take $v_i = L_h(\theta_i(v - w))$, where L_h is the Lagrangian interpolations, and the functions θ_i, $i = 1, \ldots, m$, in (8) can be taken as $\theta_i \in C(\bar{\Omega})$, $\theta_{i|\tau} \in P^1(\tau)$ for any $\tau \in \mathcal{T}_h$.

3.1.2 One- and two-level multiplicative methods (paper [15])

The multiplicative algorithm introduced in [4], in which the convex set is not decomposed as a sum of convex subsets, has been is resumed in [15]. In this paper, the convergence rates of the one- and two-level methods associated to that algorithm are given for two-obstacle problems. The general convergence result is derived here in a more general framework than that in [4], the Gâteaux derivative of the minimized functional being considered monotonous and Lipschitz continuous. Also, in order to get the convergence rate, a stability condition is added to the assumption introduced in [4].

We consider a reflexive Banach space V, $V_1, \ldots, V_m$ some closed subspaces of V, and $F : V \to \mathbf{R}$ a convex and Gâteaux differentiable functional. We also assume that there exist two positive constants α and β such that the

Gâteaux derivative of F, F', satisfies

$$\alpha||v-u||^2 \le \langle F'(v)-F'(u), v-u\rangle \text{ for any } u,v\in V \text{ and}$$
$$\sum_{i,j=1}^{m} |\langle F'(w_{ij}+u_i)-F'(w_{ij}), v_j\rangle| \le \beta(\sum_{i=1}^{m}||u_i||^2)^{\frac{1}{2}}(\sum_{j=1}^{m}||v_j||^2)^{\frac{1}{2}} \tag{9}$$
$$\text{for any } w_{ij}\in V,\ u_i\in V_i \text{ and } v_j\in V_j$$

Given a closed convex set $K \subset V$, we consider the problem

$$u \in K : \langle F'(u), v-u\rangle \ge 0 \text{ for any } v \in K, \tag{10}$$

and the following multiplicative algorithm is introduced to solve it,

Algorithm 3.2 *We start the algorithm with an arbitrary* $u^0 \in K$. *At iteration* $n+1$, *having* $u^n \in K$, $n \ge 0$, *we compute sequentially for* $i = 1, \cdots, m$, $w_i^{n+1} \in V_i$, $u^{n+\frac{i-1}{m}} + w_i^{n+1} \in K$ *satisfying*

$$\langle F'(u^{n+\frac{i-1}{m}} + w_i^{n+1}), v_i - w_i^{n+1}\rangle \ge 0, \ \textit{for any } v_i \in V_i,\ u^{n+\frac{i-1}{m}} + v_i \in K,$$

and then we update $u^{n+\frac{i}{m}} = u^{n+\frac{i-1}{m}} + w_i^{n+1}$.

To prove the convergence of this algorithm and find its convergence rate, we need

Assumption 3.3 *There exists a constant* $C_0 > 0$ *such that for any* $w, v \in K$ *and* $w_i \in V_i$ *with* $w + \sum_{j=1}^{i} w_j \in K$, $i = 1, \cdots, m$, *there exist* $v_i \in V_i$, $i = 1, \cdots, m$, *satisfying*

$$w + \sum_{j=1}^{i-1} w_j + v_i \in K, \quad v - w = \sum_{i=1}^{m} v_i, \tag{11}$$

$$\sum_{i=1}^{m} ||v_i||^2 \le C_0 \left(||v-w||^2 + \sum_{i=1}^{m} ||w_i||^2 \right). \tag{12}$$

This assumption is a little more complicate than that in the additive case, Assumption 3.1, because it contains the corrections w_i, $i = 1, \ldots, m$. The assumption containing only conditions (11) in Assumption 3.3 has been introduced in [4]. These two conditions suffice to prove the convergence, but the third condition, the stability condition, is essential in finding the convergence rate. In fact, constant C_0 is the essential parameter in the expression of this rate. We have the following convergence result.

Theorem 3.2 *Assume that the space decomposition satisfies Assumption (3.3), and assume that the functional F satisfies (9). Then for the iterative approximation $\{u^n\}_{n=1}^{\infty}$ given by Algorithm 3.2, we have*

$$\left|F(u^{n+1}) - F(u)\right| \leq \frac{\tilde{C}_1}{\tilde{C}_1 + 1}|F(u^n) - F(u)| \tag{13}$$

and

$$||u^n - u||^2 \leq \frac{2}{\alpha}\left[\frac{\tilde{C}_1}{\tilde{C}_1 + 1}\right]^n \left|F(u^0) - F(u)\right|, \tag{14}$$

where

$$\tilde{C}_1 = (\sqrt{1 + C^*} + \sqrt{C^*})^2 - 1 \textit{ with } C^* = \frac{2}{\alpha}\left((1 + C_0^{\frac{1}{2}})\beta + \frac{C_0\beta^2}{2\alpha}\right). \tag{15}$$

From the error estimation in the above theorem, we see that the convergence rate depends almost exclusively on the constant C_0. Indeed, besides the constant C_0, this rate depends on the constants α and β, ie. on the properties of the functional F. Constant β may also depend on the number m of subdomains, but it can be assimilated with the minimum number of colors needed to color the subdomains such that two subdomains of the same color do not intersect with each other. The general result of the above theorem is used to find the convergence rate of the one- and two-level methods for two-obstacle problems in linear finite element spaces.

One- and two-level method for two-obstacle problems. Let $\Omega \subset \mathbf{R}^d$, $d = 1, 2$ or 3, be an open bounded and connected domain with a polyhedral boundary. Consider the problem that seeks an unknown function $u \in H_0^1(\Omega)$ satisfying inequality (1) with the bilinear form a given in (5) and the convex set of the two-obstacle type

$$K = \{v \in H_0^1(\Omega)| : \varphi(x) \leq v(x) \leq \psi(x) \text{ a.e. in } \Omega\}, \tag{16}$$

$\varphi(x)$ and $\psi(x)$ are two obstacle functions in $H_0^1(\Omega)$. We consider that the domain Ω is decomposed as in (6), and let $\mathcal{T}_h$ be a simplicial regular mesh partition of mesh size h over this domain. We assume that $\mathcal{T}_h$ supplies a mesh partition for each subdomain Ω_i, $i = 1, \ldots, m$, and the overlapping parameter of the domain decomposition is δ. We associate to Ω and its decomposition the piecewise linear finite element spaces

$$\begin{aligned} V_h &= \{v \in C^0(\bar{\Omega}) : v|_\tau \in P_1(\tau),\ \tau \in \mathcal{T}_h,\ v = 0 \text{ on } \partial\Omega\} \text{ and} \\ V_h^i &= \{v \in V_h : v = 0 \text{ in } \Omega \backslash \Omega_i\}, \quad i = 1, \ldots, m. \end{aligned} \tag{17}$$

In the case of the two-level method, a new mesh partition $\mathcal{T}_H$ is introduced, $\mathcal{T}_h$ being a refinement of $\mathcal{T}_H$. Also, we assume that $\mathrm{diam}(\Omega_i) \leq CH$, $i = 1, \ldots, m$, C being independent of the mesh partitions. Corresponding to the coarse mesh partition, we introduce a new piecewise linear finite element,

$$V_H^0 = \left\{v \in C^0(\bar{\Omega}_0) : v|_\tau \in P_1(\tau),\ \tau \in \mathcal{T}_H,\ v = 0 \text{ on } \partial\Omega_0\right\}. \tag{18}$$

The spaces V_h V_H^0 and V_h^i, $i = 1, \ldots, m$, are considered as subspaces of $H^1(\Omega)$. The obstacle problem is approximated by a finite element function $u_h(x) \in K_h$ satisfying

$$a(u_h, v_h - u_h) \geq f(v_h - u_h) \text{ for any } v_h \in K_h,$$

where K_h is an approximation of K in (16) with two obstacles $\varphi_h,\ \psi_h \in V_h$.

The one-level Schwarz method is obtained from Algorithm 3.2 with the spaces $V = V_h$ and $V_i = V_h^i$, $i = 1, \ldots, m$, but we have $m + 1$ subspaces, $V_1 = V_H^0$, $V_2 = V_h^1, \ldots, V_{m+1} = V_h^m$, in the case of the two-level method. To prove that Assumption 3.3 holds for the one-level method, we associate to the domain decomposition (6) some functions $\theta_i \in C(\bar{\Omega})$, $\theta_{i|_\tau} \in P_1(\tau)$ for any $\tau \in \mathcal{T}_h$, $i = 1, \ldots, m$, such that

$$\begin{aligned} &0 \leq \theta_i \leq 1 \text{ on } \Omega, \\ &\theta_i = 0 \text{ on } (\cup_{j=i+1}^m \Omega_j) \backslash \Omega_i \text{ and } \theta_i = 1 \text{ on } \Omega_i \backslash (\cup_{j=i+1}^m \Omega_j). \end{aligned} \tag{19}$$

Such functions θ_i, $i = 1, \ldots, m$, with the above properties have been introduced in [4] and they are constructed using unity partitions of the domains $\cup_{j=i}^m \Omega_j$. Since the overlapping size of the domain decomposition is δ, the above functions θ_i can be chosen to satisfy

$$|\partial_{x_k} \theta_i| \leq C/\delta, \text{ a.e. in } \Omega, \text{ for any } k = 1, \ldots, d. \tag{20}$$

Conditions (11) in Assumption 3.3 have been proved in [4] for the continuous case. To prove them in the finite element case we follow a similar way. Let $v, w \in K$ and $w_i \in V_i$ satisfy $w + \sum_{j=1}^i w_j \in K_h$. We define v_i recursively by

$$v_i = L_h\left(\theta_i\left(v - w - \sum_{j=1}^{i-1} v_j\right) + (1 - \theta_i) w_i\right), \quad i = 1, \ldots, m, \tag{21}$$

where L_h is the Lagrangian interpolation. By repeating the proof in [4], we obtain the following result.

Lemma 3.2 *Let us assume that* v, w, $w + \sum_{j=1}^{i} w_j \in K_h$, $w_i \in V_i$ *for* $i = 1, \dots, m$, *and* v_i, $i = 1, \dots, m$, *are defined as in* (21). *Then we have*

$$\begin{aligned}
& v_i \in V_i, \quad w + \sum_{j=1}^{i-1} w_j + v_i \in K_h, \\
& v - w - \sum_{j=1}^{i} v_j \in H_0^1\left(\cup_{j=i+1}^{m} \Omega_j\right), \\
& v - w - \sum_{j=1}^{i} v_j = 0 \ \textit{in}\ \Omega \backslash \overline{\cup_{j=i+1}^{m} \Omega_j}, \\
& v - \sum_{j=1}^{i} v_j + \sum_{j=1}^{i} w_j \in K_h.
\end{aligned} \tag{22}$$

Using (20) and (21) we can find constant C_0, and we have

Proposition 3.1 *For the one-level method, Assumption 3.3 holds, for spaces defined in (17) and the two-obstacle convex set* K_h, *in which the constant*

$$C_0 = C(1 + \frac{1}{\delta}).$$

To prove that Assumption 3.3 holds for the two-level method, an operator $I_H : V_h \to V_H^0$, named nonlinear interpolation operator, has been used. Let us denote by x^i a node of $\mathcal{T}_H$, by ϕ_i the linear nodal basis function associated with x^i and $\mathcal{T}_H$, and by ω_i the support of ϕ_i. Given a $v \in V_h$, let us write

$$I_i^- v = \min_{x \in \omega_i} v(x)^- \text{ and } I_i^+ v = \min_{x \in \omega_i} v(x)^+, \tag{23}$$

where $v(x)^- = \max(0, -v(x))$ and $v(x)^+ = \max(0, v(x))$. Since v is piecewise linear, $I_i^- v$ or $I_i^+ v$ are attained at a node of $\mathcal{T}_h$ if they are not zero. For a $v \in V_h$, we define

$$I_H^- v := \sum_{x^i \text{ node of } \mathcal{T}_H} (I_i^- v) \phi_i(x) \text{ and } I_H^+ v := \sum_{x^i \text{ node of } \mathcal{T}_H} (I_i^+ v) \phi_i(x), \tag{24}$$

and we write

$$I_H v = I_H^+ v - I_H^- v. \tag{25}$$

This operator has not been explicitly introduced in [15], but some of its properties have been found using another nonlinear interpolation operator. We shall resume the properties of I_H in [22] in a more general framework.

To prove conditions (11) in Assumption 3.3, let us consider $v, w \in K_h$ and $w_i \in V_i$ satisfy $w + \sum_{j=0}^{i} w_j \in K_h$, for $i = 0, 1, \ldots, m$. We define

$$v_0 = w_0 + I_H(v - w - w_0), \tag{26}$$

and, similarly with (21),

$$v_i = L_h\Bigg(\theta_i\Bigg(v - w - \sum_{j=0}^{i-1} v_j\Bigg) + (1-\theta_i)w_i\Bigg) \text{ for } i = 1, \ldots, m. \tag{27}$$

Evidently, $v_0 \in V_H^0$ and we can prove that

$$w + v_0,\ v + w_0 - v_0 \in K_h. \tag{28}$$

and, similarly with (22),

$$\begin{aligned}
& v_i \in V_h^i \text{ and } w + \sum_{j=0}^{i-1} w_j + v_i \in K_h, \\
& v - w - \sum_{j=0}^{i} v_j \in H_0^1(\cup_{j=i+1}^m \Omega_j) \text{ and} \\
& v - w - \sum_{j=0}^{i} v_j = 0 \text{ in } \Omega \backslash \overline{\cup_{j=i+1}^m \Omega_j}, \\
& v - \sum_{j=0}^{i} v_j + \sum_{j=0}^{i} w_j \in K_h,
\end{aligned} \tag{29}$$

for $i = 1, \ldots, m$. Using the above result, the properties of I_H and of the functions θ_i, we can prove

Proposition 3.2 *Assumption 3.3 holds with the constant*

$$C_0 = C(1 + \frac{H}{\delta})C_d(H, h)$$

for the two-level method. Constant C is a generic constant independent of the mesh parameters, and

$$C_d(H,h) = \begin{cases} 1 & \textit{if } d = 1 \\ \left(\ln \frac{H}{h} + 1\right)^{\frac{1}{2}} & \textit{if } d = 2 \\ \left(\frac{H}{h}\right)^{\frac{1}{2}} & \textit{if } d = 3, \end{cases} \tag{30}$$

We see that, in the case of the two-level method, even if $h \to 0$ but we keep the ratios H/h and H/δ constant, the number of iterations to reach a certain error should be constant. This fact is confirmed be the numerical experiments given in the paper for a two-obstacle problem,

$$\begin{array}{l} u \in H_0^1(\Omega),\ \varphi \le u \le \psi : \\ \int_\Omega \nabla u \nabla (v-u) \ge 0 \text{ for any } v \in H_0^1(\Omega),\ \varphi \le v \le \psi, \end{array} \tag{31}$$

where $\varphi(x)$ and $\psi(x)$ are two obstacle functions in $H_0^1(\Omega)$ and $\Omega \subset \mathbf{R}^2$.

3.1.3 Convergence results in more general settings (paper [16])

The results in [15] hold for a reflexive Banach space V, but the conditions imposed to the functional F are, in general, too restrictive in comparison with the generality of the space. We shall see at the end of this subsection that $F : W^{1,\sigma}(\Omega) \to \mathbf{R}$, $F(v) = \frac{1}{\sigma}|v|_{1,\sigma}^{\sigma}$, $|\cdot|_{1,\sigma}$ being the seminorm of $W^{1,\sigma}(\Omega)$, satisfies (9) only if $\sigma = 2$. In [16], the convergence study of Algorithm 3.2 for problem (10) takes also into consideration such functionals F. First, we prove the convergence of Algorithm 3.2 for functionals F which are differentiable, strictly convex and coercive. Some properties of such functionals have been proved in [42], Lemmas 1.1 and 1.2, and used in Theorem 1.2 to prove the convergence of the methods of pointwise or block relaxation for the constrained minimization of the functionals in $\mathbf{R}^n$. The convergence rate of the algorithm is found by imposing on F a little more restrictive conditions. Finally, we prove that the assumption made to prove the general convergence result holds for convex sets in Sobolev spaces having a certain property. The convex sets of the one- or two-obstacle type have this property.

Given a convex set K of reflexive Banach space V, let us write for a $M > 0$,

$$l_M = \sup_{u,v \in K, ||u||,||v|| \le M} ||v - u||.$$

We assume that functional $F : V \to \mathbf{R}$ is Gâteaux differentiable, and that for any real number $M > 0$, there exist two functions $\delta_M,\ \gamma_M\ :\ [0, l_M] \to \mathbf{R}^+$, such that

$$\begin{array}{l} \delta_M \text{ is strictly increasing, } \frac{\delta_M(\tau)}{\tau} \to 0 \text{ as } \tau \to 0, \text{ and } \delta_M(0) = 0, \\ \gamma_M \text{ is continuous at 0 and } \gamma_M(0) = 0, \end{array} \tag{32}$$

and for any $u, v \in K$, $||u||,\ ||v|| \leq M$,

$$\begin{array}{l} \langle F'(v) - F'(u), v - u\rangle \geq \delta_M(||v - u||) \text{ and} \\ \gamma_M(||v - u||) \geq ||F'(v) - F'(u)||_{V'}. \end{array} \tag{33}$$

Also, we assume that, if the convex set K is not bounded, F is coercive in the sense that

$$F(v) \to \infty \text{ as } ||v|| \to \infty,\ v \in K$$

The first condition in (33) holds if and only if the functional F is strictly convex (see [38], Proposition 5.5 and Lemma 1.1 in [42]). Also, the second condition implies the continuous differentiability of F. It is evident that if (33) holds, then for any $u, v{\in}K$, $||u||, ||v||{\leq}M$, we have

$$\delta_M(||v - u||) \leq < F'(v) - F'(u), v - u > \leq \gamma_M(||v - u||)||v - u||. \tag{34}$$

Following the way in [42] (Lemmas 1.1 and 1.2), we can prove that for any $u, v \in K$, $||u||, ||v|| \leq M$, we have

$$\begin{array}{l} < F'(u), v - u > +\lambda_M(||v - u||) \leq F(v) - F(u) \leq \\ < F'(u), v - u > +\mu_M(||v - u||), \end{array} \tag{35}$$

where

$$\lambda_M(\tau) = \int_0^\tau \delta_M(\theta)\frac{d\theta}{\theta}, \quad \mu_M(\tau) = \int_0^\tau \gamma_M(\theta) d\theta. \tag{36}$$

A convergence result can be obtained under a weaker assumption than Assumption 3.3

Assumption 3.4 *For any $w, v \in K$ and $w_i \in V_i$ with $w + \sum_{j=1}^{i} w_j \in K$, $i = 1, \cdots, m$, there exist $v_i \in V_i$, $i = 1, \cdots, m$, satisfying*

$$\begin{array}{l} w + \displaystyle\sum_{j=1}^{i-1} w_j + v_i \in K, \quad v - w = \sum_{i=1}^{m} v_i \quad \textit{and} \\ \textit{application } V \times V_1 \times \cdots \times V_m \to V_1 \times \cdots \times V_m, \\ (v - w, w_1, \cdots, w_m) \to (v_1, \cdots, v_m) \textit{ is bounded,} \\ \textit{i.e. it transforms the bounded sets in some bounded sets.} \end{array}$$

With this assumption, we have

Theorem 3.3 *We consider that V is a reflexive Banach, $V_1, \cdots, V_m$ are some closed subspaces of V, K is a non empty closed convex subset of V, and F is Gâteaux differentiable functional on K which is assumed to be coercive if K is not bounded. If Assumption 3.4 holds, and for any $M > 0$ there exist two functions δ_M and γ_M satisfying (32) then, for any $i = 1, \cdots, m$, $u^{n+\frac{i}{m}} \to u$, strongly in V, as $n \to \infty$, where u is the solution of problem (10) and $u^{n+\frac{i}{m}}$ are given by Algorithm 3.2 starting from an arbitrary u^0.*

The error estimate essentially stands on the convergence order of the functions $\delta_M(\tau)$ and $\gamma_M(\tau)$ to zero as $\tau \to 0$. In the following we take these functions of polynomial form

$$\delta_M(\tau) = \alpha_M \tau^p, \quad \gamma_M(\tau) = \beta_M \tau^{q-1}, \tag{37}$$

where $\alpha_M > 0$, $\beta_M > 0$, $p > 1$ and $q > 1$ are some real constants. Consequently, we assume that for any $u, v \in K$, $||u||$, $||v|| \le M$,

$$\begin{array}{l} \alpha_M ||v-u||^p \le \langle F'(v) - F'(u), v-u\rangle \text{ and} \\ ||F'(v) - F'(u)||_{V'} \le \beta_M ||v-u||^{q-1}. \end{array} \tag{38}$$

Also, in view of (36) we get

$$\lambda(\tau) = \tfrac{\alpha_M}{p}\tau^p, \quad \mu(\tau) = \tfrac{\beta_M}{q}\tau^q, \tag{39}$$

and using (37), equations (34) and (35) can be written as

$$\alpha_M ||v-u||^P \le < F'(v) - F'(u), v-u > \le \beta_M ||v-u||^q, \tag{40}$$

and

$$\begin{array}{l} < F'(u), v-u > + \frac{\alpha_M}{p}||v-u||^p \le F(v) - F(u) \le \\ < F'(u), v-u > + \frac{\beta_M}{q}||v-u||^q, \end{array} \tag{41}$$

respectively. We have marked here that the constants α_M and β_M may depend on M, and we see from (40) that $p \ge q$. Naturally, the convergence rate will depend on the spaces $V_1, \cdots, V_m$, and we shall consider the following form of Assumption 2.1 having the third condition strengthened, like in Assumption 3.3 in [15],

Assumption 3.5 *There exists a constant C_0 such that for any $w, v \in K$ and $w_i \in V_i$ with $w + \sum_{j=1}^{i} w_j \in K$, $i = 1, \cdots, m$, there exist $v_i \in V_i$, $i = 1, \cdots, m$, satisfying*

$$w + \sum_{j=1}^{i-1} w_j + v_i \in K, \quad v - w = \sum_{i=1}^{m} v_i \quad \textit{and}$$

$$\sum_{i=1}^{m} ||v_i||^p \leq C_0 \left(||v - w||^p + \sum_{i=1}^{m} ||w_i||^p \right). \tag{42}$$

The following theorem is a generalization to nonlinear inequalities of the result in [79] concerning the convergence of the method for nonlinear equations.

Theorem 3.4 *On the conditions of Theorem 3.3 we consider the functions δ_M and γ_M defined in (37) and we make Assumption 3.5. If u is the solution of problem (10) and u^n, $n \geq 0$, are its approximations obtained from Algorithm 3.2, then we have the following error estimations:*

(i) if $p = q$ we have

$$\begin{array}{l} F(u^n) - F(u) \leq \left(\frac{\tilde{C}_1}{\tilde{C}_1+1} \right)^n \left[F(u^0) - F(u) \right], \\ ||u^n - u||^p \leq \frac{\tilde{C}_1+1}{\tilde{C}_3} \left(\frac{\tilde{C}_1}{\tilde{C}_1+1} \right)^n \left[F(u^0) - F(u) \right]. \end{array} \tag{43}$$

(ii) if $p > q$ we have

$$\begin{array}{l} F(u^n) - F(u) \leq \frac{F(u^0)-F(u)}{\left[1+n\tilde{C}_2 (F(u^0)-F(u))^{\frac{p-q}{q-1}} \right]^{\frac{q-1}{p-q}}}, \\ ||u - u^n||^p \leq \frac{\tilde{C}_1}{\tilde{C}_3} \frac{\left(F(u^0)-F(u) \right)^{\frac{q-1}{p-1}}}{\left[1+(n-1)\tilde{C}_2 (F(u^0)-F(u))^{\frac{p-q}{q-1}} \right]^{\frac{(q-1)^2}{(p-1)(p-q)}}}. \end{array} \tag{44}$$

Constants $\tilde{C}_1$, $\tilde{C}_3$ and $\tilde{C}_2$ can be written as

$$\tilde{C}_1 = \beta_M (\tfrac{p}{\alpha_M})^{\frac{q}{p}} m^{2-\frac{q}{p}} \left[(1 + 2C_0^{\frac{1}{p}}) \left(F(u^0) - F(u) \right)^{\frac{p-q}{p(p-1)}} + \right.$$
$$\left. \left(\beta_M (\tfrac{p}{\alpha_M})^{\frac{q}{p}} m^{2-\frac{q}{p}} \right)^{\frac{1}{p-1}} C_0^{\frac{1}{p-1}} / \eta^{\frac{1}{p-1}} \right] / (1 - \eta),$$

$$\tilde{C}_3 = \frac{(2-\eta)\alpha_M}{(1-\eta)p},$$

$$\tilde{C}_2 = \frac{p-q}{(p-1)\left(F(u^0)-F(u)\right)^{\frac{p-q}{q-1}} + (q-1)\tilde{C}_1^{\frac{p-1}{q-1}}},$$

where $\eta \in (0,1)$ is a fixed constant.

Multiplicative Schwarz method Let Ω be an open bounded domain in $\mathbf{R}^d$ with Lipschitz continuous boundary $\partial\Omega$. We take $V = W_0^{1,\sigma}(\Omega)$, $1 < \sigma < \infty$, and a convex closed set $K \subset V$ satisfying

Property 3.1 *If $v, w \in K$, and if $\theta \in C^1(\Omega)$ with $0 \le \theta \le 1$, then $\theta v + (1-\theta)w \in K$.*

Evidently, the one- and two-obstacle convex sets have the above property. In general, the convex sets defined by the values of the functions, but not by the values of their derivatives, have this property. We associate to the domain decomposition (6) the subspaces $V_i = W_0^{1,\sigma}(\Omega_i)$, $i = 1, \cdots, m$. In this case, Algorithm 3.2 represents a multiplicative Schwarz method for inequality (10). Using the functions θ_i, $i = 1, \ldots, m$, defined in (19) and (20) we have

Proposition 3.3 *Assumption 3.5 holds for any convex set K having Property 3.1.*

Now, let us consider the inequality

$$u \in K : \int_\Omega |\nabla u|^{\sigma-2}\nabla u \nabla(v-u) \ge f(v-u), \text{ for any } v \in K. \tag{45}$$

where $1 < \sigma < \infty$, $K \subset V \equiv W_0^{1,\sigma}(\Omega)$ be a closed and convex set having Property 3.1 and $f \in V'$. This inequality is of the form (10) with

$$F(v) = \frac{1}{\sigma}\int_\Omega |\nabla v|^\sigma - f(v).$$

We know (see [43]) that if $1 < \sigma \le 2$, then there exist two positive constants α and β such that

$$< F'(v) - F'(u), v-u > \ge \alpha \frac{||v-u||_{1,\sigma}^2}{(||v||_{1,\sigma} + ||u||_{1,\sigma})^{2-\sigma}},$$

and

$$\beta||v-u||_{1,\sigma}^{\sigma-1} \geq ||F'(v)-F'(u)||_{V'},$$

for any $v,u \in W_0^{1,\sigma}(\Omega)$. Consequently, the functions introduced in (37) can be written as

$$\delta_M(\tau) = \frac{\alpha}{(2M)^{2-\sigma}}\tau^2 \text{ and } \gamma_M(\tau) = \beta\tau^{\sigma-1},$$

and therefore,

$$\alpha_M = \frac{\alpha}{(2M)^{2-\sigma}},\ \beta_M = \beta,\ p = 2 \text{ and } q = \sigma$$

in (37).

If $\sigma \geq 2$, then there exist two positive constants α and β such that (see [34])

$$< F'(v)-F'(u), v-u >\geq \alpha||v-u||_{1,\sigma}^{\sigma},$$

and

$$\beta(||v||_{1,\sigma}+||u||_{1,\sigma})^{\sigma-2}||v-u||_{1,\sigma} \geq ||F'(v)-F'(u)||_{V'},$$

for any $v,u \in W_0^{1,\sigma}(\Omega)$. Therefore, for a given $M > 0$, we have

$$\delta_M(\tau) = \alpha\tau^{\sigma} \text{ and } \gamma_M(\tau) = \beta(2M)^{\sigma-2}\tau,$$

and therefore,

$$\alpha_M = \alpha,\ \beta_M = \beta(2M)^{\sigma-2},\ p = \sigma \text{ and } q = 2$$

in (37). We can conclude from the above comments that Algorithm 3.2 can be applied for the solving of problem (45) if the convex set K has Property 3.1. Naturally, the error estimations in Theorem 3.4 hold. Solution of problem (45) for $\sigma = 1.5$, $\sigma = 2.$ and $\sigma = 3$ have been given in this paper. The domain Ω was a rectangle in $\mathbf{R}^2$ and the convex set has been of two-obstacle type, $K = [\varphi, \psi]$, with $\varphi,\ \psi \in W_0^{1,\sigma}(\Omega)$, $\varphi \leq \psi$.

3.1.4 One-, two- and multilevel methods for problems in $W^{1,\sigma}(\Omega)$ (paper [22])

Paper [22] is a continuation of [16]. We prove that the one-, two-level and multilevel multiplicative Schwarz methods obtained from Algorithm 3.2 converge for a problem (10) in which F satisfies the general properties (38) and the convex set K has a similar property with Property 3.1. To this end, we prove that Assumption 3.5 also holds for any closed convex K satisfying that property. In these cases the dependence of C_0 on the domain decomposition and mesh parameters can be explicitly written. Consequently, provided that functional F satisfies (38), Algorithm 3.2 converges and we can apply Theorem 3.4 to get the convergence rate.

As in [15], we consider a domain $\Omega \subset \mathbf{R}^d$ which is decomposed as in (6) with an overlapping parameter of size δ. Let $\mathcal{T}_h$ be a simplicial regular mesh partition of mesh size h over Ω, and we assume that $\mathcal{T}_h$ supplies a mesh partition for each subdomain Ω_i, $i = 1, \ldots, m$. The convex K_h is defined as a subset of V_h satisfying

Property 3.2 *If $v, w \in K_h$, and if $\theta \in C^1(\Omega)$ with $0 \leq \theta \leq 1$, then $L_h(\theta v + (1-\theta)w) \in K_h$.*

Above, L_h is the Lagrangian interpolation. Evidently, the one- and two-obstacle convex sets have this property. The finite element spaces we shall use will be considered as subspaces of $W^{1,\sigma}$, for some fixed $1 \leq \sigma \leq \infty$. We denote by $||\cdot||_{0,\sigma}$ the norm in L^σ, and by $||\cdot||_{1,\sigma}$ and $|\cdot|_{1,\sigma}$ the norm and seminorm in $W^{1,\sigma}$, respectively.

One-level method. In the case of the one-level method, we use the linear finite element space V_h and, associated to the domain decomposition, the subspaces $V_h^1, \ldots, V_h^m$ defined in (17). Using the functions θ_i, $i = 1, \ldots, m$ introduced in (19) and (20), with a proof similar with that of Proposition 3.1, we get

Proposition 3.4 *Assumption 3.5 holds for the piecewise linear finite element spaces, $V = V_h$ and $V_i = V_h^i$, $i = 1, \cdots, m$, and for any convex set $K = K_h \subset V_h$ having Property 3.2. The constant in (42) of Assumption 3.5 can be taken of the form*

$$C_0 = C(m+1)(1 + \frac{m-1}{\delta}), \tag{46}$$

where C is independent of the mesh parameter and the domain decomposition.

Two-level method. In the case of the two-level method, we consider in addition another coarse simplicial regular mesh partition of mesh size H of Ω, $\mathcal{T}_H$, besides the fine one, $\mathcal{T}_h$. We assume that $\mathcal{T}_h$ is a refinement of $\mathcal{T}_H$. As for the one-level method, we assume that $\mathcal{T}_h$ supplies a mesh partition for each Ω_i, $1 \leq i \leq m$, and the overlapping parameter of the domain decomposition is δ. Also, we assume that $\operatorname{diam}(\Omega_i) \leq CH$, $i = 1, \ldots, m$, C being independent of the mesh partitions. The domain Ω may be different from

$$\Omega_0 = \cup_{\tau \in \mathcal{T}_H} \tau, \tag{47}$$

but we assume that if a node of $\mathcal{T}_H$ lies on $\partial\Omega_0$, then it lies on $\partial\Omega$, too, and $\operatorname{dist}(x, \Omega_0) \leq CH$ for any node x of $\mathcal{T}_h$, C being independent of both meshes.

We consider the linear finite element space V_h and its subspaces associated to the domain decomposition, $V_h^1, \ldots, V_h^m$, defined in (17), and also, the finite element space associated to the coarse mesh, V_H^0, defined in (18). Since these finite element spaces are considered as subspaces of $W^{1,\sigma}$, we need some results similar to those obtained for H^1. First, we have the following lemma in which inequality (48) can be viewed as one of Friedrichs-Poincaré type for the finite element spaces.

Lemma 3.3 *Let $\omega \subset \mathbf{R}^d$ be a domain of diameter H, and ω_i, $i = 0, 1, \cdots, N$, be an overlapping decomposition of it, $\omega = \cup_{i=0}^N \omega_i$. We consider a simplicial regular mesh partition $\mathcal{T}_h$ of ω and assume that it supplies a mesh partition for each ω_i, $i = 0, 1, \cdots, N$, too. Let $x^0 \in \bar{\omega}_0$ be a node of $\mathcal{T}_h$. We assume that the overlapping partition of ω satisfies:*

(i) for any $x \in \bar{\omega}_0$, the line segment $[x^0, x]$ lies in $\bar{\omega}_0$,

(ii) for $N > 0$, if $\omega_i \cap \omega_j \neq \emptyset$, $0 \leq i \neq j \leq N$, then for any $x \in \bar{\omega}_i$, $y \in \bar{\omega}_j$ and $z \in \bar{\omega}_i \cap \bar{\omega}_j$, the line segments $[x, z]$ and $[y, z]$ lie in $\bar{\omega}_i$ and $\bar{\omega}_j$, respectively.

On these conditions, if v is a continuous function which is linear on each $\tau \in \mathcal{T}_h$, and $v(x^0) = 0$, then

$$||v||_{0,\sigma,\omega} \leq C(N, \sigma) C(d, \sigma) H C_{d,\sigma}(H, h) |v|_{1,\sigma,\omega}, \tag{48}$$

where

$$C_{d,\sigma}(H, h) = \begin{cases} 1 & \text{if } d = \sigma = 1 \text{ or } 1 \leq d < \sigma \leq \infty \\ \left(\ln \frac{H}{h} + 1\right)^{\frac{d-1}{d}} & \text{if } 1 < d = \sigma < \infty \\ \left(\frac{H}{h}\right)^{\frac{d-\sigma}{\sigma}} & \text{if } 1 \leq \sigma < d < \infty, \end{cases} \tag{49}$$

$$C(d,\sigma)=\begin{cases} C & \textit{if } d=\sigma=1 \textit{ or } 1=\sigma<d<\infty \\ C\left(d\frac{\sigma-1}{\sigma-d}\right)^{\frac{\sigma-1}{\sigma}} & \textit{if } 1\le d<\sigma\le\infty \\ Cd^{\frac{d-1}{d}} & \textit{if } 1<d=\sigma<\infty \\ C(d\frac{\sigma-1}{d-\sigma})^{\frac{\sigma-1}{\sigma}} & \textit{if } 1<\sigma<d<\infty. \end{cases} \tag{50}$$

and

$$C(N,\sigma)=\begin{cases} 1 & \textit{if } N=0 \\ \textit{if } (N+1)\frac{C_\omega^{(N+1)/\sigma}-1}{C_\omega^{1/\sigma}-1} & \textit{if } N\neq 0 \end{cases} \tag{51}$$

with

$$C_\omega=\max_{\omega_i\cap\omega_j\neq\emptyset}\frac{|\omega_i|}{|\omega_i\cap\omega_j|} \tag{52}$$

In (52) we have denoted by $|\quad|$ the measure of a set, and we have marked in (48) that the norm in L^σ and the semi-norm in $W^{1,\sigma}$, $1\le\sigma\le\infty$, refer to the domain ω. The constant C in (50) is independent of H, h, d, σ and the decomposition of ω.

Remark 3.1 *In general, since the mesh $\mathcal{T}_h$ is regular, the overlapping decomposition of ω in Lemma 3.3 can be taken such that the number N and the constant C_ω in (52) are bounded and independent of H and h. In this point of view, the constants $C(d,\sigma)$, $C(N,\sigma)$ and C_ω, written in (50)–(52), can be considered as independent of H and h, and assimilated to the generic constant C. In the following we write (48) as*

$$||v||_{0,\sigma,\omega}\le CHC_{d,\sigma}(H,h)|v|_{1,\sigma,\omega}, \tag{53}$$

where $C=C(N,\sigma)C(d,\sigma)$ and $C_{d,\sigma}(H,h)$ is given in (49). We point out that, for $\sigma=2$, $C_{d,\sigma}(H,h)$ is given in (49) coincides with $C_d(H,h)$ in (30) and the above inequality is well-known in H^1.

The above lemma can be very useful in various error estimations. The following result, for instance, extends to $W^{1,s}$ a known result in H^1.

Corollary 3.1 *Let ω be a domain of diameter H and having a simplicial regular mesh partition $\mathcal{T}_h$. If v is a continuous function which is linear on each $\tau\in\mathcal{T}_h$, and $v=0$ on $\partial\omega$, then for any $1\le\sigma\le\infty$ we have*

$$||v||_{0,\infty,\omega}\le CH^{\frac{\sigma-d}{\sigma}}C_{d,\sigma}(H,h)|v|_{1,\sigma,\omega}, \tag{54}$$

where $C_{d,\sigma}(H,h)$ is given in (49), and C is independent of H and h.

As in [15], the nonlinear interpolation operator defined in (25) is used to prove that Assumption 3.5 holds. Its properties, using the norm and seminorm in $W^{1,\sigma}$, are proved using Lemma 3.3. In this way, we have

Lemma 3.4 *For any $v \in V_h$ we have*

$$||I_H v - v||_{0,\sigma,\Omega_0} \leq CHC_{d,\sigma}(H,h)|v|_{1,\sigma,\Omega_0} \tag{55}$$

and

$$||I_H v||_{0,\sigma,\Omega_0} \leq C||v||_{0,\sigma,\Omega_0} \text{ and } |I_H v|_{1,\sigma,\Omega_0} \leq CC_{d,\sigma}(H,h)|v|_{1,\sigma,\Omega_0} \tag{56}$$

where Ω_0 is the union of the simplexes in $\mathcal{T}_H$ written in (47), $C_{d,\sigma}(H,h)$ is defined in (49), and C is independent of H, h and δ. Equations (392) and (387) also hold if Ω_0 is replaced by Ω. Moreover, if $\mathcal{K}$ is a convex and closed set in V_h having Property 3.2, with $0 \in \mathcal{K}$, then for any $v \in \mathcal{K}$ we have $I_H v \in \mathcal{K} \cap V_H^0$.

Now, we can prove the following proposition which shows that the constant C_0 in Assumption 3.5 is independent of the mesh and domain decomposition parameters if H/δ and H/h are constant. This result is similar to that given in [15] for the inequalities coming from minimization of the quadratic functionals. In the first part of the proof, the construction of v_i, $i = 1, \cdots, m$, is similar to that given for the one-level method. In the second part we define an appropriate v_0 using the previous lemma.

Proposition 3.5 *Assumption 3.5 is verified for the piecewise linear finite element spaces, $V = V_h$ $V_i = V_h^i$, $i = 1, \cdots, m$, and $V_0 = V_H^0$, defined in (17), and (18), respectively, and any convex set $K = K_h$ satisfying Property 3.2. The constant in (42) of Assumption 3.5 can be taken of the form*

$$C_0 = C(m+2)^{1-\frac{1}{p}} \left(1 + (m-1)\frac{H}{\delta}\right) C_{d,\sigma}(H,h), \tag{57}$$

where C is independent of the mesh and domain decomposition parameters, and $C_{d,\sigma}(H,h)$ is given in (49).

Multilevel method. In the case of the multilevel method, we consider a family of regular meshes $\mathcal{T}_{h_j}$ of mesh sizes h_j, $j = 1, \dots, J$ over the domain $\Omega \subset \mathbf{R}^d$. We write

$$\Omega_j = \cup_{\tau \in \mathcal{T}_{h_j}} \tau \tag{58}$$

and we assume that $\mathcal{T}_{h_{j+1}}$ is a refinement of $\mathcal{T}_{h_j}$ on Ω_j, $j = 1, \dots, J-1$, and $\Omega_1 \subset \Omega_2 \subset \dots \subset \Omega_J = \Omega$. Also, we assume that, if a node of $\mathcal{T}_{h_j}$ lies on $\partial\Omega_j$, then it lies on $\partial\Omega_{j+1}$, too, that is, it lies on $\partial\Omega$. Besides, we suppose that $\mathrm{dist}_{x_{j+1} \text{node of } \mathcal{T}_{h_{j+1}}}(x_{j+1}, \Omega_j) \leq C h_j,\ j = 1, \dots, J-1$. Here, and in the following, C denotes a generic positive constant independent of the mesh sizes, the number of meshes, as well as of the overlapping parameters and the number of subdomains in the domain decompositions. Since the mesh $\mathcal{T}_{h_{j+1}}$ is a refinement of $\mathcal{T}_{h_j}$, we have $h_{j+1} \leq h_j$, and assume that there exists a constant γ, independent of the number of meshes or their sizes, such that $1 < \gamma \leq \frac{h_j}{h_{j+1}} \leq C\gamma, \quad j = 1, \dots, J-1$. At each level $j = 1, \dots, J$, we consider an overlapping decomposition $\{\Omega_j^i\}_{1 \leq i \leq I_j}$ of Ω, and assume that the mesh partition $\mathcal{T}_{h_j}$ of Ω_j supplies a mesh partition for each Ω_j^i, $1 \leq i \leq I_j$. Also, we assume that the overlapping size for the domain decomposition at the level $1 \leq j \leq J$ is δ_j. Since $h_{j+1} \leq \delta_{j+1}$, we have $\frac{h_j}{\delta_{j+1}} \leq C\gamma, \quad j = 1, \dots, J-1$. In addition, we suppose that there exists a constant C such that if ω_{j+1}^i is a connected component of Ω_{j+1}^i, $j = 1, \dots, J-1$, $i = 1, \dots, I_j$, then $\mathrm{diam}(\omega_{j+1}^i) \leq C h_j$. Finally, we assume that $I_1 = 1$ and write $I = \max_{1 \leq J} I_j$.

At each level $j = 1, \dots, J$, we introduce the linear finite element spaces,

$$V_{h_j} = \{v \in C(\bar{\Omega}_j) : v|_\tau \in P_1(\tau),\ \tau \in \mathcal{T}_{h_j},\ v = 0 \text{ on } \partial\Omega_j\}, \tag{59}$$

and, for $i = 1, \dots, I_j$, we write

$$V_{h_j}^i = \{v \in V_{h_j} : v = 0 \text{ in } \Omega_j \backslash \Omega_j^i\}. \tag{60}$$

The functions in V_{h_j} $j = 1, \dots, J-1$, will be extended with zero outside Ω_j. Since $\mathcal{T}_{h_{j+1}}$ is a refinement of $\mathcal{T}_{h_j}$, $j = 1, \dots, J-1$, we have $V_{h_1} \subset V_{h_2} \subset \dots \subset V_{h_J}$.

In order to prove that Assumption 3.5 holds for the convex set $K = K_{h_J}$ and the spaces $V = V_{h_J}$, $V_j^i = V_{h_j}^i$, $j = 1, \cdots, L$, $i = 1, \cdots, m_j$, and to find the constant C_0 in (42) as a function of the domain decomposition and mesh parameters, we need the following lemma. This result generalizes to more than two levels the second inequality (56) in Lemma 3.4. To this end, we introduce operators $I_{h_k} : V_{h_{j+1}} \to V_{h_j}$, $j = 1, \cdots, J-1$, which are similar to the operator $I_H : V_h \to V_H$ defined in (25).

Lemma 3.5 *For a given* $1 \le j < J-1$, *let* $v_k,\ w_k \in V_{h_k}$, $k = j+1, \cdots, J-1$, *such that*

$$v_k = w_k + I_{h_k}(v_{k+1}). \tag{61}$$

Then,

$$\begin{aligned} |I_{h_j} v_{j+1}|_{1,s,\Omega_j} \le C(J-j)^{\frac{s-1}{s}} \{ \sum_{k=j+1}^{J-1} C_{d,s}(h_j, h_k)^s |w_k|^s_{1,s,\Omega_j} + \\ C_{d,s}(h_j, h_J)^s |v_J|^s_{1,s,\Omega_j} \}^{\frac{1}{s}}. \end{aligned} \tag{62}$$

Moreover, (62) also holds if its seminorms over Ω_j *are replaced with seminorms over* Ω_k, *for any* $k = j+1, \cdots, J$.

The following proposition proves that Assumption 3.5 in the case of the multilevel method, too. Therefore, we can apply Theorem 3.4 to get the convergence rate of the multilevel method deduced from Algorithm 3.2. Evidently, in this case the number of subspaces is $m = I_1 + \ldots + I_J$.

Proposition 3.6 *Assumption 3.5 is verified for the piecewise linear finite element spaces,* $V = V_{h_J}$ *and* $V_j^i = V_{h_j}^i$, $j = 1, \cdots, J$, $i = 1, \cdots, I_j$ *defined in (59) and (60), respectively, and any convex set* $K = K_{h_J} \subset V_{h_J}$ *with Property 3.2. The constant in (42) of Assumption 3.5 can be taken of the form*

$$C_0 = CI^2(J+1)^{2-\frac{1}{p}-\frac{1}{s}} \sum_{j=1}^{J} [1 + (I-1)\frac{h_{j-1}}{\delta_j}] C_{d,s}(h_{j-1}, h_J) \tag{63}$$

in which we take $h_0 = h_1$, C *is independent of the mesh and domain decomposition parameters, and* $C_{d,s}(H, h)$ *is given in (49).*

In the above multilevel method a mesh is the refinement of that one on the previous level, but the domain decompositions are almost independent from a level to another one. The multigrid method is obtained from the multilevel method by taking the subsets Ω_j^i of a particular form: we associate at each node x_j^i of $\mathcal{T}_{h_j}$, $j = 1, \cdots, J$, $i = 1, \cdots, I_j$, an Ω_j^i defined as the union of the simplexes in $\mathcal{T}_{h_j}$ having x_j^i as a vertex. Consequently, the subspaces $V_{h_j}^i$ will be direct sums of some one-dimensional spaces generated by the nodal basis functions associated with the nodes of $\mathcal{T}_{h_j}$. Evidently, all the previous assumptions on the domain decompositions are satisfied and we can take $\delta_j = h_j$. In this case, if we write $h = h_1$ and denote by H the diameter of Ω, then the constant C_0 in (63) can be taken as

$$C_0 = CJ^{3-\frac{1}{p}-\frac{1}{\sigma}}\gamma C_{d,\sigma}(H,h). \tag{64}$$

3.1.5 Additive method in $W^{1,\sigma}$ (paper [23])

In [23], the additive method in [11] is resumed in the general background in [16]. Moreover, the new proof of convergence theorem does not use Assumption 3.2.

We consider a reflexive Banach space V, some closed subspaces of V, $V_1, \cdots, V_m$, and $K \subset V$ a non empty closed convex subset, and we make the following

Assumption 3.6 *There exists a constant $C_0 > 0$ such that for any $w, v \in K$ there exist $v_i \in V_i$, $i = 1, \cdots, m$, which satisfy*

$$v - w = \sum_{i=1}^{m} v_i, \quad w + v_i \in K \text{ and } \sum_{i=1}^{m} ||v_i|| \leq C_0 ||v - w||.$$

We consider a Gâteaux differentiable functional $F : V \to \mathbf{R}$, which is assumed to be coercive on K, in the sense that $F(v) \to \infty$, as $||v|| \to \infty$, $v \in K$, if K is not bounded, and its derivative satisfies (38). The following additive algorithm is introduced to solve problem (10).

Algorithm 3.3 *We start the algorithm with an arbitrary $u^0 \in K$. At iteration $n+1$, having $u^n \in K$, $n \geq 0$, we solve the inequalities*

$$\begin{aligned} & w_i^{n+1} \in V_i,\ u^n + w_i^{n+1} \in K : \ < F'(u^n + w_i^{n+1}), v_i - w_i^{n+1} > \geq 0, \\ & \text{for any } v_i \in V_i,\ u^n + v_i \in K, \end{aligned} \tag{65}$$

for $i = 1, \cdots, m$, and then we update $u^{n+1} = u^n + \rho \sum_{i=1}^{m} w_i^{n+1}$, where $\rho >$ is chosen such that $u^{n+1} \in K$ for any $n \geq 0$.

The general convergence result is similar with that in Theorem 3.4,

Theorem 3.5 *We consider that V is a reflexive Banach, $V_1, \cdots, V_m$ are some closed subspaces of V, K is a non empty closed convex subset of V satisfying Assumption 3.6, and F is a Gâteaux differentiable functional on V which is supposed to be coercive if K is not bounded, and satisfies (38). On these conditions, if u is the solution of problem (10) and u^n, $n \geq 0$, are its approximations obtained from Algorithm 3.3, then there exists $M > 0$ such that the following error estimations hold:*

(i) if $p = q$ we have

$$F(u^n) - F(u) \leq \left(\frac{\tilde{C}_1}{\tilde{C}_1 + 1} \right)^n \left[F(u^0) - F(u) \right], \tag{66}$$

$$||u^n - u||^p \leq \frac{p}{\alpha_M} \left(\frac{\tilde{C}_1}{\tilde{C}_1 + 1} \right)^n \left[F(u^0) - F(u) \right], \tag{67}$$

where

$$\tilde{C}_1 = \frac{1}{\rho} \left(1 - \rho + m^{p-1} \frac{\beta_M (1 + C_0)}{\frac{\alpha_M}{p}} + m^{p-1} \left(\frac{\beta_M C_0}{\frac{\alpha_M}{p}} \right)^{\frac{p}{p-1}} \right),$$

(ii) if $p > q$ we have

$$F(u^n) - F(u) \leq \frac{F(u^0) - F(u)}{\left[1 + n \tilde{C}_2 \left(F(u^0) - F(u) \right)^{\frac{p-q}{q-1}} \right]^{\frac{q-1}{p-q}}}, \tag{68}$$

$$||u - u^n||^p \leq \frac{p}{\alpha_M} \frac{F(u^0) - F(u)}{\left[1 + n \tilde{C}_2 \left(F(u^0) - F(u) \right)^{\frac{p-q}{q-1}} \right]^{\frac{q-1}{p-q}}}, \tag{69}$$

where

$$\tilde{C}_2 = \frac{p - q}{(p-1) \left(F(u^0) - F(u) \right)^{\frac{p-q}{q-1}} + (q-1) \tilde{C}_3^{\frac{p-1}{q-1}}},$$

with

$$\tilde{C}_3 = \tfrac{1-\rho}{\rho}\left(F(u^0)-F(u)\right)^{\frac{p-q}{p-1}} + \frac{m^{\frac{(p-1)q}{p}}}{\rho^{\frac{q}{p}}}\frac{\beta_M(1+C_0)}{\left(\frac{\alpha_M}{p}\right)^{\frac{q}{p}}}\cdot$$
$$\left(F(u^0)-F(u)\right)^{\frac{p-q}{p(p-1)}} + \frac{m^{q-1}}{\rho^{\frac{q-1}{p-1}}}\frac{(\beta_M C_0)^{\frac{p}{p-1}}}{\left(\frac{\alpha_M}{p}\right)^{\frac{q}{p-1}}}$$

For the one- and two-level methods, we can prove that if the convex set has Property 3.2, then Assumption 3.6 holds with

$$C_0 = Cm\left(1+1/\delta\right), \tag{70}$$

in the case of the one-level method, and

$$C_0 = C(m+1)\left(1+H/\delta\right)C_{d,\sigma}(H,h), \tag{71}$$

in the case of the two-level method.

3.1.6 Inequalities with contraction operators (paper [24])

In [24], the convergence of the additive and multiplicative methods is studied for inequalities containing an extra term given by an operator. The framework of the general convergence result is that in [16] or [23], and it is applied to prove the convergence of the one- and two-level methods. Besides the direct use of the algorithms for the inequalities with contraction operators, we can use these results to obtain the convergence rate of the Schwarz method for other types of inequalities or nonlinear equations. In this way, we prove the convergence and estimate the error of the one- and two-level Schwarz methods for some inequalities in Hilbert spaces which are not of the variational type. Also, the general convergence result can be applied to prove the convergence of the Schwarz method for the Navier-Stokes problem. Finally, we give conditions of existence and uniqueness of the solution for all problems we consider. We point out that these conditions and the convergence conditions of the proposed algorithms are of the same type.

A reflexive Banach space V is considered, $V_1,\ldots,V_m$ are closed subspaces of V, and $K\subset V$ is a non empty closed convex set. We suppose that Assumptions 3.5, with $p=1$, and 3.6 are satisfied in the case of the multiplicative and additive algorithms, respectively. Let a $F : V \to \mathbf{R}$ be Gâteaux differentiable functional which is coercive in the sense that $F(v) \to \infty$, as $||v|| \to \infty$, $v \in V$, if K is not bounded, and its derivative satisfies (38) with $p = q = 2$. Finally, let $T : V \to V'$ be an operator with the property that for any $M > 0$ there exists $0 < \rho_M$ such that

$$||T(v) - T(u)||_{V'} \leq \rho_M ||v - u|| \tag{72}$$

for any $v, u \in V$, $||v||, ||u|| \leq M$, and we consider the problem

$$u \in K : \langle F'(u), v - u\rangle - \langle T(u), v - u\rangle \geq 0, \text{ for any } v \in K. \tag{73}$$

Concerning the existence and the uniqueness of the solution of problem (73) we have the following result.

Proposition 3.7 *Let V be a reflexive Banach space and K a closed convex non empty subset of V. We assume that the operator T satisfies (72), and F is coercive, Gâteaux differentiable on K and satisfies (38). If there exists a constant $0 < \theta < 1$ such that*

$$\frac{\rho_M}{\alpha_M} \leq \theta, \text{ for any } M > 0, \tag{74}$$

then problem (73) has a unique solution.

Depending on the argument of the operator, three multiplicative algorithms are proposed. In the first algorithm, T depends on the correction that we are looking for.

Algorithm 3.4 *We start the algorithm with an arbitrary $u^0 \in K$. At iteration $n+1$, having $u^n \in K$, $n \geq 0$, we compute $w_i^{n+1} \in V_i$, $u^{n+\frac{i-1}{m}} + w_i^{n+1} \in K$, the solution of the inequality*

$$\begin{aligned} &\langle F'(u^{n+\frac{i-1}{m}} + w_i^{n+1}), v_i - w_i^{n+1}\rangle - \langle T(u^{n+\frac{i-1}{m}} + w_i^{n+1}), v_i - w_i^{n+1}\rangle \geq 0, \\ &\textit{for any } v_i \in V_i, \ u^{n+\frac{i-1}{m}} + v_i \in K, \end{aligned} \tag{75}$$

and then we update

$$u^{n+\frac{i}{m}} = u^{n+\frac{i-1}{m}} + w_i^{n+1},$$

for $i = 1, \cdots, m$.

A simplified variant of Algorithm 3.4, can be written as

Algorithm 3.5 *We start the algorithm with an arbitrary* $u^0 \in K$*. At iteration* $n+1$*, having* $u^n \in K$*,* $n \geq 0$*, we compute* $w_i^{n+1} \in V_i$*,* $u^{n+\frac{i-1}{m}} + w_i^{n+1} \in K$*, the solution of the inequality*

$$\begin{array}{l} \langle F'(u^{n+\frac{i-1}{m}} + w_i^{n+1}), v_i - w_i^{n+1} \rangle - \langle T(u^{n+\frac{i-1}{m}}), v_i - w_i^{n+1} \rangle \geq 0, \\ \text{for any } v_i \in V_i,\ u^{n+\frac{i-1}{m}} + v_i \in K, \end{array} \quad (76)$$

and then we update

$$u^{n+\frac{i}{m}} = u^{n+\frac{i-1}{m}} + w_i^{n+1},$$

for $i = 1, \cdots, m$*.*

We can simplify more Algorithm 3.4 if we assume that the operator T depends in the current iteration only on the solution in the previous iteration,

Algorithm 3.6 *We start the algorithm with an arbitrary* $u^0 \in K$*. At iteration* $n+1$*, having* $u^n \in K$*,* $n \geq 0$*, we compute* $w_i^{n+1} \in V_i$*,* $u^{n+\frac{i-1}{m}} + w_i^{n+1} \in K$*, the solution of the inequality*

$$\begin{array}{l} \langle F'(u^{n+\frac{i-1}{m}} + w_i^{n+1}), v_i - w_i^{n+1} \rangle - \langle T(u^n), v_i - w_i^{n+1} \rangle \geq 0, \\ \text{for any } v_i \in V_i,\ u^{n+\frac{i-1}{m}} + v_i \in K, \end{array} \quad (77)$$

and then we update

$$u^{n+\frac{i}{m}} = u^{n+\frac{i-1}{m}} + w_i^{n+1},$$

for $i = 1, \cdots, m$*.*

Also, two additive algorithms are proposed for the solution of problem (73). In the first one, T depends on the current correction,

Algorithm 3.7 *We start the algorithm with an arbitrary* $u^0 \in K$*. At iteration* $n+1$*, having* $u^n \in K$*,* $n \geq 0$*, we solve the inequalities*

$$\begin{array}{l} w_i^{n+1} \in V_i,\ u^n + w_i^{n+1} \in K : \\ \langle F'(u^n + w_i^{n+1}), v_i - w_i^{n+1} \rangle - \langle T(u^n + w_i^{n+1}), v_i - w_i^{n+1} \rangle \geq 0, \\ \text{for any } v_i \in V_i,\ u^n + v_i \in K, \end{array} \quad (78)$$

for $i = 1, \cdots, m$*, and then we update* $u^{n+1} = u^n + \varrho \sum_{i=1}^{m} w_i^{n+1}$*, where* $\varrho > 0$ *is chosen such that* $u^{n+1} \in K$ *for any* $n \geq 0$*.*

A simplified form of Algorithm 3.7 can be written as

Algorithm 3.8 *We start the algorithm with an arbitrary $u^0 \in K$. At iteration $n+1$, having $u^n \in K$, $n \ge 0$, we solve the inequalities*

$$\begin{array}{l} w_i^{n+1} \in V_i,\ u^n + w_i^{n+1} \in K: \\ \langle F'(u^n + w_i^{n+1}), v_i - w_i^{n+1}\rangle - \langle T(u^n), v_i - w_i^{n+1}\rangle \ge 0, \\ \textit{for any } v_i \in V_i,\ u^n + v_i \in K, \end{array} \tag{79}$$

for $i = 1, \cdots, m$, and then we update $u^{n+1} = u^n + \varrho \sum_{i=1}^{m} w_i^{n+1}$, where $\varrho > 0$ is chosen such that $u^{n+1} \in K$ for any $n \ge 0$.

The following theorem proves that the above algorithms are geometrically convergent.

Theorem 3.6 *Let V be a reflexive Banach space, $V_1, \cdots, V_m$ some closed subspaces of V, and K a closed convex non empty subset of V. We assume that the operator T satisfies (72), and F is coercive, Gâteaux differentiable and satisfies (38). Let u be the solution of problem (73), and u^n, $n \ge 0$, be its approximations which are obtained from one of the above multiplicative or additive algorithms. We suppose that Assumption 3.5 is satisfied in the case of Algorithms 3.4-3.6, and Assumption 3.6 holds for Algorithms 3.7 and 3.8. Then, there exists a function $\theta_{\max} : \mathbf{R}^+ \to \mathbf{R}^+$, and we assume that*

$$\frac{\rho_M}{\alpha_M} < \theta_{\max}(M), \ \textit{for any } M > 0. \tag{80}$$

On these conditions, Algorithms 3.4-3.8 are convergent for any $u^0 \in K$, and the error estimates

$$\begin{array}{l} F(u^n) - \langle T(u), u^n\rangle - F(u) + \langle T(u), u\rangle \le \\ \left(\frac{\tilde{C}_1}{\tilde{C}_1+1}\right)^n [F(u^0) - \langle T(u), u^0\rangle - F(u) + \langle T(u), u\rangle] \end{array} \tag{81}$$

and

$$||u^n - u||^2 \le \frac{2}{\alpha_{M_0}} \left(\frac{\tilde{C}_1}{\tilde{C}_1+1}\right)^n [F(u^0) - \langle T(u), u^0\rangle - F(u) + \langle T(u), u\rangle] \tag{82}$$

hold for any $n \ge 1$. Constant $\tilde{C}_1$ depends on F, T, m,

$$M_0 = \max(\ ||u||,\ \sup\{||v|| : F(v) - \langle T(u), v\rangle \le F(u^0) - \langle T(u), u^0\rangle\}\ ),$$

and is an increasing function of C_0.

Functions $\theta_{\max}$ are arise from convergence conditions. More precisely, writing

$$\theta_M = \frac{\rho_M}{\alpha_M} \text{ and } \tau_M = \frac{\beta_M}{\alpha_M}, \tag{83}$$

in the case of Algorithms 3.4-3.6, to get error estimate (81) we should find some $\varepsilon_1, \varepsilon_2, \varepsilon_3 > 0$ such that

$$\varepsilon_1 < \frac{1}{\tau_M C_0} \tag{84}$$

and

$$\begin{aligned} &\theta_M^2 \left[\left(C_0 + \frac{1+3C_0}{2}\varepsilon_2 \right) \left(1 + \frac{1}{2\varepsilon_3} \right) m - \frac{m\varepsilon_3}{2} \left(1 + 2C_0 + \frac{1+3C_0}{2\varepsilon_2} \right) \right] - \\ &\theta_M \left[\frac{1}{2}\left(1 - \tau_M C_0 \varepsilon_1\right) \left(1 + \frac{1}{2\varepsilon_3} \right) m + \frac{1}{2} \left(C_0 + \frac{1+3C_0}{2}\varepsilon_2 \right) + \right. \\ &\left. \frac{m\varepsilon_3}{2}\tau_M \left(1 + C_0 (2 + \frac{1}{2\varepsilon_1}) \right) \right] + \frac{1}{4}\left(1 - \tau_M C_0 \varepsilon_1\right) > 0, \end{aligned} \tag{85}$$

We can simply verify that for any ε_1 satisfying (84) and for any $\varepsilon_2, \varepsilon_3 > 0$ there exists $\theta_{\varepsilon_1\varepsilon_2\varepsilon_3} > 0$ such that any $0 < \theta_M < \theta_{\varepsilon_1\varepsilon_2\varepsilon_3}$ is a solution of (85). We can verify that

$$\theta_{\varepsilon_1\varepsilon_2\varepsilon_3} < \frac{\varepsilon_3}{m(2\varepsilon_3 + 1)}, \tag{86}$$

and we can define an upper bound for θ_M as

$$\theta_{\max}(M) = \sup_{0<\varepsilon_1<\frac{1}{\tau_M C_0},\, 0<\varepsilon_2,\, 0<\varepsilon_3} \theta_{\varepsilon_1\varepsilon_2\varepsilon_3} \tag{87}$$

ie. we get condition (80). A similar reasoning is made in the case of the additive Algorithms 3.7 and 3.8 to get function $\theta_{\max}(M)$ and condition (80).

Evidently, in view of the results in [22] and [23], we know that for the one- and two-level methods, Assumption 3.5 holds with the constants C_0 in given in (46) and (57), and Assumption 3.6 holds with C_0 in (70) and (71).

Inequalities in Hilbert spaces. A first application of the above theory is the convergence study of the one- and two-level methods for inequalities which do not arise from the minimization of a functional. For a Hilbert space

V and a non empty convex set $K \subset V$, we are interested in the solution of problem

$$u \in K : \langle A(u), v-u \rangle \geq 0 \text{ for any } v \in K. \tag{88}$$

To this end, let us consider a linear symmetric operator $S : V \to V'$ which is V-elliptic and continuous, ie. there exist κ, $\sigma > 0$ such that

$$\kappa ||v||^2 \leq \langle S(v), v \rangle \text{ and } ||S(v)||_{V'} \leq \varsigma ||v|| \text{ for any } v \in V. \tag{89}$$

We associate to this operator the inner product $(v, w)_{V_S} = \langle S(v), w \rangle$, $v, w \in V$, and the norm $||v||_{V_S} = (v, v)_{V_S}^{\frac{1}{2}}$, $v \in V$. We have

$$||S(v)||_{V_S'} = ||v||_{V_S} \text{ for any } v \in V, \text{ and } ||S^{-1}(f)||_{V_S} = ||f||_{V_S'} \text{ for any } f \in V', \tag{90}$$

where $||\cdot||_{V_S'}$ is the norm in V' associated to the norm $||\cdot||_{V_S}$ in V. We assume that for any real number $M > 0$ there exist two constants λ_M^S, $\mu_M^S > 0$ for which

$$\lambda_M^S ||v-u||_{V_S}^2 \leq \langle A(v) - A(u), v-u \rangle \tag{91}$$

and

$$||A(v) - A(u)||_{V_S'} \leq \mu_M^S ||v-u||_{V_S}, \tag{92}$$

for any $u, v \in K$ with $||u||_{V_S}, ||v||_{V_S} \leq M$.

Now, we write problem (88) as,

$$u \in K : \langle S(u), v-u \rangle \geq \langle T(u), v-u \rangle \text{ for any } v \in K, \tag{93}$$

where

$$T = S - \delta A \tag{94}$$

for a fixed $\delta > 0$. In this way, we have written problem (88) as a problem of the form (73) in which $F(v) = \frac{1}{2}(S(v), v)$, for any $v \in V$, and, evidently, $F' = S$. The following subspace correction algorithm for problem (88) is in fact Algorithm 3.4 for problem (93) with operator T in (94).

Algorithm 3.9 *We start the algorithm with an arbitrary $u^0 \in K$. At iteration $n+1$, having $u^n \in K$, $n \geq 0$, we compute sequentially for $i = 1, \cdots, m$, $w_i^{n+1} \in V_i$ satisfying $u^{n+\frac{i-1}{m}} + w_i^{n+1} \in K$, as the solution of the inequality*

$$\begin{aligned} &\langle A(u^{n+\frac{i-1}{m}} + w_i^{n+1}), v_i - w_i^{n+1} \rangle \geq 0, \\ &\textit{for any } v_i \in V_i, \ u^{n+\frac{i-1}{m}} + v_i \in K, \end{aligned} \tag{95}$$

and then we update

$$u^{n+\frac{i}{m}} = u^{n+\frac{i-1}{m}} + w_i^{n+1}.$$

Also, additive Algorithm 3.7 for problem (93) with operator T in (94), becomes the following algorithm for problem (88).

Algorithm 3.10 *We start the algorithm with an arbitrary $u^0 \in K$. At iteration $n+1$, having $u^n \in K$, $n \geq 0$, we solve the inequalities*

$$\begin{array}{l} w_i^{n+1} \in V_i,\ u^n + w_i^{n+1} \in K : \langle A(u^n + w_i^{n+1}), v_i - w_i^{n+1} \rangle \geq 0, \\ \textit{for any } v_i \in V_i,\ u^n + v_i \in K, \end{array} \tag{96}$$

for $i = 1, \cdots, m$, and then we update $u^{n+1} = u^n + \varrho \sum_{i=1}^{m} w_i^{n+1}$, with $0 < \varrho \leq 1/m$.

By a simple calculus, we get

$$||T(u) - T(v)||_{V_S'}^2 \leq \left(\delta^2 (\mu_M^S)^2 - 2\delta\lambda_M^S + 1\right) ||v - u||_{V_S}^2 \tag{97}$$

For problem (93), with the notations in (38) and (72), we get from (90) and (97) that

$$\alpha_M = \beta_M = 1 \text{ and } \rho_M = \sqrt{\delta^2 (\mu_M^S)^2 - 2\delta\lambda_M^S + 1}, \tag{98}$$

where $\delta > 0$ can be arbitrary. The following proposition is a direct consequence of Proposition 3.7.

Proposition 3.8 *Let V be a Hilbert space and K a closed convex non empty subset of V. If there exists a linear symmetric continuous and V-elliptic operator $S : V \to V'$ such that operator $A : V \to V'$ satisfies (91) and (92), and in addition, there exists a constant $0 < \theta < 1$ such that*

$$\sqrt{1 - \theta^2} \leq \frac{\lambda_M^S}{\mu_M^S}, \quad \textit{for any } M > 0,$$

then problem (88) has a unique solution.

The following theorem, which proves the convergence of Algorithms 3.9 and 3.10, is a corollary of Theorems 3.6.

Theorem 3.7 *Let V be a Hilbert space, $V_1, \cdots, V_m$ some closed subspaces of V, and K a closed convex non empty subset of V. We assume that Assumptions 3.5 and 3.6 hold, and there exists a linear symmetric continuous and V-elliptic operator $S : V \to V'$ such that the operator $A : V \to V'$ satisfies (91) and (92). Let u be the solution of problem (88) and u^n, $n \geq 0$, be its approximations obtained either from Algorithm 3.9 or Algorithm 3.10. Then, there exists a constant $\theta_{\max} \in (0,1)$, and we assume that*

$$\sqrt{1-\theta_{\max}^2} < \frac{\lambda_M^S}{(\mu_M^S)^2} \text{ and } \mu_M^S \geq 1, \text{ for any } M > 0. \tag{99}$$

On these conditions, Algorithms 3.9 and 3.10 are convergent for any $u^0 \in K$, and the error estimate

$$||u^n - u||_{V_S}^2 \leq \left(\frac{\tilde{C}_1}{\tilde{C}_1+1}\right)^n \left[||u^0 - u||_{V_S}^2 + 2\sqrt{1-\theta_{\max}^2}\langle A(u), u^0 - u\rangle\right] \tag{100}$$

holds for any $n \geq 1$. Constant $\tilde{C}_1$ in which we use α_M, β_M and ρ_M in (98) where

$$M = \max(\ ||u||_{V_S},\ \sup\{||v||_{V_S} : ||v-u||_{V_S}^2 + 2\sqrt{1-\theta_{\max}^2}\langle A(u), v-u\rangle \leq ||u^0 - u||_{V_S}^2 + 2\sqrt{1-\theta_{\max}^2}\langle A(u), u^0 - u\rangle\}\ ),$$

and

$$\delta = \sqrt{1-\theta_{\max}^2} \tag{101}$$

Remark 3.2 *From Theorem 3.7, it follows that the multiplicative and additive Schwarz methods in the Hilbert spaces converge for inequalities whose operator A satisfies (91) and (92) in which the two constants λ_M^S and $(\mu_M^S)^2$ are close enough for some precondition operator S, even if the operator A is not the Gâteaux derivative of a convex functional. Evidently, the dependence of the convergence rate on the overlapping and mesh parameters we have given in the previous section for one- and two-level methods also hold for these inequalities.*

Navier-Stokes problem. For a bounded domain $\Omega \subset \mathbf{R}^d$, $d = 2, 3$, with a Lipschitz continuous boundary Γ, the weak form of the steady-state Navier-Stokes problem consists in finding $(\mathbf{u}, p) \in H^1(\Omega)^d \times L_0^2(\Omega)$ which satisfies

$$\begin{aligned} &a(\mathbf{u}; \mathbf{u}, \mathbf{v}) - (p, \mathrm{div}\mathbf{v}) = \langle \mathbf{f}, \mathbf{v} \rangle \text{ for any } \mathbf{v} \in H_0^1(\Omega)^d \\ &\mathrm{div}\mathbf{u} = 0 \text{ in } \Omega \\ &\mathbf{u} = \mathbf{g} \text{ on } \Gamma, \end{aligned} \tag{102}$$

where $\mathbf{u}$ and p are the fluid velocity and pressure, respectively, $\mathbf{f} \in H^{-1}(\Omega)^d$, $\mathbf{g} \in H^{1/2}(\Gamma)^d$ such that $\int_\Gamma \mathbf{g} \cdot \mathbf{n} = 0$, and

$$\begin{aligned} &a(\mathbf{w}; \mathbf{u}, \mathbf{v}) = \nu a_0(\mathbf{u}, \mathbf{v}) + a_1(\mathbf{w}; \mathbf{u}, \mathbf{v}), \\ &a_0(\mathbf{u}, \mathbf{v}) = \int_\Omega \nabla \mathbf{u} \cdot \nabla \mathbf{v},\ a_1(\mathbf{w}; \mathbf{u}, \mathbf{v}) = \int_\Omega ((\mathbf{w} \cdot \nabla)\mathbf{u}) \cdot \mathbf{v}, \end{aligned} \tag{103}$$

$\nu > 0$ being the viscosity of the fluid. It is well known that there exists a constant $\tilde{C}$ such that, for any $\mathbf{u}, \mathbf{v}, \mathbf{w} \in H^1(\Omega)^d$

$$|a_1(\mathbf{w}; \mathbf{u}, \mathbf{v})| \le \tilde{C} |\mathbf{u}|_1 ||\mathbf{v}||_1 ||\mathbf{w}||_1,$$

where $||\cdot||_1$ and $|\cdot|_1$ are the norm and seminorm on $H^1(\Omega)^d$, respectively.

Let us introduce the space

$$V = \{\mathbf{v} \in H_0^1(\Omega)^d : \mathrm{div}\mathbf{v} = 0\} \tag{104}$$

and let

$$\mathcal{N} = \sup_{\mathbf{u},\mathbf{v},\mathbf{w} \in V} \frac{a_1(\mathbf{w}; \mathbf{u}, \mathbf{v})}{|\mathbf{u}|_1 |\mathbf{v}|_1 |\mathbf{w}|_1}$$

be the norm of the trilinear form a_1. We know (see [40], for instance) that if

$$\frac{\mathcal{N}}{\nu^2} ||\mathbf{f}||_{V'} < 1, \tag{105}$$

then problem (102) has a unique solution. Using the space V, we can write problem (102) in a velocity formulation as, find $\mathbf{u} \in V$ such that

$$a(\mathbf{u}; \mathbf{u}, \mathbf{v}) = \langle \mathbf{f}, \mathbf{v} \rangle \text{ for any } \mathbf{v} \in V. \tag{106}$$

Evidently, all results we have obtained for inequalities hold also for equations. We prove in the following that we can derive specific algorithms to problem (106) from Algorithms 3.4–3.8. To this end, following the way in

[40], we write this problem in an appropriate form. First we define an operator $A : V \to \mathcal{L}(V, V')$ by

$$\langle (A(\mathbf{w}))\mathbf{u}, \mathbf{v}\rangle = a(\mathbf{w}; \mathbf{u}, \mathbf{v}) \text{ for any } \mathbf{u}, \mathbf{v}, \mathbf{w} \in V, \tag{107}$$

and let us introduce on V the norm $||v|| = |v|_1$. Operator A is Lipschitz continuous of constant $\mathcal{N}$,

$$||A(\mathbf{w}_1) - A(\mathbf{w}_2)||_{\mathcal{L}(V,V')} \leq \mathcal{N}||\mathbf{w}_1 - \mathbf{w}_2|| \tag{108}$$

Also, since $a_1(\mathbf{w}; \mathbf{v}, \mathbf{v}) = 0$ for any $\mathbf{v},\ \mathbf{w} \in V$, we have

$$\langle (A(\mathbf{w}))\mathbf{v}, \mathbf{v}\rangle = \nu||\mathbf{v}||^2$$

for any $\mathbf{v},\ \mathbf{w} \in V$, ie. the bilinear form $\langle (A(\mathbf{w}))\cdot, \cdot\rangle$ is uniformly $V-$elliptic on V. According to Lax-Milgram lemma, operator $A(\mathbf{w})$ is invertible for any $\mathbf{w} \in V$. Moreover, we have

$$||A(\mathbf{w})^{-1}||_{\mathcal{L}(V',V)} \leq \frac{1}{\nu} \tag{109}$$

for any $\mathbf{w} \in V$. Now, in view of (107), problem (106) is equivalent with the finding of $\mathbf{u} \in V$ such that $(A(\mathbf{u}))\mathbf{u} = \mathbf{f}$ in V', or with

$$\mathbf{u} \in V : \langle I(\mathbf{u}), \mathbf{v}\rangle = \langle T(\mathbf{u}), \mathbf{v}\rangle \text{ for any } \mathbf{v} \in V \tag{110}$$

where $T(\mathbf{v}) = A(\mathbf{v})^{-1}\mathbf{f}$ and $I(\mathbf{v}) = \mathbf{v}$ for any $\mathbf{v} \in V$. In the above equation, $\langle \cdot, \cdot\rangle$ means the inner product associated with the norm $||\cdot|| = |\cdot|_1$. In this way, we have written problem (106) as a problem (73) in which $F' = I$. Evidently, for problem (110), the constants in (38) are

$$\alpha_M = \beta_M = 1 \tag{111}$$

To find ρ_M in (72), we use (108) and (109), and get

$$||T(\mathbf{w}) - T(\mathbf{v})||_{\mathcal{L}(V',V)} = ||(A(\mathbf{w})^{-1} - A(\mathbf{v})^{-1})\mathbf{f}||_{\mathcal{L}(V',V)} \leq \frac{\mathcal{N}}{\nu^2}||\mathbf{f}||_{V'}||\mathbf{v} - \mathbf{w}||.$$

Therefore, we have

$$||T(\mathbf{w}) - T(\mathbf{v})||_{\mathcal{L}(V',V)} = ||(A(\mathbf{w})^{-1} - A(\mathbf{v})^{-1})\mathbf{f}||_{\mathcal{L}(V',V)} \leq \frac{\mathcal{N}}{\nu^2}||\mathbf{f}||_{V'}||\mathbf{v} - \mathbf{w}||$$

ie.

$$\rho_M = \frac{\mathcal{N}}{\nu^2}||\mathbf{f}||_{V'} \tag{112}$$

Remark 3.3 Using (111) and (112), we find again the condition of existence and uniqueness of the solution for the Navier-Stokes problem, (105), from Proposition 3.7.

Now, if the domain Ω is decomposed as in (6), we associate with the subdomain Ω_i the subspace of V,

$$V_i = \{\mathbf{v}_i \in H_0^1(\Omega_i)^d : \operatorname{div}\mathbf{v}_i = 0\}. \tag{113}$$

From the equivalence of problems (106) and (110), multiplicative Algorithms 3.4, 3.5 and 3.6 for problem (106) can be written as: we start the algorithms with an arbitrary initial guess $\mathbf{u}^0 \in V$, and, at each iteration $n \geq 1$ and on each subdomain $i = 1, \cdots, m$, we solve

$$\mathbf{w}_i^{n+1} \in V_i : a(\mathbf{u}^{n+\frac{i-1}{m}} + \mathbf{w}_i^{n+1}; \mathbf{u}^{n+\frac{i-1}{m}} + \mathbf{w}_i^{n+1}, \mathbf{v}_i) = \langle \mathbf{f}, \mathbf{v}_i \rangle \text{ for any } \mathbf{v}_i \in V_i, \tag{114}$$

$$\mathbf{w}_i^{n+1} \in V_i : a(\mathbf{u}^{n+\frac{i-1}{m}}; \mathbf{u}^{n+\frac{i-1}{m}} + \mathbf{w}_i^{n+1}, \mathbf{v}_i) = \langle \mathbf{f}, \mathbf{v}_i \rangle \text{ for any } \mathbf{v}_i \in V_i, \tag{115}$$

$$\mathbf{w}_i^{n+1} \in V_i : a(\mathbf{u}^n; \mathbf{u}^{n+\frac{i-1}{m}} + \mathbf{w}_i^{n+1}, \mathbf{v}_i) = \langle \mathbf{f}, \mathbf{v}_i \rangle \text{ for any } \mathbf{v}_i \in V_i, \tag{116}$$

respectively, and then we update

$$\mathbf{u}^{n+\frac{i}{m}} = \mathbf{u}^{n+\frac{i-1}{m}} + \mathbf{w}_i^{n+1}.$$

Also, additive Algorithms 3.7 and 3.8 for problem (106) can be written as: we start the algorithms with an arbitrary initial guess $\mathbf{u}^0 \in V$, and, at each iteration $n \geq 1$ and on each subdomain $i = 1, \cdots, m$, we solve

$$\mathbf{w}_i^{n+1} \in V_i : a(\mathbf{u}^n + \mathbf{w}_i^{n+1}; \mathbf{u}^n + \mathbf{w}_i^{n+1}, \mathbf{v}_i) = \langle \mathbf{f}, \mathbf{v}_i \rangle \text{ for any } \mathbf{v}_i \in V_i, \tag{117}$$

$$\mathbf{w}_i^{n+1} \in V_i : a(\mathbf{u}^n; \mathbf{u}^n + \mathbf{w}_i^{n+1}, \mathbf{v}_i) = \langle \mathbf{f}, \mathbf{v}_i \rangle \text{ for any } \mathbf{v}_i \in V_i, \tag{118}$$

respectively, and then we update

$$\mathbf{u}^{n+1} = \mathbf{u}^n + \varrho \sum_{i=1}^m \mathbf{w}_i^{n+1},$$

with $0 < \varrho \leq 1/m$.

As we have already mentioned, we can apply Theorem 3.6, and we have

Theorem 3.8 *Let* $\mathbf{u}$ *be the solution of problem (106) and* $\mathbf{u}^n$, $n \geq 0$, *be its approximations obtained from one of algorithms (114), (115), (116), (117) or (118). Then, there exists a constant* $\theta_{\max} \in (0,1)$, *and we assume that*

$$\frac{\mathcal{N}}{\nu^2}||\mathbf{f}||_{V'} < \theta\mathrm{max} \tag{119}$$

On these conditions, for any $\mathbf{u}^0 \in V$, *algorithms (114)–(118) are convergent and the error estimate*

$$||\mathbf{u}^n - \mathbf{u}||^2 \leq \tilde{C}_2 \left(\frac{\tilde{C}_1}{\tilde{C}_1+1}\right)^n C(\mathbf{u}_0) \tag{120}$$

holds for any $n \geq 1$, *where* $C(\mathbf{u}_0)$ *is a constant depending on the initial guess. The constants* $\tilde{C}_1$ *and* $\tilde{C}_2$ *are similar with the constants in Theorems 3.6 in which we take* $\alpha_M = \beta_M = 1$ *and* $\rho_M = \frac{\mathcal{N}}{\nu^2}||\mathbf{f}||_{V'}$.

In the case of equations, Assumptions 3.5 and 3.6 made for the convergence of the multiplicative or additive algorithms in Theorem 3.6 reduce to

Assumption 3.7 *There exists a constant* $C_0 > 0$ *such that for any* $v \in V$ *there exist* $v_i \in V_i$, $i = 1, \cdots, m$, *which satisfy*

$$v = \sum_{i=1}^{m} v_i \text{ and } \sum_{i=1}^{m} ||v_i||^2 \leq C_0 ||v||^2.$$

We can prove that the spaces V and V_i, $i = 1, \ldots, m$, defined in (104) and (113), respectively, satisfy this assumption. However, we can not use the same techniques as for the spaces in H^1 or in $W^{1,\sigma}$ to prove that this assumption holds if we use one- or two-multilevel methods. To our knowledge, this has not been studied so far.

Remark 3.4 *Since* $\theta\mathrm{max} < 1$, *convergence condition (119) is stronger than the existence and uniqueness condition (105), but they are of the same type and hold for enough large viscosities* ν *of the fluid.*

3.1.7 Variational inequalities of the second kind and quasi - variational inequalities. Contact problems with friction (paper [27])

In [27], we present and analyze subspace correction methods for the solution of variational inequalities of the second kind and quasi-variational inequalities, and apply these theoretical results to non smooth contact problems in linear elasticity with Tresca and non-local Coulomb friction. As in [24], we introduce these methods in a reflexive Banach space, prove that they are globally convergent and give error estimates. In the context of finite element discretization, where our methods turn out to be one- and two-level Schwarz methods, we specify their convergence rate and its dependence on the discretization parameters and conclude that our methods converge optimally. Transferring this results to frictional contact problems, we thus can overcome the mesh dependence of some fixed-point schemes which are commonly employed for contact problems with Coulomb friction.

We consider a reflexive Banach space V, $V_1, \dots, V_m$ some closed subspaces of V, and $K \subset V$ a non empty closed convex set. We suppose that Assumptions 3.5, with $p = 1$, and 3.6 is satisfied. Also, let a $F : V \to \mathbf{R}$ be Gâteaux differentiable functional which satisfies (38).

Variational inequalities of the second kind. Let $\varphi : K \to \mathbf{R}$ be a convex, lower semi-continuous (l.s.c.) functional. If K is not bounded, let $F + \varphi$ furthermore be coercive in the sense that $F(v) + \varphi(v) \to \infty$ as $\|v\| \to \infty$ for $v \in K$.

In addition to Assumption 3.5 we also suppose that

$$\sum_{i=1}^{m}[\varphi(w + \sum_{j=1}^{i-1} w_j + v_i) - \varphi(w + \sum_{j=1}^{i-1} w_j + w_i)] \leq \varphi(v) - \varphi(w + \sum_{i=1}^{m} w_i) \quad (121)$$

for $v, w \in K$ and $v_i, w_i \in V_i$ as in Assumption 3.5.

Now, let us consider the variational inequality of the second kind: find $u \in K$ such that

$$\langle F'(u), v - u\rangle + \varphi(v) - \varphi(u) \geq 0\,, \qquad v \in K\,. \quad (122)$$

The proposed algorithm corresponding to the subspaces $V_1, \dots, V_m$ and the convex set K is now given by

Algorithm 3.11 *Choose an arbitrary* $u^0 \in K$. *At iteration step* $n+1$, *having* $u^n \in K$, $n \geq 0$, *compute sequentially for* $i = 1, \ldots, m$, *the local corrections* $w_i^{n+1} \in V_i$, $u^{n+\frac{i-1}{m}} + w_i^{n+1} \in K$, *as the solution of the variational inequalities*

$$\begin{aligned} &\langle F'(u^{n+\frac{i-1}{m}} + w_i^{n+1}), v_i - w_i^{n+1}\rangle + \varphi(u^{n+\frac{i-1}{m}} + v_i) \\ &-\varphi(u^{n+\frac{i-1}{m}} + w_i^{n+1}) \ \geq \ 0 \quad v_i \in V_i, \ u^{n+\frac{i-1}{m}} + v_i \in K, \end{aligned} \tag{123}$$

and then update $u^{n+\frac{i}{m}} = u^{n+\frac{i-1}{m}} + w_i^{n+1}$.

We have the following general convergence result:

Theorem 3.9 *Let* V, $V_1, \ldots, V_m \subset V$ *and* $K \subset V$ *be as introduced above and let* K *fulfill Assumption 3.5. Assume that* F *is Gâteaux differentiable on* K *and that it satisfies (38), that the convex and l.s.c. functional* φ *satisfies (121), and that* $F + \varphi$ *is coercive if* K *is not bounded. If* u *solves (73) and* u^n, $n \geq 0$, *are its approximations from Algorithm 3.3, then there exists an* $M > 0$ *s.t.* $\max(\|u\|, \|u^0\|, \max\limits_{n\geq 0, 1\leq i\leq m} \|u^{n+\frac{i}{m}}\|) \leq M$ *and the following error estimates hold:*

(i) if $p = q$ *we have*

$$\begin{aligned} & F(u^n) + \varphi(u^n) - F(u) - \varphi(u) \\ \leq & \left(\tfrac{\tilde{C}_1}{\tilde{C}_1+1}\right)^n [F(u^0) + \varphi(u^0) - F(u) - \varphi(u)], \end{aligned} \tag{124}$$

$$\|u^n - u\|^p \leq \frac{p}{\alpha_M} \left(\frac{\tilde{C}_1}{\tilde{C}_1 + 1}\right)^n \left[F(u^0) + \varphi(u^0) - F(u) - \varphi(u)\right]. \tag{125}$$

(ii) if $p > q$ *we have*

$$\begin{aligned} & F(u^n) + \varphi(u^n) - F(u) - \varphi(u) \\ \leq & \frac{F(u^0)+\varphi(u^0)-F(u)-\varphi(u)}{\left[1+n\tilde{C}_2(F(u^0)+\varphi(u^0)-F(u)-\varphi(u))^{\frac{p-q}{q-1}}\right]^{\frac{q-1}{p-q}}}, \end{aligned} \tag{126}$$

$$\|u - u^n\|^p \ \leq \ \frac{p}{\alpha_M} \frac{F(u^0) + \varphi(u^0) - F(u) - \varphi(u)}{\left[1 + n\tilde{C}_2 \left(F(u^0) + \varphi(u^0) - F(u) - \varphi(u)\right)^{\frac{p-q}{q-1}}\right]^{\frac{q-1}{p-q}}}. \tag{127}$$

Constants $\tilde{C}_1$ and $\tilde{C}_2$ depend on F, m, C_0 and u^0, and can be written as

$$\begin{aligned}\tilde{C}_1 = \beta_M(1+2C_0)m^{2-\frac{q}{p}}(\frac{p}{\alpha_M})^{\frac{q}{p}}\big(F(u^0)-F(u)\\ +\varphi(u^0)-\varphi(u))^{\frac{p-q}{p(p-1)}}+\beta_M C_0 m^{\frac{p-q+1}{p}}\frac{1}{\varepsilon^{\frac{1}{p-1}}}(\frac{p}{\alpha_M})^{\frac{q-1}{p-1}}.\end{aligned} \tag{128}$$

and

$$\tilde{C}_2 = \frac{p-q}{(p-1)(F(u^0)+\varphi(u^0)-F(u)-\varphi(u))^{\frac{p-q}{q-1}}+(q-1)\tilde{C}_1^{\frac{p-1}{q-1}}}. \tag{129}$$

with

$$\varepsilon = \alpha_M/\left(p\beta_M C_0 m^{\frac{p-q+1}{p}}\right). \tag{130}$$

Quasi-variational inequalities. Let $\varphi : K\times K \to \mathbf{R}$ be a functional such that, for any $u \in K$, $\varphi(u,\cdot) : K \to \mathbf{R}$ is convex, l.s.c. and coercive in the sense that

$$F(v)+\varphi(u,v)\to\infty, \text{ as } \|v\|\to\infty, \ v\in K, \tag{131}$$

if K is not bounded.

In this section we assume that $p = q = 2$ in (38). Also, we assume that for any $M > 0$ there exists $c_M > 0$ such that

$$\begin{aligned}|\varphi(v_1,w_2)+\varphi(v_2,w_1)-\varphi(v_1,w_1)-\varphi(v_2,w_2)|\\ \le\ c_M\|v_1-v_2\|\|w_1-w_2\|\end{aligned} \tag{132}$$

for any $v_1,\ v_2,\ w_1\ w_2 \in K$, $\|v_1\|,\ \|v_2\|,\ \|w_1\|\ \|w_2\| \le M$. In addition to the hypotheses of Assumption 3.5, we suppose that

$$\begin{aligned}\sum_{i=1}^{m}[\varphi(u,w+\sum_{j=1}^{i-1}w_j+v_i)-\varphi(u,w+\sum_{j=1}^{i-1}w_j+w_i)]\\ \le\ \varphi(u,v)-\varphi(u,w+\sum_{i=1}^{m}w_i)\end{aligned} \tag{133}$$

for any $u \in K$ and for $v, w \in K$ and $v_i, w_i \in V_i$, $i = 1,\dots,m$, in Assumption 3.5. As is (121), this condition is of technical nature, and in the case of the one- and the two-level methods, it holds for certain numerical approximations of the convex functional φ in the second variable.

Now, we consider the quasi-variational inequality: find $u \in K$ such that

$$\langle F'(u), v-u\rangle + \varphi(u,v) - \varphi(u,u) \ge 0, \qquad v \in K. \tag{134}$$

As in Proposition 3.7, we can show that problem (134) has a unique solution if there exists a constant $\theta < 1$ such that $c_M/\alpha_M \le \theta$, for any $M > 0$.

Our first subspace correction algorithm based on $V_1, \ldots, V_m$ and the convex set K is now given by

Algorithm 3.12 *Choose an arbitrary $u^0 \in K$. At iteration $n+1$, having $u^n \in K$, $n \ge 0$, compute sequentially for $i = 1, \ldots, m$, the local corrections $w_i^{n+1} \in V_i$, $u^{n+\frac{i-1}{m}} + w_i^{n+1} \in K$, satisfying*

$$\begin{array}{r} \langle F'(u^{n+\frac{i-1}{m}} + w_i^{n+1}), v_i - w_i^{n+1}\rangle \\ +\varphi(u^{n+\frac{i-1}{m}} + w_i^{n+1}, u^{n+\frac{i-1}{m}} + v_i) \\ -\varphi(u^{n+\frac{i-1}{m}} + w_i^{n+1}, u^{n+\frac{i-1}{m}} + w_i^{n+1}) \ \ge \ 0 \end{array} \tag{135}$$

for $v_i \in V_i$, $u^{n+\frac{i-1}{m}} + v_i \in K$. Then update: $u^{n+\frac{i}{m}} = u^{n+\frac{i-1}{m}} + w_i^{n+1}$.

A simplified variant of Algorithm 3.12 is given by

Algorithm 3.13 *As Algorithm 3.12, only inequality (135) is replaced by*

$$\begin{array}{r} \langle F'(u^{n+\frac{i-1}{m}} + w_i^{n+1}), v_i - w_i^{n+1}\rangle \\ +\varphi(u^{n+\frac{i-1}{m}}, u^{n+\frac{i-1}{m}} + v_i) - \varphi(u^{n+\frac{i-1}{m}}, u^{n+\frac{i-1}{m}} + w_i^{n+1}) \ \ge \ 0 \end{array} \tag{136}$$

for $v_i \in V_i$, $u^{n+\frac{i-1}{m}} + v_i \in K$.

We can simplify Algorithm 3.12 even more to

Algorithm 3.14 *Inequality (135) in Algorithm 3.12 is replaced by*

$$\begin{array}{r} \langle F'(u^{n+\frac{i-1}{m}} + w_i^{n+1}), v_i - w_i^{n+1}\rangle + \varphi(u^n, u^{n+\frac{i-1}{m}} + v_i) \\ -\varphi(u^n, u^{n+\frac{i-1}{m}} + w_i^{n+1}) \ \ge \ 0\,, \end{array} \tag{137}$$

for $v_i \in V_i$, $u^{n+\frac{i-1}{m}} + v_i \in K$.

The following theorem proves that if c_M is small enough in comparison with α_M and β_M, then Algorithms 3.12, 3.13 and 3.14 are convergent.

Theorem 3.10 *Let V, $V_1, \ldots, V_m \subset V$ and $K \subset V$ be as introduced above and let K satisfy Assumption 3.5. We assume also that F is Gâteaux differentiable on K and satisfies (38) with $p = q = 2$ and that the functional φ is convex and l.s.c. in the second variable and satisfies (132), (133), and the coerciveness condition (131), if K is not bounded. Then, if u is the solution of problem (134), $u^{n+\frac{i}{m}}$, $n \geq 0$, $i = 1, \ldots, m$, are its approximations obtained from one of Algorithms 3.12, 3.13 or 3.14, and*

$$\frac{\alpha_M}{2} \geq mc_M + \sqrt{2m(25C_0 + 8)\beta_M c_M}, \text{ for any } M > 0\,, \tag{138}$$

then there exists an $M > 0$ such that $\max(\|u\|, \|u^0\|, \max\limits_{n\geq 0, 1\leq i\leq m} \|u^{n+\frac{i}{m}}\|) \leq$ *M and we have the error estimate*

$$\begin{aligned} & F(u^n) + \varphi(u, u^n) - F(u) - \varphi(u,u) \\ \leq & \left(\tfrac{\tilde{C}_1}{\tilde{C}_1+1}\right)^n [F(u^0) + \varphi(u,u^0) - F(u) - \varphi(u,u)] \end{aligned} \tag{139}$$

$$\|u^n - u\|^2 \leq \frac{2}{\alpha_M}\left(\frac{\tilde{C}_1}{\tilde{C}_1+1}\right)^n \left[F(u^0) + \varphi(u^0) - F(u) - \varphi(u)\right]. \tag{140}$$

The constant $\tilde{C}_1$ depends on F, φ, m, u^0 and is an increasing function of C_0, and can be written as

$$\begin{aligned} \tilde{C}_1 &= \tilde{C}_2/\tilde{C}_3 \\ \tilde{C}_2 &= \beta_M m(1 + 2C_0 + \frac{C_0}{\varepsilon_1}) + c_M m(1 + 2C_0 + \frac{1+3C_0}{\varepsilon_2}) \\ \tilde{C}_3 &= \frac{\alpha_M}{2} - c_M(1+\varepsilon_3)m, \end{aligned} \tag{141}$$

with

$$\varepsilon_1 = \varepsilon_2 = \frac{2c_M m}{\frac{\alpha_M}{2} - c_M m}, \quad \varepsilon_3 = \frac{\frac{\alpha_M}{2} - c_M m}{2c_M m}. \tag{142}$$

One- and two-level methods. In comparison with Theorem 3.6, the proofs of the above theorems are more complicated because of the introduction of the functionals φ instead of the operator T. The two conditions (121) and (133) have been introduced for technical reasons. The one- and two level methods corresponding to Algorithms 3.11 and 3.12-3.14 are obtained as in [22] and [23]. Consequently, Assumption 3.5 holds for convex sets having Property 3.2, with the constant C_0 which can be written in the terms of the mesh and domain decomposition parameters. In general functionals φ do not satisfy (121) and (133), and, for this reason they have been replaced by some numerical approximations in V_h. In this way, φ in the case of the variational inequalities of the second kind has been considered of the form

$$\varphi(v) = \sum_{\kappa\in\mathcal{N}_h} s_\kappa(h)\phi(v(x_\kappa)) = \sum_{k\in\mathcal{N}_h} s_\kappa(h)\phi_\kappa(v) \tag{143}$$

where $\phi\colon \mathbf{R}\to\mathbf{R}$ is a continuous convex function, $\mathcal{N}_h$ is the set of nodes of the mesh $\mathcal{T}_h$, and $s_\kappa(h)\geq 0$, $\kappa\in\mathcal{N}_h$, are non-negative real numbers which may depend on the mesh size h. For ease of notation we set $\phi_\kappa(v)=\phi(v(x_\kappa))$ We see that the ϕ_κ, $\kappa\in\mathcal{N}_h$, can be viewed as some functionals $\phi_\kappa\colon K_h\to\mathbf{R}$ which satisfy

$$\phi_\kappa(L_h(\theta v+(1-\theta)w)) \leq \theta(x_\kappa)\phi_\kappa(v)+(1-\theta(x_\kappa))\phi_\kappa(w) \tag{144}$$

for any $v,w\in K_h$, and any function $\theta\colon\bar{\Omega}\to\mathbf{R}$ as in Property 3.2.

For the case of quasi-variational inequalities we assume that the functional φ is of the form

$$\varphi(u,v) = \sum_{\kappa\in\mathcal{N}_h} I_\kappa(\phi\,(u,v(x_\kappa))) = \sum_{k\in\mathcal{N}_h} \phi_\kappa(u,v)\,. \tag{145}$$

Here, $I_\kappa\colon L^2(\Omega)\to\mathbf{R}$ and $\phi\colon K_h\times\mathbf{R}\to L^2(\Omega)$ are assumed to be continuous, and we furthermore assume that, for any $u\in K_h$, $I_\kappa(\phi\,(u,\cdot)):\mathbf{R}\to\mathbf{R}$, $\kappa\in\mathcal{N}_h$, are convex functions. Again, for ease of notation, we set $\phi_\kappa(u,v)=I_\kappa(\phi(u,v(x_\kappa)))$, $\kappa\in\mathcal{N}_h$, and assume as above that φ satisfies the coerciveness condition (131). Again, one can see that the ϕ_κ can be viewed as functionals $\phi_\kappa\colon K_h\times K_h\to\mathbf{R}$, which satisfy

$$\phi_\kappa(u,L_h(\theta v+(1-\theta)w)) \leq \theta(x_\kappa)\phi_\kappa(u,v)+(1-\theta(x_\kappa))\phi_\kappa(u,w) \tag{146}$$

for any $u,v,w\in K_h$, and any function $\theta\colon\bar{\Omega}\to\mathbf{R}$ as in Property 3.2.

In view of the results in [22] and [23], we know that for the one- and two-level methods, Assumption 3.5 holds with the constants C_0 in given in (46) and (57). Also, the functionals φ defined in (143) and (145) satisfy (121) and (133), respectively. Therefore, all the conditions of Theorem 3.9 are satisfied. Also, Theorem 3.10 holds provided that the functional φ satisfies (132). This condition has to be checked for each particular problem.

Contact problems with friction. We consider a deformable body in linear elasticity, and let $\Omega \subset \mathbf{R^d}$, $d = 2, 3$, be its reference domain which has a Lipschitz continuous boundary Γ. This boundary is decomposed according to $\Gamma = \bar{\Gamma}_D \cup \bar{\Gamma}_N \cup \bar{\Gamma}_C$ into the three open and disjoint parts Γ_D, where the displacement $\mathbf{u} = \mathbf{g} \in \mathbf{H}^{1/2}(\Gamma_D)$ is given, Γ_N, where the stress $\sigma(\mathbf{u})\mathbf{n} = \mathbf{p} \in \mathbf{L}^2(\Gamma_N)$ is given, and Γ_C, the possible contact boundary. The actual contact boundary γ_C is not known in advance, but we assume that $\bar{\gamma}_C \subset \Gamma_C$. Also, we assume $\mathrm{vol}_{d-1}(\Gamma_D) > 0$. For ease of notation, by $\mathbf{H}^{1/2}(\Gamma_C)$ we denote the space $\mathbf{H}^{1/2}_{00}(\Gamma_C)$ (see [67] for definitions), and its dual by $\mathbf{H}^{-1/2}(\Gamma_C)$. The classical formulation of the contact problem with local Coulomb's law of friction, between a linear elastic body and a rigid one, can be found in [37], for instance, and its weak form is given by the quasi-variational inequality: find $\mathbf{u} \in \mathbf{K}$ such that

$$a(\mathbf{u}, \mathbf{v} - \mathbf{u}) + j(\mathbf{u}, \mathbf{v}) - j(\mathbf{u}, \mathbf{u}) \ \geq \ f(\mathbf{v} - \mathbf{u}), \qquad \mathbf{v} \in \mathbf{K}, \tag{147}$$

where

$$\mathbf{K} \ = \ \{\mathbf{v} \in \mathbf{H}^1(\Omega) \colon v_n \leq 0 \text{ a.e. on } \Gamma_C \text{ and } \mathbf{v} = \mathbf{g} \text{ a.e. on } \Gamma_D\} \tag{148}$$

$$\begin{aligned} a(\mathbf{u}, \mathbf{v}) &= \int_\Omega E_{ijlm} \epsilon_{ij}(\mathbf{v}) \epsilon_{ml}(\mathbf{u}) = \int_\Omega \sigma(\mathbf{u}) \colon \epsilon(\mathbf{v}), \\ j(\mathbf{u}, \mathbf{v}) &= -\textstyle\int_{\Gamma_C} \mathcal{F} \, \sigma_n(\mathbf{u}) \, |\mathbf{v}_t| \,, \quad f(\mathbf{v}) = (\mathbf{f}, \mathbf{v})_{\mathbf{L}^2(\Omega)} + (\mathbf{p}, \mathbf{v})_{\mathbf{L}^2(\Gamma_N)}, \end{aligned} \tag{149}$$

$\mathbf{f} \in \mathbf{L}^2(\Omega)$ represents the volume force density, $0 \leq \mathcal{F} \in L^\infty(\Gamma_C)$ is the coefficient of friction, and the normal stresses $\sigma_n(\mathbf{u}) \in H^{-1/2}(\Gamma_C)$ satisfy

$$\int_{\Gamma_C} \sigma_n(\mathbf{u}) v_n \equiv \langle \sigma_n(\mathbf{u}), v_n \rangle_{H^{-1/2}(\Gamma_C) \times H^{1/2}(\Gamma_C)} \ = \ a(\mathbf{u} \, , \, \mathbf{v}) - f(\mathbf{v}),$$

for any $\mathbf{v} \in \mathbf{H}^1(\Omega)$ with $\mathbf{v} = 0$ on Γ_D, and $\mathbf{v}_t = 0$ on Γ_C.

Clearly, $a(\cdot, \cdot)$ is a symmetric bilinear form, and there exist α, $\beta \ > 0$ such that

$$\alpha \|\mathbf{v}\|^2_{\mathbf{H}^1(\Omega)} \leq a(\mathbf{v}, \mathbf{v}) \text{ and } a(\mathbf{v}, \mathbf{w}) \leq \beta \|\mathbf{v}\|_{\mathbf{H}^1(\Omega)} \|\mathbf{w}\|_{\mathbf{H}^1(\Omega)} \tag{150}$$

for any $\mathbf{v}$, $\mathbf{w} \in \mathbf{H}^1(\Omega)$.

Now, for a $\tau \in H^{-1/2}(\Gamma_C)$, $\tau \leq 0$ (i.e., $\langle \tau, v \rangle_{H^{-1/2}(\Gamma_C) \times H^{1/2}(\Gamma_C)} \leq 0$ for any $v \in H^{1/2}(\Gamma_C)$, $v \geq 0$) let us consider the following problem: find $\mathbf{u} = \mathbf{u}(\tau) \in \mathbf{K}$ s.t.

$$a(\mathbf{u}, \mathbf{v} - \mathbf{u}) + j_\tau(\mathbf{v}) - j_\tau(\mathbf{u}) \ \geq \ f(\mathbf{v} - \mathbf{u}) \,, \qquad \mathbf{v} \in \mathbf{K}, \tag{151}$$

where $j_\tau(\mathbf{v}) = -\int_{\Gamma_C} \mathcal{F}\,\tau|\mathbf{v}_t|$. Problem (151) is called *contact problem with Tresca friction*, and the reduced friction functional j_τ is convex, l.s.c., proper, and subdifferentiable, cf. [37]. Thus, there exists a unique solution of (151), see [38].

The weak form of the *contact problem with non-local Coulomb friction* is given by a quasi-variational inequality of the form (147): find $\mathbf{u} \in \mathbf{K}$ s.t.

$$a(\mathbf{u}, \mathbf{v}-\mathbf{u}) + j_{\mathrm{QP}}(\mathbf{u}, \mathbf{v}) - j_{\mathrm{QP}}(\mathbf{u}, \mathbf{u}) \;\geq\; f(\mathbf{v}-\mathbf{u}), \qquad \mathbf{v} \in \mathbf{K}\,. \tag{152}$$

Here, the non-differentiable functional is defined as

$$j_{\mathrm{QP}}(\mathbf{u}, \mathbf{v}) = -\langle Q(\sigma_n(P\mathbf{u})), \mathcal{F}|\mathbf{v}_t|\rangle_{L^2(\Gamma_C)} = -\int_{\Gamma_C} \mathcal{F}\, Q(\sigma_n(P\mathbf{u}))\, |\mathbf{v}_t|\,,$$

where $Q\colon H^{-1/2}(\Gamma_C) \to L^2(\Gamma_C)$ and $P\colon \mathbf{K} \to \mathbf{K}_f$ are two Lipschitz continuous operators. Here, the convex set $\mathbf{K}$ is from (148) and

$$\mathbf{K}_f = \{\mathbf{v} \in \mathbf{H}^1(\Omega)\colon a(\mathbf{v}, \varphi) = \int_\Omega \mathbf{f}\cdot\varphi \text{ for any } \varphi \in \mathcal{D}(\Omega)^d\}\,.$$

For $\mathbf{v} \in \mathbf{K}_f$, we have (cf. [67]), $\|\sigma_n(\mathbf{v})\|_{H^{-1/2}(\partial\Omega)} \leq C(\|\mathbf{v}\|_{\mathbf{H}^1(\Omega)} + \|\mathbf{f}\|_{\mathbf{L}^2(\Omega)})$. The operator P is usually defined as a projection operator in $\mathbf{H}^1$ and we see that if $\mathbf{u}$ is a solution of problem (147) or (152), then $P\mathbf{u} = \mathbf{u}$. Consequently, problem (152) is equivalent to: find $\mathbf{u} \in \mathbf{K}$ s.t.

$$a(\mathbf{u}, \mathbf{v}-\mathbf{u}) + j_{\mathrm{Q}}(\mathbf{u}, \mathbf{v}) - j_{\mathrm{Q}}(\mathbf{u}, \mathbf{u}) \;\geq\; f(\mathbf{v}-\mathbf{u}), \qquad \mathbf{v} \in \mathbf{K}, \tag{153}$$

where

$$j_{\mathrm{Q}}(\mathbf{u}, \mathbf{v}) = -\int\limits_{\Gamma_C} \mathcal{F}\, Q(\sigma_n(\mathbf{u}))\, |\mathbf{v}_t|\,.$$

We can prove that there exists a constant $\mathcal{C} > 0$ such that

$$\begin{aligned} &|j_{\mathrm{QP}}(\mathbf{u}_1, \mathbf{v}_2) + j_{\mathrm{QP}}(\mathbf{u}_2, \mathbf{v}_1) - j_{\mathrm{QP}}(\mathbf{u}_1, \mathbf{v}_1) - j_{\mathrm{QP}}(\mathbf{u}_2, \mathbf{v}_2)| \\ &\quad \leq\; \mathcal{C}\|\mathcal{F}\|_{L^\infty(\Gamma_C)}\|\mathbf{u}_1 - \mathbf{u}_2\|_{\mathbf{H}^1(\Omega)}\|\mathbf{v}_1 - \mathbf{v}_2\|_{\mathbf{H}^1(\Omega)} \end{aligned} \tag{154}$$

for any $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{v}_1$, $\mathbf{v}_2$ $\in \mathbf{K}$. Usually, Q is taken as a convolution, $Q(\tau) = \omega * \tau$, $\tau \in H^{-1/2}(\Gamma_C)$, where $\omega \in \mathcal{D}(-\eta, \eta)$, $\int_{-\eta}^{\eta} \omega = 1$, with $\eta \in \mathbf{R}$, $\eta > 0$, but other choices of this operator can be made. We have the following existence and uniqueness result.

Proposition 3.9 *If* $\mathbf{f} \in \mathbf{L}^2(\Omega)$, $\mathbf{g} \in \mathbf{H}^{1/2}(\Gamma_D)$, $\mathbf{p} \in \mathbf{L}^2(\Gamma_N)$, $\mathcal{F} \in L^\infty(\Gamma_C)$, $\mathcal{F} \geq 0$ *a.e. on* Γ_C, *and* $\mathcal{C}\|\mathcal{F}\|_{L^\infty(\Gamma_C)} < \alpha$, *with* $\mathcal{C}$ *as in (154) and* α *as in (150), then problem (152) has a unique solution. Moreover, this solution is also the unique solution of problem (153).*

To introduce the one- and two-level methods, let us denote by $\mathcal{N}_h$ the nodes of $\mathcal{T}_h$ lying in Γ_C. For a node $x_\kappa \in \mathcal{N}_h$, we denote by $\mathcal{E}_\kappa$ the set of the triangle edges, for $d = 2$, or tetrahedron sides (i.e. triangles), for $d = 3$, which have x_κ as a vertex. We shall write $\mathcal{E} = \{e \in \mathcal{E}_\kappa \colon x_\kappa \in \mathcal{N}_h\}$. Also, we denote by $\mathbf{N}_\kappa$ the set of the outward normals to Ω associated to the elements in $\mathcal{E}_\kappa$. With these notations, we define the convex set

$$\mathbf{K}_h = \{\mathbf{v} \in \mathbf{V}_h \colon \mathbf{v} = \mathbf{g} \text{ on } \Gamma_D, \text{ and } \mathbf{v}(x_\kappa)\cdot\mathbf{n} \leq 0 \text{ for any } x_\kappa \in \mathcal{N}_h \text{ and } \mathbf{n} \in \mathbf{N}_\kappa\}\,.$$

Inequality (151) is a variational inequality of the second kind. The fixed $\tau \leq 0$, will be chosen constant on each $e \in \mathcal{E}$, and, for $\mathcal{F}$ continuous, the term under the integral of j_τ can be linearly approximated (trapezoidal rule for $d = 2$), i.e.,

$$\begin{aligned} \varphi(\mathbf{v}) &= -\sum_{e\in\mathcal{E}} \int_e L_{eh}(\mathcal{F}\,\tau|\mathbf{v}_t|) \\ &= -\sum_{x_\kappa\in\mathcal{N}_h}\sum_{e\in\mathcal{E}_\kappa} \mathcal{F}(x_\kappa)\tau|\mathbf{v}(x_\kappa) - (\mathbf{v}(x_\kappa)\cdot\mathbf{n})\mathbf{n}| \int_e \lambda_\kappa, \end{aligned} \tag{155}$$

where λ_κ is the trace on Γ_C of the basis function associated to the node x_κ, and L_{eh} is the P_1-Lagrangian interpolation over the edge e. Consequently, we consider the finite element variant of problem (151) with φ as defined in (155). The functional φ in (155) is of the form of that given in (143). Consequently, we can apply Theorem 3.9 to get the convergence and error estimate of the one- and two-level methods corresponding to Algorithm 3.11 for the contact problem with Tresca friction.

In the case of the contact problem with non-local Coulomb friction, we consider the quasi-variational inequality (152) in which the non-differentiable term j_{QP} is approximated by

$$\begin{aligned} \varphi(\mathbf{u},\mathbf{v}) &= -\sum_{e\in\mathcal{E}} \int_e Q(\sigma_n(P\mathbf{u}))L_{eh}(\mathcal{F}\,|\mathbf{v}_t|) \\ &= -\sum_{x_\kappa\in\mathcal{N}_h}\sum_{e\in\mathcal{E}_\kappa} \mathcal{F}(x_\kappa)|\mathbf{v}(x_\kappa) - (\mathbf{v}(x_\kappa)\cdot\mathbf{n})\mathbf{n}| \int_e Q(\sigma_n(P\mathbf{u}))\lambda_\kappa\,. \end{aligned} \tag{156}$$

This φ is of the form and has all asked properties of that in (145). Also, we can prove that (154) holds if we replace j_{QP} by its above approximation φ. Therefore, we can conclude that the one- and two-level methods corresponding to Algorithms 3.12–3.14 are convergent for the contact problem with non-local Coulomb friction, with the convergence rate given in Theorem 3.6. We furthermore point out that the convergence condition (138) in Theorem 3.6 and the existence and uniqueness condition in Theorem 3.9 are similar.

Let us conclude with some remarks on the application of Algorithms 3.12–3.14 to the problem with non-local friction. Since, in general, $\mathbf{u}^{n+\frac{i}{m}} \notin \mathbf{K}_f$, we kept in the definition of $\varphi(\mathbf{u}, \mathbf{v})$ in (156) the projection P. Thus, in point of view of the computing time, it seems that Algorithm 3.14 is more favorable than Algorithms 3.12 or 3.13. In this case, we have to compute the projection P only once in each iteration. Moreover, if we take in Algorithm 3.14 the m-th correction space as

$$\mathbf{V}_h^m = \{\mathbf{v} \in \mathbf{C}^0(\bar{\Omega})\colon \mathbf{v}|_t \in \mathbf{P}_1(t), t \in \mathcal{T}_h,\ \mathbf{v} = \mathbf{0} \text{ on } \Gamma_C\},$$

then $\mathbf{u}^n \in \mathbf{K}_f$, and we can omit P in the definitions of φ in (156),

$$\begin{aligned} \varphi(\mathbf{u}, \mathbf{v}) &= -\sum_{e \in \mathcal{E}} \int_e Q(\sigma_n(\mathbf{u})) L_{eh}(\mathcal{F}\,|\mathbf{v}_t|) \\ &= -\sum_{x_\kappa \in \mathcal{N}_h} \sum_{e \in \mathcal{E}_\kappa} \mathcal{F}(x_\kappa) |\mathbf{v}(x_\kappa) - (\mathbf{v}(x_\kappa) \cdot \mathbf{n})\mathbf{n}| \int_e Q(\sigma_n(\mathbf{u})) \lambda_\kappa. \end{aligned}$$

In this particular case, φ in Algorithm 3.14 approximates j_Q, the non-differentiable term of problem (153). We point out that the subproblem associated to the above subspace $\mathbf{V}_h^m$ is linear.

3.1.8 Application of the Schwarz method in geophysical problems (papers [20] and [25])

Papers [20] and [25] deal with numerical modeling of the nucleation (initiation) phase of rupture in a earthquake. This model is a dynamic faulting under slip-dependent friction in a linear elastic domain. In [20], a anti-plane shearing problem is considered, and a in-plane or 3D problem is studied in [25]. In both cases we have to solve contact problems with friction written as quasi-variational inequalities. The domain Ω contains a finite number of cuts. Its boundary $\partial\Omega$ is divided into two disjoint parts : the exterior boundary $\Gamma_d = \partial\bar{\Omega}$ and the internal one, Γ, composed by bounded connected arcs, called cracks or faults.

The use of an implicit time-stepping scheme (Newmark method) allows much larger values of the time step than the critical CFL (Courant - Friedrichs - Lewy) time step, and higher physical consistency with respect to the friction law. The finite element form of the quasi-variational inequality is solved by a Schwarz domain decomposition method, by separating the inner nodes of the domain from the nodes on the fault. In this way, the quasi-variational inequality splits in two subproblems. The first one is a large linear system of equations, and its unknowns are related to the mesh nodes of the first subdomain (i.e. lying inside the domain). The unknowns of the second subproblem are the degrees of freedom of the mesh nodes of the second subdomain (i.e. lying on the domain boundary where the conditions of contact and friction are imposed). This nonlinear subproblem is solved by the same Schwarz algorithm, leading to some local nonlinear subproblems of a very small size. Numerical experiments are performed to illustrate convergence in time and space, instability capturing, energy dissipation and the influence of normal stress variations.

Anti-plane shearing problem. The anti-plane shearing problem in [20] is stated as: find $w : \mathbf{R}_+ \times \Omega \to \mathbf{R}$, solution of the wave equation:

$$\rho \partial_{tt} w(t) = \mathrm{div}(G \nabla w(t)) \text{ in } \Omega$$

with boundary conditions on the exterior boundary

$$w(t) = 0 \text{ on } \Gamma_d$$

with boundary conditions of the Signorini type on the interior boundary

$$\begin{aligned}&[G\partial_n w(t)] = 0,\ [\partial_t w(t)] \geq 0,\ G\partial_n w(t) + f([w(t)]) \geq 0,\\ &[\partial_t w(t)](G\partial_n w(t) + f([w(t)])) = 0 \text{ on } \Gamma\end{aligned}$$

and with the initial conditions

$$w(0) = w_0,\ \partial_t w(0) = w_1 \text{ in } \Omega.$$

Above, we have denoted by $[\cdot]$ the jump across Γ, (i.e. $[v] = v^+ - v^-$), $\rho \in L^\infty(\Omega),\ \rho(x) \geq \rho_0 > 0$, a.e. in Ω, is the density, and $G \in L^\infty(\Omega),\ G(x) \geq G_0 > 0$, a.e. in Ω, is the shear rigidity.

The weak form can be written as: find the displacement $w : [0,T] \to W$ such that

$$\begin{aligned}&\partial_t w(t) \in W_+ : \ \int_\Omega \rho \partial_{tt} w(v - \partial_t w) + \int_\Omega G \nabla w \cdot \nabla (v - \partial_t w) + \\ &\int_\Gamma f([w])([v] - [\partial_t w]) \geq 0 \quad \forall v \in W_+\end{aligned} \tag{157}$$

where

$$W = \{v \in H^1(\Omega) : v = 0 \text{ on } \Gamma_d\}, \quad W_+ = \{v \in W : [v] \geq 0 \text{ on } \Gamma\}$$

We consider the Newmark method, with parameters $\beta = 1/4$ and $\gamma = 1/2$, for the time discretization of the above dynamic problem. To this end, let $\Delta t > 0$ be the time step, N the maximum number of steps, and $T = N\Delta t$. We denote by w^n, $\dot{w}^n$ and $\ddot{w}^n$ the discretization of the solution at time $t = n\Delta t$. Then, the numerical solution w^{n+1}, $\dot{w}^{n+1}$ and $\ddot{w}^{n+1}$ at time $t = (n+1)\Delta t$ is obtained from

$$w^{n+1} = w^n + \Delta t \dot{w}^n + \frac{(\Delta t)^2}{4}(\ddot{w}^{n+1} + \ddot{w}^n), \quad \dot{w}^{n+1} = \dot{w}^n + \frac{\Delta t}{2}(\ddot{w}^{n+1} + \ddot{w}^n)$$

Using these equations we get the time discretized form of (157) in terms of the velocity $\dot{w}^n$: find $\dot{w}^{n+1} \in W_+$ such that

$$\begin{aligned}&\int_\Omega \rho \dot{w}^{n+1}(v - \dot{w}^{n+1}) + \frac{(\Delta t)^2}{4} \int_\Omega G \nabla \dot{w}^{n+1} \cdot \nabla (v - \dot{w}^{n+1}) + \\ &\int_\Gamma h_n([\dot{w}^{n+1}])([v] - [\dot{w}^{n+1}]) \geq F_n(v - \dot{w}^{n+1}) \quad \forall v \in W_+\end{aligned} \tag{158}$$

where h_n and F_n are given by

$$\begin{aligned}h_n(x,s) &= \frac{\Delta t}{2} f(x, [w^n](x) + ([\dot{w}^n](x) + s)\frac{\Delta t}{2}) \\ F_n(v) &= \int_\Omega \rho(\dot{w}^n + \frac{\Delta t}{2}\ddot{w}^n)v - \frac{\Delta t}{2}\int_\Omega G\nabla(w^n + \frac{\Delta t}{2}\dot{w}^n) \cdot \nabla v\end{aligned}$$

If $\dot{w}^{n+1}$ is obtained, then one can deduce w^{n+1} and $\ddot{w}^{n+1}$ through

$$w^{n+1} = w^n + \frac{\Delta t}{2}(\dot{w}^n + \dot{w}^{n+1}), \quad \ddot{w}^{n+1} = \frac{2}{\Delta t}(\dot{w}^{n+1} - \dot{w}^n) - \ddot{w}^n$$

The use of an implicit scheme for the wave equation with frictional type conditions on the faults will imply that we have to solve a nonlinear problem, given by a variational inequality, at each time step. Let us put

$$\gamma = \frac{(\Delta t)^2}{4} G$$

and let us introduce the energy function $J_n : W \to \mathbf{R}$ given by

$$J_n(v) = \frac{1}{2}\int_\Omega \rho v^2 + \frac{1}{2}\int_\Omega \gamma |\nabla v|^2 + \int_\Gamma H_n([v]) - F_n(v)$$

where H_n, which is the antiderivative of h_n, represents the density of energy dissipated on the fault during the time interval $[n\Delta t, (n+1)\Delta t]$,

$$H_n(x, u) = \int_0^u h_n(x, s)ds \text{ a.e. } x \in \Gamma, \ \forall\, u \geq 0$$

Writing $u_{n+1} = \dot{w}^{n+1}$, problem (158) becomes the following elliptic quasi-variational problem: Find $u_{n+1} \in W_+$ such that

$$\begin{aligned} &\int_\Omega \rho u_{n+1}(v - u_{n+1}) + \int_\Omega \gamma \nabla u_{n+1} \cdot \nabla(v - u_{n+1}) + \\ &\int_\Gamma h_n([u_{n+1}])([v] - [u_{n+1}]) \geq F_n(v - u_{n+1}) \quad \forall v \in W_+ \end{aligned} \tag{159}$$

The following result proves the existence of the solution of problem (159).

Theorem 3.11 *If $u_{n+1} \in W_+$ is a local minimum for J_n, then u_{n+1} is a solution of (159). Moreover there exists at least a global minimum for J_n, i.e. there exists $u_{n+1} \in W_+$ such that*

$$J_n(u_{n+1}) \leq J_N(v) \quad \forall v \in W_+$$

In order to prove that J_n satisfies (38), let us analyze here what are the conditions to be imposed on the parameters Δt, G, ρ and $\partial_s f$, such that the functional J_n would be strongly coercive. To this end, we have to consider the following eigenvalue problem connected to (158) : Find $\Phi \in W$, $\Phi \neq 0$ and $\lambda^2 \in \mathbf{R}$ such that

$$\begin{aligned} &\text{div } (G\nabla(\Phi)) = \lambda^2 \rho \Phi \quad \text{ in } \Omega \\ &\Phi = 0 \text{ on } \Gamma_d, \quad [G\partial_n \Phi] = 0, \ G\partial_n\Phi = g[\Phi] \text{ on } \Gamma \end{aligned} \tag{160}$$

where $g(x) = -\inf_{s\in\mathbf{R}_+} \partial_s f(x, s) = -S(x) \inf_{s\in\mathbf{R}_+} \partial_s \mu(x, s)$. The above eigenvalue problem played a key role in the study of the nucleation phase of earthquakes. Through the first eigenvalue, important physical properties (characteristic time, critical fault length, etc.) were deduced.

The variational formulation of the eigenvalue problem is

$$\Phi \in W : \int_\Omega G(\nabla\Phi)\cdot(\nabla v) + \lambda^2 \int_\Omega \rho\Phi v = \int_\Gamma g[\Phi][v], \quad \forall v \in W$$

and we have the following result

Theorem 3.12 *Let Ω be bounded. Then*

i) The eigenvalues and eigenfunctions of (160) consist of a sequence $(\lambda_n^2, \Phi_n)_{n\in\mathbf{N}}$ with $\lambda_0^2 \geq \lambda_1^2 \geq \ldots \lambda_n^2 \geq \ldots$, and $\lambda_n^2 \to -\infty$

ii) Let $\beta > 0$ and let us denote by $\lambda_0^2(\beta)$ the first eigenvalue (160) of in which g was replaced by βg. Then $\beta \to \lambda_0^2(\beta)$ is a convex increasing function and the following inequality holds

$$\int_\Omega G|\nabla v|^2 + \lambda_0^2(\beta)\int_\Omega \rho v^2 \geq \int_\Gamma g[v], \quad \forall v \in W.$$

Note that, in general, λ_0^2 is not negative, hence there exist at most a finite number of positive eigenvalues.

Theorem 3.13 *Let Ω be bounded. Then*

i) J_n' is a Lipschitz functional, i.e. there exists a real constant b such that

$$||J_n'(v_1) - J_n'(v_2)||_{W'} \leq b||v_1 - v_2||_W$$

ii) If

$$\frac{(\Delta t)^2}{4}\lambda_0^2 < 1 \tag{161}$$

where λ_0^2 is given by the above theorem, then J_n is an uniformly convex functional, i.e. there exists $a > 0$ such that

$$J_n'(v_1)(v_1 - v_2) - J_n'(v_2)(v_1 - v_2) \geq a||v_1 - v_2||_W^2$$

The above condition (161) on the time step Δt is not a CFL-type condition. If the process is stable, i.e. $\lambda_0^2 \leq 0$, then there is no condition (in terms of convergence and stability) on the time step. If the process is unstable, i.e. $\lambda_0^2 > 0$, then (161), which is equivalent to

$$\Delta t < \Delta t_{cr} \equiv \frac{2}{\lambda_0}$$

is just a convergence criterion for the domain decomposition method which solves the non quadratic minimization problem at each time step. In all the physical applications we have considered, the critical time step Δt_{cr} as found to be very large (10 up to 100 times larger than the critical CFL time step).

In-plane or 3D problems. In the in-plane or 3D problem in [25], the exterior boundary of Ω consists of Γ_D and Γ_N, where the measure of Γ_D does not vanish. The elastodynamic problem consists in finding the displacement field $\mathbf{u} : [0,T] \times \Omega \to \mathbf{R}^d$ satisfying:

$$\begin{aligned}
&\mathbf{div}\, \boldsymbol{\sigma}(\mathbf{u}(t)) = \rho \ddot{\mathbf{u}}(t) \text{ in } \Omega,\\
&\boldsymbol{\sigma}(\mathbf{u}(t)) = \mathcal{C}\boldsymbol{\varepsilon}(\mathbf{u}(t)) \text{ in } \Omega,\\
&\mathbf{u}(t) = 0 \text{ on } \Gamma_D,\\
&\boldsymbol{\sigma}(\mathbf{u}(t))\mathbf{n} = 0 \text{ on } \Gamma_N.
\end{aligned}$$

The contact on Γ is assumed to be frictional, without separation, and the stick and slip zones are not known in advance:

$$\begin{aligned}
&[\dot{u}_n(t)] = 0, \quad [\boldsymbol{\sigma}(\mathbf{u}(t))\mathbf{n}] = 0,\\
&[\dot{\mathbf{u}}_t(t)] = 0 \;\Rightarrow\; |\boldsymbol{\sigma}_t(\mathbf{u}(t)) + \boldsymbol{\sigma}_t^p| \leq -\mu(s(t))(\sigma_n(\mathbf{u}(t)) + \sigma_n^p),\\
&[\dot{\mathbf{u}}_t(t)] \neq 0 \;\Rightarrow\; \boldsymbol{\sigma}_t(\mathbf{u}(t)) + \boldsymbol{\sigma}_t^p = \mu(s(t))(\sigma_n(\mathbf{u}(t)) + \sigma_n^p)\frac{[\dot{\mathbf{u}}_t(t)]}{|[\dot{\mathbf{u}}_t(t)]|}.
\end{aligned}$$

The initial conditions are

$$\mathbf{u}(0) = \mathbf{u}_0 \text{ and } \dot{\mathbf{u}}(0) = \mathbf{u}_1.$$

Above, $\mathcal{C}$ is the fourth order symmetric and elliptic tensor of linear elasticity,

$$s(t) = \int_0^t |[\dot{\mathbf{u}}_t(\xi)]| d\xi$$

is the total slip, μ is the friction coefficient, $\mu : \mathbf{R}_+ \to \mathbf{R}_+$ a decreasing Lipschitz function with respect to the slip, and $\boldsymbol{\sigma}^p$ is a pre-stress.

Writing $\mathbf{H} = L^2(\Omega)^d$, $\Sigma = H^{-\frac{1}{2}}(\Gamma)$, $\mathbf{V} = \{\mathbf{v} \in H^1(\Omega)^d : \mathbf{v} = 0 \text{ on } \Gamma_D,\ [v_n] = 0 \text{ on } \Gamma\}$, $\mathbf{W} = \{\mathbf{v} \in H^1(\Omega)^d : \mathbf{v} = 0 \text{ on } \Gamma_D,\ [\mathbf{v}_t] = 0 \text{ on } \Gamma\}$, the weak form can be written as: find the displacement $\mathbf{u} \in \mathbf{V}$ with $\dot{\mathbf{u}} \in \mathbf{V}$, $\ddot{\mathbf{u}} \in \mathbf{H}$ and $\sigma_n \in \Sigma$ verifying

$$\begin{aligned}
&b(\ddot{\mathbf{u}}, \mathbf{v} - \dot{\mathbf{u}}) + a(\mathbf{u}, \mathbf{v} - \dot{\mathbf{u}}) -\\
&\int_\Gamma \mu(s)(\sigma_n + \sigma_n^p)(||\mathbf{v}_t|| - ||\dot{\mathbf{u}}_t||) + \int_\Gamma \boldsymbol{\sigma}_t^p \cdot [\mathbf{v}_t - \dot{\mathbf{u}}_t] \geq 0 \quad \forall \mathbf{u} \in \mathbf{V}\\
&\int_\Gamma \sigma_n [w_n] = \int_\Omega \rho \ddot{\mathbf{u}} \cdot \mathbf{w} + \int_\Omega (\mathcal{C}\boldsymbol{\varepsilon}(\mathbf{u})) : (\boldsymbol{\varepsilon}(\mathbf{w}))
\end{aligned} \tag{162}$$

where

$$b(\mathbf{u},\mathbf{v}) = \int_\Omega \rho \mathbf{u}\cdot\mathbf{v} \text{ and } a(\mathbf{u},\mathbf{v}) = \int_\Omega (\mathcal{C}\boldsymbol{\varepsilon}(\mathbf{u})) : \boldsymbol{\varepsilon}(\mathbf{v})$$

With a time step Δt, as in [20], we consider a time discretization of (162) at $t_0, t_1, \ldots, t^N = T$ using the Newmark method, with parameters $\beta = 1/4$ and $\gamma = 1/2$. The numerical solution $\mathbf{u}^{n+1}$, $\dot{\mathbf{u}}^{n+1}$ and $\ddot{\mathbf{u}}^{n+1}$ at time $t = (n+1)\Delta t$ is obtained from

$$\mathbf{u}^{n+1} = \mathbf{u}^n + \Delta t \dot{\mathbf{u}}^n + \frac{(\Delta t)^2}{4}(\ddot{\mathbf{u}}^{n+1} + \ddot{\mathbf{u}}^n), \quad \dot{\mathbf{u}}^{n+1} = \dot{\mathbf{u}}^n + \frac{\Delta t}{2}(\ddot{\mathbf{u}}^{n+1} + \ddot{\mathbf{u}}^n)$$

Using these equations we get the time discretized form of (162) in terms of the velocity $\dot{\mathbf{u}}^n$: find $\dot{\mathbf{u}}^{n+1} \in \mathbf{V}$ and $\sigma_n^{n+1} \in \Sigma$ such that

$$\begin{aligned}
&b(\dot{\mathbf{u}}^{n+1}, \mathbf{v} - \dot{\mathbf{u}}^{n+1}) + \left(\frac{\Delta t}{2}\right)^2 a(\dot{\mathbf{u}}^{n+1}, \mathbf{v} - \dot{\mathbf{u}}^{n+1}) \\
&-\frac{\Delta t}{2}\int_\Gamma \mu(s^{n+1})(\sigma_n^{n+1} + \sigma_n^p)(|[\mathbf{v}_t]| - |[\dot{\mathbf{u}}_t^{n+1}]|) \geq F_n(\mathbf{v} - \dot{\mathbf{u}}^{n+1}) \quad \forall \mathbf{v} \in \mathbf{V} \\
&\frac{\Delta t}{2}\int_\Gamma \sigma_n^{n+1}[w_n] = b(\dot{\mathbf{u}}^{n+1}, \mathbf{w}) + \left(\frac{\Delta t}{2}\right)^2 a(\dot{\mathbf{u}}^{n+1}, \mathbf{w}) - F_n(\mathbf{w}) \quad \forall \mathbf{w} \in \mathbf{W}
\end{aligned}$$

where s^{n+1} and F_n are given by

$$\begin{aligned}
&s^{n+1} = s^n + \frac{\Delta t}{2}(|[\dot{\mathbf{u}}_t^{n+1}]| + |[\dot{\mathbf{u}}_t^n]|) \\
&F_n(\mathbf{v}) = b(\dot{\mathbf{u}}^n + \frac{\Delta t}{2}\ddot{\mathbf{u}}^n, \mathbf{v}) - \frac{\Delta t}{2} a(\mathbf{u}^n + \frac{\Delta t}{2}\dot{\mathbf{u}}^n, \mathbf{v}) - \frac{\Delta t}{2}\int_\Gamma \boldsymbol{\sigma}_t^p \cdot [\mathbf{v}_t]
\end{aligned}$$

If $\dot{\mathbf{u}}^{n+1}$ is found, then one can deduce $\mathbf{u}^{n+1}$ and $\ddot{\mathbf{u}}^{n+1}$ through

$$\mathbf{u}^{n+1} = \mathbf{u}^n + \frac{\Delta t}{2}(\dot{\mathbf{u}}^n + \dot{\mathbf{u}}^{n+1}) \text{ and } \ddot{\mathbf{u}}^{n+1} = \frac{2}{\Delta t}(\dot{\mathbf{u}}^{n+1} - \dot{\mathbf{u}}^n) - \ddot{\mathbf{u}}^n$$

Hence, the use of an implicit scheme for the wave equation with frictional type conditions on the faults will imply the resolution of a nonlinear problem, given by a quasi-variational inequality, at each time step.

One-level method. For the space discretization we use the linear finite elements. The domain Ω is decomposed in two overlapping subdomains, Ω_1 and Ω_2. Subdomain Ω_1 contains all the inner nodes, $\Omega_1 = \Omega$. Subdomain Ω_2

contains all nodes on Γ and is constructed as follows. Let us denote by x_i^+ and x_i^-, $i = 1, \ldots, n_\Gamma$, the pairs of nodes on the two sides of Γ having the same coordinates, and by ϕ_i^+ and ϕ_i^- the pair of the nodal basis functions corresponding to this nodes in the finite element space associated to the problem. Writing $O_i^+ = \mathrm{Int}(\mathrm{supp}\phi_i^+)$ and $O_i^- = \mathrm{Int}(\mathrm{supp}\phi_i^-)$, we define $\Omega_2 = \cup_{i=1}^{n_\Gamma}(O_i^+ \cup O_i^-)$. The Schwarz method (one-level method) is applied with these subdomains to solve the problem. The subproblem on Ω_1 is a linear one. The nonlinear subproblem on Ω_2 is iteratively solved by the same Schwarz method using the decomposition given by O_i^+, O_i^-, $i = 1, \ldots, n_\Gamma$, of Ω_2. Numerical experiments are given in both papers.

3.1.9 Multigrid methods with constraint level decomposition for variational inequalities (paper [28])

In [28], we introduce four multigrid algorithms for the constrained minimization of non-quadratic functionals. These algorithms are combinations of additive or multiplicative iterations on levels with additive or multiplicative ones over the levels. The convex set is decomposed as a sum of convex level subsets, and consequently, the algorithms have an optimal computing complexity. The methods are described as multigrid V-cycles, but the results hold for other iteration types, the W-cycle iterations, for instance. We estimate the global convergence rates of the proposed algorithms as functions of the number of levels, and compare them with the convergence rates of other existing multigrid methods. Even if the general convergence theory holds for convex sets which can be decomposed as a sum of convex level subsets, our algorithms are applied to the one-obstacle problems because, for these problems, we are able to construct optimal decompositions. But, in this case, the convergence rates of the methods introduced in this paper are better than those of the methods we know in the literature.

Abstract convergence result. We consider a reflexive Banach space V and $V_1, \ldots, V_J$, are some closed subspaces of V, where $V_J = V$. Let $K \subset V$ be a nonempty closed convex set, and we assume that there exist some convex sets $K_j \subset V_j$, $j = 1, \ldots, J$ such that

$$K = K_1 + \ldots + K_J \tag{163}$$

The algorithms we introduce will be combinations of additive or multiplicative algorithms over levels with additive or multiplicative algorithms on each level. To this end, we assume that at each level $1 \le j \le J$ we have I_j closed subspaces of V_j, V_{ji}, $i = 1, \ldots, I_j$, and we shall write $I = \max_{j \in J} I_j$. Also, for a fixed $\sigma > 1$, we assume that there exists a constant C_1 such that

$$||\sum_{j=1}^{J}\sum_{i=1}^{I_j} w_{ji}|| \le C_1 (\sum_{j=1}^{J}\sum_{i=1}^{I_j} ||w_{ji}||^{\sigma})^{\frac{1}{\sigma}} \tag{164}$$

for any $w_{ji} \in V_{ji}$, $j = J, \ldots, 1$, $i = 1, \ldots, I_j$. Evidently, we can take, for instance,

$$C_1 = (IJ)^{\frac{\sigma-1}{\sigma}} \tag{165}$$

but sharper estimations can be available in certain cases. In the case when we use multiplicative algorithms on the levels $1 \le j \le J$, we make the following

Assumption 3.8 *We assume that there exist two positive constants C_2 and C_3, and that any $w \in K$ can be written as $w = \sum_{j=1}^{J} w_j$, with $w_j \in K_j$, $j = 1, \ldots, J$, such that*

- for any $v \in K$,

- and any $w_{ji} \in V_{ji}$ satisfying $w_j + \sum_{k=1}^{i} w_{jk} \in K_j$, $j = 1, \ldots, J$, $i = 1, \ldots, I_j$, there exist $v_{ji} \in V_{ji}$, $j = 1, \ldots, J$, $i = 1, \ldots, I_j$, which satisfy

$$w_j + \sum_{k=1}^{i-1} w_{jk} + v_{ji} \in K_j \text{ for } j = 1, \ldots, J,\ i = 1, \ldots, I_j,$$

$$v - w = \sum_{j=1}^{J}\sum_{i=1}^{I_j} v_{ji} \text{ and } \sum_{j=1}^{J}\sum_{i=1}^{I_j} ||v_{ji}||^{\sigma} \le C_2^{\sigma} ||v - w||^{\sigma} + C_3^{\sigma} \sum_{j=1}^{J}\sum_{i=1}^{I_j} ||w_{ji}||^{\sigma}.$$

If we use additive algorithms on the levels $1 \le j \le J$, we assume

Assumption 3.9 *We assume that there exists a constant $C_2 > 0$, and that any $w \in K$ can be written as $w = \sum_{j=1}^{J} w_j$, with $w_j \in K_j$, $j = 1, \ldots, J$, such that for any $v \in K$,*

there exist $v_{ji} \in V_{ji}$, $j = 1, \ldots, J$, $i = 1, \ldots, I_j$, which satisfy

$$w_j + v_{ji} \in K_j \text{ for } j = 1, \ldots, J,\ i = 1, \ldots, I_j,$$

$$v - w = \sum_{j=1}^{J}\sum_{i=1}^{m} v_{ji} \textit{ and } \sum_{j=1}^{J}\sum_{i=1}^{I_j} ||v_{ji}||^\sigma \le C_2^\sigma ||v-w||^\sigma.$$

Remark 3.5 *In the proofs, for the writing union, we shall consider in Assumption 3.9 a constant* $C_3 = 0$ *and inequality* $\sum_{j=1}^{J}\sum_{i=1}^{I_j} ||v_{ji}||^\sigma \le C_2^\sigma ||v-w||^\sigma$ *will be written like in Assumption 3.8,* $\sum_{j=1}^{J}\sum_{i=1}^{I_j} ||v_{ji}||^\sigma \le C_2^\sigma ||v-w||^\sigma + C_3^\sigma \sum_{j=1}^{J}\sum_{i=1}^{I_j} ||w_{ji}||^\sigma$, *for any* $w_{ji} \in V_{ji}$.

We consider problem (10) where, as in [16], [22] or in [23], $F : V \to \mathbf{R}$ is a Gâteaux differentiable functional, which is assumed to be coercive on K, in the sense that $F(v) \to \infty$, as $||v|| \to \infty$, $v \in K$, if K is not bounded, and its derivative satisfies (38) for $p,\ q > 1$ such that

$$\frac{p}{p-q+1} \le \sigma \le p \tag{166}$$

We can prove that if F satisfies (38), then

$$1 < q \le 2 \le p$$

In certain cases, the second equation in (38) can be refined, and we assume that there exist some constants $0 < \beta_{jk} \le 1$, $\beta_{jk} = \beta_{kj}$, $j,\ k = J, \ldots, 1$, such that

$$\langle F'(v + v_{ji}) - F'(v), v_{kl}\rangle \le \beta_M \beta_{jk} ||v_{ji}||^{q-1} ||v_{kl}|| \tag{167}$$

for any $v \in V$, $v_{ji} \in V_{ji}$, $v_{kl} \in V_{kl}$ with $||v||, ||v+v_{ji}||, ||v_{kl}|| \le M$, $i = 1, \ldots, I_j$ and $l = 1, \ldots, I_k$. Evidently, in view of (38), the above inequality holds for

$$\beta_{jk} = 1,\ j,\ k = J, \ldots, 1 \tag{168}$$

To solve problem (10), we propose four algorithms which are either of additive or multiplicative type from a level to another one, in combination with additive or multiplicative iterations on the levels. We first define the algorithm which is of the multiplicative type over the levels as well as on each level.

Algorithm 3.15 *We start the algorithm with a* $u^0 \in K$ *and decompose it as in Assumption 3.8 with* $w = u^0$, $u^0 = u_1^0 + \ldots + u_J^0$, $u_j^0 \in K_j$, $j = 1, \ldots, J$.

At iteration $n+1$, $n \geq 0$, assuming that we have $u^n \in K$, we decompose it as in Assumption 3.8 with $w = u^n$, $u^n = u_1^n + \ldots + u_J^n$, $u_j^n \in K_j$, $j = 1, \ldots, J$. Then, for $j \in J, \ldots, 1$,

- we successively calculate, the corrections $w_j^{n+1} \in V_j$, $u_j^n + w_j^{n+1} \in K_j$, by the multiplicative algorithm: we first write $w_j^n = 0$, and for $i = 1, \ldots, I_j$, successively calculate $w_{ji}^{n+1} \in V_{ji}$, $u_j^n + w_j^{n+\frac{i-1}{I_j}} + w_{ji}^{n+1} \in K_j$, the solution of the inequality

$$\langle F'\left(u^n + \sum_{k=j+1}^{J} w_k^{n+1} + w_j^{n+\frac{i-1}{I_j}} + w_{ji}^{n+1}\right), v_{ji} - w_{ji}^{n+1}\rangle \geq 0 \tag{169}$$

for any $v_{ji} \in V_{ji}$, $u_j^n + w_j^{n+\frac{i-1}{I_j}} + v_{ji} \in K_j$, and write $w_j^{n+\frac{i}{I_j}} = w_j^{n+\frac{i-1}{I_j}} + w_{ji}^{n+1}$,

- then, we write, $u^{n+\frac{J-j+1}{J}} = u^{n+\frac{J-j}{J}} + w_j^{n+1}$.

The algorithm which is of multiplicative type over the levels and of the additive type on levels is written as,

Algorithm 3.16 *We start the algorithm with an $u^0 \in K$ and decompose it as in Assumption 3.9 with $w = u^0$, $u^0 = u_1^0 + \ldots + u_J^0$, $u_j^0 \in K_j$, $j = 1, \ldots, J$. At iteration $n+1$, $n \geq 0$, assuming that we have $u^n \in K$, we decompose it as in Assumption 3.9 with $w = u^n$, $u^n = u_1^n + \ldots + u_J^n$, $u_j^n \in K_j$, $j = 1, \ldots, J$. Then, for $j = J, \ldots, 1$,*

- we successively calculate, the corrections $w_j^{n+1} \in V_j$, $u_j^n + w_j^{n+1} \in K_j$, by the additive algorithm: we simultaneously calculate $w_{ji}^{n+1} \in V_{ji}$, $u_j^n + w_{ji}^{n+1} \in K_j$, the solution of the inequality

$$\langle F'\left(u^n + \sum_{k=j+1}^{J} w_k^{n+1} + w_{ji}^{n+1}\right), v_{ji} - w_{ji}^{n+1}\rangle \geq 0 \tag{170}$$

for any $v_{ji} \in V_{ji}$, $u_j^n + v_{ji} \in K_j$, and write $w_j^{n+1} = \frac{r}{I} \sum_{i=1}^{I_j} w_{ji}^{n+1}$, with a fixed $0 < r \leq 1$.

- then, we write, $u^{n+\frac{J-j+1}{J}} = u^{n+\frac{J-j}{J}} + w_j^{n+1}$.

Now, the additive algorithm over levels and which is of multiplicative type on each level reads,

Algorithm 3.17 *We start the algorithm with an* $u^0 \in K$ *and decompose it as in Assumption 3.8 with* $w = u^0$, $u^0 = u_1^0 + \ldots + u_J^0$, $u_j^0 \in K_j$, $j = 1, \ldots, J$. *At iteration* $n+1$, $n \geq 0$, *assuming that we have* $u^n \in K$, *we decompose it as in Assumption 3.8 with* $w = u^n$, $u^n = u_1^n + \ldots + u_J^n$, $u_j^n \in K_j$, $j = 1, \ldots, J$. *Then we simultaneously calculate, for* $j = 1, \ldots, J$, *the corrections* $w_j^{n+1} \in V_j$, $u_j^n + w_j^{n+1} \in K_j$, *by the multiplicative algorithm:*

– we first write $w_j^n = 0$, *and for* $i = 1, \ldots, I_j$, *successively calculate* $w_{ji}^{n+1} \in V_{ji}$, $u_j^n + w_j^{n+\frac{i-1}{I_j}} + w_{ji}^{n+1} \in K_j$, *the solution of the inequality*

$$\langle F' \left(u^n + w_j^{n+\frac{i-1}{I_j}} + w_{ji}^{n+1} \right), v_{ji} - w_{ji}^{n+1} \rangle \geq 0 \tag{171}$$

for any $v_{ji} \in V_{ji}$, $u_j^n + w_j^{n+\frac{i-1}{I_j}} + v_{ji} \in K_j$, *and write* $w_j^{n+\frac{i}{I_j}} = w_j^{n+\frac{i-1}{I_j}} + w_{ji}^{n+1}$,
Then, we write $u^{n+1} = u^n + \frac{s}{J} \sum_{j=1}^{J} w_j^{n+1}$, *with a fixed* $0 < s \leq 1$.

Finally, the algorithm which is of additive type over the levels as well as on each level is written as,

Algorithm 3.18 *We start the algorithm with an* $u^0 \in K$ *and decompose it as in Assumption 3.9 with* $w = u^0$, $u^0 = u_1^0 + \ldots + u_J^0$, $u_j^0 \in K_j$, $j = 1, \ldots, J$. *At iteration* $n+1$, $n \geq 0$, *assuming that we have* $u^n \in K$, *we decompose it as in Assumption 3.9 with* $w = u^n$, $u^n = u_1^n + \ldots + u_J^n$, $u_j^n \in K_j$, $j = 1, \ldots, J$. *Then we simultaneously calculate, for* $j = 1, \ldots, J$, *the corrections* $w_j^{n+1} \in V_j$, $u_j^n + w_j^{n+1} \in K_j$, *by the additive algorithms:*

– we simultaneously calculate $w_{ji}^{n+1} \in V_{ji}$, $u_j^n + w_{ji}^{n+1} \in K_j$, *the solution of the inequality*

$$\langle F' \left(u^n + w_{ji}^{n+1} \right), v_{ji} - w_{ji}^{n+1} \rangle \geq 0 \tag{172}$$

for any $v_{ji} \in V_{ji}$, $u_j^n + v_{ji} \in K_j$, *and write* $w_j^{n+1} = \frac{r}{I} \sum_{i=1}^{I_j} w_{ji}^{n+1}$, *with a fixed* $0 < r \leq 1$.
Then, we write $u^{n+1} = u^n + \frac{s}{J} \sum_{j=1}^{J} w_j^{n+1}$, *with a fixed* $0 < s \leq 1$.

The convergence result is given by

Theorem 3.14 *We consider that* V *is a reflexive Banach,* V_j, $j = 1, \ldots, J$, *are closed subspaces of* V, *and* V_{ji}, $i = 1, \ldots, I_j$, *are some closed subspaces of*

V_j, $j = 1, \dots, J$. Let K be a non empty closed convex subset of V decomposed as in (163) where K_j are closed convex subsets of V_j, $j = 1, \dots, J$, and F be a Gâteaux differentiable functional on V which is supposed to be coercive if K is not bounded, and satisfies (38). Also, we assume that Assumption 3.8 holds for Algorithms 3.15 and 3.17, and Assumption 3.9 holds for Algorithms 3.16 and 3.18. On these conditions, if u is the solution of problem (10) and u^n, $n \geq 0$, are its approximations obtained from the above described algorithms, then there exists $M > 0$ such that $||u||$, $||u^n|| \leq M$, $n \geq 0$, and the following error estimations hold:

(i) if $p = q = 2$ we have

$$F(u^n) - F(u) \leq \left(\frac{\tilde{C}_1}{\tilde{C}_1 + 1} \right)^n \left[F(u^0) - F(u) \right], \tag{173}$$

$$||u^n - u||^2 \leq \frac{2}{\alpha_M} \left(\frac{\tilde{C}_1}{\tilde{C}_1 + 1} \right)^n \left[F(u^0) - F(u) \right], \tag{174}$$

and

(ii) if $p > q$ we have

$$F(u^n) - F(u) \leq \frac{F(u^0) - F(u)}{\left[1 + n\tilde{C}_2 \left(F(u^0) - F(u) \right)^{\frac{p-q}{q-1}} \right]^{\frac{q-1}{p-q}}}, \tag{175}$$

$$||u - u^n||^p \leq \frac{p}{\alpha_M} \frac{F(u^0) - F(u)}{\left[1 + n\tilde{C}_2 \left(F(u^0) - F(u) \right)^{\frac{p-q}{q-1}} \right]^{\frac{q-1}{p-q}}}. \tag{176}$$

Constants $\tilde{C}_1$ and $\tilde{C}_2$ are given by

$$\tilde{C}_1 = \frac{1-t}{t} + \frac{1}{C_2 t \varepsilon} \left[\frac{C_2}{\varepsilon} + 1 + C_1 C_2 + C_3 \right] \tag{177}$$

and

$$\tilde{C}_2 = \frac{p-q}{(p-1)(F(u^0) - F(u))^{\frac{p-q}{q-1}} + (q-1)\tilde{C}_3^{\frac{p-1}{q-1}}}. \tag{178}$$

where

$$\tilde{C}_3 = \frac{1-t}{t}(F(u^0)-F(u))^{\frac{p-q}{p-1}} + \frac{\frac{\alpha_M}{p}}{C_2\varepsilon}\left[\frac{C_2}{\varepsilon^{\frac{1}{p-1}}(t\frac{\alpha_M}{p})^{\frac{q-1}{p-1}}} + \right.$$
$$\left. \frac{(1+C_1C_2+C_3)(IJ)^{\frac{p-\sigma}{p\sigma}}}{(t\frac{\alpha_M}{p})^{\frac{q}{p}}}(F(u^0)-F(u))^{\frac{p-q}{p(p-1)}}\right] \tag{179}$$

with

$$t = \begin{cases} 1 & \textit{for Algorithm } 3.15 \\ \frac{r}{I} & \textit{for Algorithm } 3.16 \\ \frac{s}{J} & \textit{for Algorithm } 3.17 \\ \frac{s}{J}\frac{r}{I} & \textit{for Algorithm } 3.18 \end{cases} \tag{180}$$

and

$$\varepsilon = \frac{\alpha_M}{p}\frac{1}{2C_2\beta_M I^{\frac{\sigma-1}{\sigma}+\frac{p-q+1}{p}} J^{\frac{\sigma-1}{\sigma}-\frac{q-1}{p}}(\max_{k=1,\cdots,J}\sum_{j=1}^{J}\beta_{kj})} \tag{181}$$

Multilevel methods. We consider a family of regular meshes $\mathcal{T}_{h_j}$ of mesh sizes h_j, $j=1,\ldots,J$ over the domain $\Omega \subset \mathbf{R}^d$, and make the same assumptions on them as in Section 3.1.4 (paper [22]). Also, we introduce the same linear finite element spaces V_{h_j}, $j=1,\ldots,J$, corresponding to the levels, and their subspaces $V_{h_j}^i$, $i=1,\ldots,I_j$ associated with the domain decompositions $\{\Omega_j^i\}_{1\le i\le I_j}$, at each level $j=1,\ldots,J$. As in [22], these finite element spaces will be considered as subspaces of $W^{1,\sigma}$, $1\le\sigma\le\infty$. We denote by $||\cdot||_{0,\sigma}$ the norm in L^σ, and by $||\cdot||_{1,\sigma}$ and $|\cdot|_{1,\sigma}$ the norm and seminorm in $W^{1,\sigma}$, respectively.

In V_{h_J}, we consider the one-obstacle problem given by inequality (10) with the convex set

$$K = \{v \in V_{h_J} : \varphi \le v\}, \tag{182}$$

with $\varphi \in V_{h_J}$. We shall prove that Assumptions 3.8 and 3.9 hold for this type of convex set, and explicitly write the constants C_2 and C_3 as functions of the mesh and overlapping parameters. We can then conclude from Theorem 3.3 that if the functional F has the asked properties, then Algorithms 3.15–3.18 are globally convergent. To this end, we use again nonlinear interpolation operators $I_{h_k} : V_{h_{j+1}} \to V_{h_j}$, $j = 1, \cdots, J-1$, which are similar to the operator $I_H : V_h \to V_H$ defined in (25).

Now, for a $v \in V_{h_J}$, we recursively define

$$v^J = v \text{ and } v^j = I_{h_j} v^{j+1}, \; j = J-1, \dots, 1 \tag{183}$$

and we have

Lemma 3.6 *Let $v^j, w^j \in V_{h_j}$, $j = J, \dots, 1$ defined as in (183) for some $v, w \in V_{h_J}$, respectively. Then, for $j = J, \dots, 1$, we have*

$$|v^j - w^j|_{1,\sigma} \le C C_{d,\sigma}(h_j, h_J) |v - w|_{1,\sigma} \tag{184}$$

Another property of the nonlinear interpolation operators I_{h_j} is given by the following lemma.

Lemma 3.7 *For any $v, w \in V_{h_{j+1}}$, $j = J-1, \dots, 1$, we have*

$$||v - w - I_{h_j} v + I_{h_j} w||_{0,\sigma} \le C h_j C_{d,\sigma}(h_j, h_{j+1}) |v - w|_{1,\sigma} \tag{185}$$

We consider a decomposition of $\varphi = \varphi_J + \ldots + \varphi_1$ with $\varphi_j \in V_{h_j}$, $j = J, \dots, 1$, and define

$$K_j = \{v \in V_{h_j} : \varphi_j \le v\}, \; j = J, \dots, 1 \tag{186}$$

In this way, we get a decomposition of K as in (163). For a $v \in K$, with the notation in 183, we write

$$\begin{aligned} &v_j = \varphi_j + (v - \varphi)^j - (v - \varphi)^{j-1}, \; j = J, \dots, 2 \\ &v_1 = \varphi_1 + (v - \varphi)^1 \end{aligned} \tag{187}$$

Evidently,

$$v_j \in K_j, \; j = J, \dots, 1, \text{ and } v = v_J + \ldots + v_1 \tag{188}$$

We have the following

Lemma 3.8 *If $v_j, w_j \in K_j$, $j = J, \dots, 1$, are defined as in (187) for some $v, w \in K$, respectively, then*

$$|v_j - w_j|_{1,\sigma} \le C C_{d,\sigma}(h_{j-1}, h_J) |v - w|_{1,\sigma} \tag{189}$$

and

$$||v_j - w_j||_{0,\sigma} \le C h_{j-1} C_{d,\sigma}(h_j, h_J) |v - w|_{1,\sigma} \tag{190}$$

where we take $h_0 = h_1$ for $j = 1$.

To prove that Assumption 3.8 holds, we associate to the decomposition $\{\Omega_j^i\}_{1\le i\le I_j}$ of Ω_j, some functions $\theta_j^i \in C(\bar{\Omega}_j)$, $\theta_j^i|_\tau \in P_1(\tau)$ for any $\tau \in \mathcal{T}_{h_j}$, $i = 1, \cdots, I_j$, such that

$$
\begin{aligned}
&0 \le \theta_j^i \le 1 \text{ on } \Omega_j, \\
&\theta_j^i = 0 \text{ on } \cup_{l=i+1}^{I_j} \Omega_j^l \backslash \Omega_j^i,\ \theta_j^i = 1 \text{ on } \Omega_j^i \backslash \cup_{l=i+1}^{I_j} \Omega_j^l
\end{aligned}
\tag{191}
$$

Also, for Assumption 3.9, we associate a unity partition to each domain decomposition $\{\Omega_j^i\}_{1\le i\le I_j}$, $j = J, \ldots, 1$,

$$
0 \le \theta_j^i \le 1 \text{ and } \sum_{i=1}^{I_j} \theta_j^i = 1 \text{ on } \Omega_j \tag{192}
$$

with $\theta_j^i \in C(\bar{\Omega}_j)$, $\theta_j^i|_\tau \in P_1(\tau)$ for any $\tau \in \mathcal{T}_{h_j}$, $i = 1, \cdots, I_j$. Moreover, since the overlapping size of the domain decomposition on a level $j = J, \ldots, 1$ is δ_j, the above functions θ_j^i can be chosen to satisfy

$$
|\partial_{x_k} \theta_j^i| \le C/\delta_j, \text{ a.e. in } \Omega_j, \text{ for any } k = 1, \ldots, d \tag{193}
$$

The above three equations are similar with (19), (8) and (20), respectively, which have been introduced for the one-level method.

Finally, we recall some interpolation properties. For a $v \in V_{h_j}$ and a continuous functions θ which is of polynomial form on the elements of $\tau \in \mathcal{T}_{h_j}$, we have,

$$
||\theta v - L_{h_j}(\theta v)||_{0,\sigma} \le C h_j |\theta v|_{1,\sigma} \text{ and } |L_{h_j}(\theta v)|_{1,\sigma} \le C|\theta v|_{1,\sigma}
$$

where L_{h_j} is the P_1-Lagrangian interpolation operator which uses the function values at the nodes of the mesh $\mathcal{T}_{h_j}$. Therefore, we have

$$
||L_{h_j}(\theta v)||_{1,\sigma} \le C||\theta v||_{1,\sigma} \tag{194}
$$

Using these results, we can prove

Proposition 3.10 *For the convex sets K_j, $j = J, \ldots, 1$, defined in (186), Assumption 3.8 holds with the constants C_2 and C_3,*

$$
C_2 = C I^{\frac{\sigma+1}{\sigma}} \left[\sum_{j=2}^{J} C_{d,\sigma}(h_{j-1}, h_J)^\sigma \right]^{\frac{1}{\sigma}} \text{ and } C_3 = C I^{\frac{\sigma+1}{\sigma}} \tag{195}
$$

Also,

Proposition 3.11 *For the convex sets K_j, $j = J, \ldots, 1$, defined in (186), Assumption 3.9 holds with the constants C_2 and C_3,*

$$C_2 = CI^{\frac{1}{\sigma}}[\sum_{j=2}^{J} C_{d,\sigma}(h_{j-1}, h_J)^\sigma]^{\frac{1}{\sigma}} \text{ and } C_3 = 0 \tag{196}$$

The constants C_1 and β_{jk}, $j,\ k = J, \ldots, 1$, can be taken as in (165) and (168), but better choices are available in the case of the multigrid methods. As we see form the above estimations, the convergence rates given in Theorem 3.3 depend on the functional F, the maximum number of the subdomains on each level, I, and the number J of levels. The number of subdomains on levels can be associated with the number of colors needed to mark the subdomains such that the subdomains with the same color do not intersect with each other. Since this number of colors depends in general on the dimension of the Euclidean space where the domain lies, we can conclude that our convergence rate essentially depends on the number J of levels.

We first estimate the constants C_1–C_3 as functions of J. To this end, in the remainder of this section, C will be a generic constant which does not depend on J. Writing $S_{d,\sigma}(J) = \left[\sum_{j=2}^{J} C_{d,\sigma}(h_{j-1}, h_J)^\sigma\right]^{\frac{1}{\sigma}}$ from (49), we can consider

$$S_{d,\sigma}(J) = \begin{cases} (J-1)^{\frac{1}{\sigma}} & \text{if } d = \sigma = 1 \\ & \text{or } 1 \le d < \sigma < \infty \\ CJ & \text{if } 1 < d = \sigma < \infty \\ C^J & \text{if } 1 \le \sigma < d < \infty \end{cases} \tag{197}$$

in our estimations. In this general framework, we take C_1, and β_{jk}, $j,\ k = J, \ldots, 1$, as in (165) and (168),

$$C_1 = CJ^{\frac{\sigma-1}{\sigma}} \text{ and } \max_{k=1,\cdots,J} \sum_{j=1}^{J} \beta_{kj} = J \tag{198}$$

Also, from (195) and (196), we get

$$C_2 = CS_{d,\sigma}(J) \text{ and } C_3 = \begin{cases} C & \text{for Algorithms 3.15 and 3.17} \\ 0 & \text{for Algorithms 3.16 and 3.18} \end{cases} \tag{199}$$

As a consequence of Theorem 3.14 and Propositions 3.10 and 3.11 we have

Corollary 3.2 *Let us consider the finite element spaces V_{h_j} defined in (59) which are associated with the levels $j = 1, \dots, J$, and their subspaces $V_{h_j}^i$, $i = 1, \dots, I_j$, given in (60), which are associated with the level domain decompositions. Also, let K be the closed convex subset of $V = V_J$ given in (182), which is decomposed as a sum of the level closed convex sets $K_j \subset V_{h_j}$, $j = J, \dots, 1$, defined in (186). If F is a Gâteaux differentiable functional on V which is supposed to be coercive and to satisfy (154), then the approximation sequences u^n, $n \geq 0$ obtained from Algorithms 3.15–3.18 converge to the solution u of the one-obstacle problem (10) and the error estimations in Theorem 3.14 hold. The constants $\tilde{C}_1$ and $\tilde{C}_2$ in these error estimations depend on the number of levels J through the constants C_1–C_3 given in (198) and (199).*

Remark 3.6 1) The results of this section have referred to problems in $W^{1,\sigma}$ with Dirichlet boundary conditions, and the functions corresponding to the coarse levels have been extended with zero outside the domains Ω_j, $j = J-1, \dots, 1$. Let us assume that the problem has mixed boundary conditions: $\partial\Omega_J = \Gamma_d \cup \Gamma_n$, with Dirichlet conditions on Γ_d and Neumann conditions on Γ_n. In this case, if a node of $\mathcal{T}_{h_j}$, $j = J-1, \dots, 1$, lies in $\mathrm{Int}(\Gamma_n)$, we have to assume that all the sides of the elements $\tau \in \mathcal{T}_{h_j}$ having that node are included in Γ_n.

2) Similar convergence results with those ones presented in this section can be obtained for problems in $(W^{1,s})^d$.

Multigrid methods. In the above multilevel methods a mesh is the refinement of that one on the previous level, but the domain decompositions are almost independent from one level to another. We obtain similar multigrid methods by decomposing the level domains by the supports of the nodal basis functions. Consequently, the subspaces $V_{h_j}^i$, $i = 1, \dots, I_j$, are one-dimensional spaces generated by the nodal basis functions associated with the nodes of $\mathcal{T}_{h_j}$, $j = J, \dots, 1$. In this case Algorithms 3.15–3.18 are V-cycle multigrid iterations in which the smoothing steps are performed by a combination of multiplicative methods with additive ones. Evidently, similar results can be given for the W-cycle multigrid iterations.

Concerning the constants β_{jk}, $j, k = J, \dots, 1$, in (167), we can prove that, in the case of the multigrid methods, there exist such constants such that

$$\max_{k=1,\dots,J}\sum_{j=1}^{J}\beta_{kj}=C \tag{200}$$

where C is a constant independent of the meshes and their number. Also, the constant C_1 in (129) can be as,

$$C_1=(n!)^{\frac{1}{\sigma}}C^{\frac{n-1}{n}}\left(I\frac{\gamma^{\frac{d}{n}}}{\gamma^{\frac{d}{n}}-1}\right)^{\frac{n-1}{\sigma}} \tag{201}$$

where $n\in\mathbf{N}$, $n-1<\sigma\le n$, and C is a constant independent of the meshes and their number.

Now, we shall write the convergence rate of the multigrid Algorithms 3.15–3.18 in function of the number J of levels. To this end, we write the error estimations in Theorem 3.14 of the four algorithms using the above estimations of C_1 and $\max_{k=J,\dots,1}\sum_{j=1}^{J}\beta_{kj}$, and C_2 and C_3 given in (199). In order to be more conclusive, we limit ourselves to the typical example in Section 3.1.3 (paper [16]),

$$F(v)=\frac{1}{\sigma}|v|_{1,\sigma}^{\sigma}-L(v),\quad v\in W^{1,\sigma}(\Omega) \tag{202}$$

where L is a linear and continuous functional on $W^{1,\sigma}(\Omega)$, $\sigma>1$. In this case,

$$p=2,\ q=\sigma \text{ if } \sigma<2;\quad p=2,\ q=2 \text{ if } \sigma=2;\quad p=\sigma,\ q=2 \text{ if } \sigma>2$$

Evidently, we can use the same procedure for other problems, too.

For $\sigma=2$ and $p=q=2$, in view of (177) and (199), we get

$$\tilde{C}_1(J)=\begin{cases} CS_{d,2}(J)^2 & \text{for Algorithms 3.15 and 3.16} \\ CJS_{d,2}(J)^2 & \text{for Algorithms 3.17 and 3.18}\end{cases} \tag{203}$$

and, from Theorem 3.3, we have

$$||u^n-u||_{1,2}^2\le\tilde{C}_0\left(1-\frac{1}{1+\tilde{C}_1(J)}\right)^n \tag{204}$$

where $\tilde{C}_0$ is a constant independent of J.

For $1 < q = \sigma < 2$ and $p = 2$, in view of (179) and (199), we get

$$\tilde{C}_3(J) = \begin{cases} CJ^{\frac{(\sigma-1)(2-\sigma)}{\sigma}} S_{d,\sigma}(J)^2 & \text{for Algorithms 3.15 and 3.16} \\ CJ^{\frac{2(\sigma-1)}{\sigma}} S_{d,\sigma}(J)^2 & \text{for Algorithms 3.17 and 3.18} \end{cases} \tag{205}$$

From Theorem 3.14, we get that

$$||u^n - u||_{1,\sigma}^2 \le \tilde{C}_0 \frac{1}{\left(1 + n\tilde{C}_2(J)\right)^{\frac{\sigma-1}{2-\sigma}}} \tag{206}$$

where, in view of (178), we can take

$$\tilde{C}_2(J) = \frac{1}{1 + \tilde{C}_3(J)^{\frac{1}{\sigma-1}}} \tag{207}$$

For $p = \sigma > 2$ and $q = 2$, we get

$$\tilde{C}_3(J) = \begin{cases} CJ^{\frac{\sigma-2}{\sigma-1}} S_{d,\sigma}(J)^{\frac{\sigma}{\sigma-1}} & \text{for Algorithms 3.15 and 3.16} \\ CJS_{d,\sigma}(J)^{\frac{\sigma}{\sigma-1}} & \text{for Algorithms 3.17 and 3.18} \end{cases} \tag{208}$$

Finally, in this case, we have

$$||u^n - u||_{1,\sigma}^{\sigma} \le \tilde{C}_0 \frac{1}{\left(1 + n\tilde{C}_2(J)\right)^{\frac{1}{\sigma-2}}} \tag{209}$$

where

$$\tilde{C}_2(J) = \frac{1}{1 + \tilde{C}_3(J)^{\sigma-1}} \tag{210}$$

We make now some remarks on the above error estimations of the four algorithms. First, we point out that the above convergence results give global rate estimations. As we have expected, the multiplicative (over the levels) Algorithms 3.15 and 3.16 converge better than their additive variants, Algorithms 3.17 and 3.18. For the complementarity problems, we can compare the convergence rates of the four multigrid algorithms with the similar ones in the literature. In this case, $p = q = \sigma = d = 2$ in the above example, from (204) and (203), we get that the convergence rate of Algorithms 3.15 and

3.16 is of $1 - \frac{1}{1+CJ^2}$, and that of Algorithms 3.17 and 3.18 is of $1 - \frac{1}{1+CJ^3}$. For the truncated monotone multigrid method, an asymptotic convergence rate of $1 - \frac{1}{1+CJ^4}$, and under some conditions, of $1 - \frac{1}{1+CJ^3}$, is found in [53] and [44]. An estimate of $1 - \frac{1}{1+CJ^3}$ is also obtained in [53] for the asymptotic convergence rate of the standard monotone multigrid methods. In [44], it is mentioned that this asymptotic rate may be of $1 - \frac{1}{1+CJ^2}$, or even of $1 - \frac{1}{1+CJ}$, under some conditions.

3.1.10 Navier-Stokes/Darcy coupling (paper [26])

In [26], the coupling of the Navier-Stokes and Darcy equations is considered for modeling the interaction between surface and porous-media flows. The problem is formulated as an interface equation by means of the associated (nonlinear) Steklov-Poincaré operators, and the well-posedness is proved. Iterative methods to solve a conforming finite element approximation of the coupled problem are proposed and analyzed. Finally, numerical examples are given to illustrate the convergence of the proposed methods.

Let $\Omega \subset R^d$ $(d = 2, 3)$ be a bounded domain, decomposed into two non intersecting subdomains Ω_f and Ω_p separated by an interface Γ. The motion of the fluid is described by the Navier-Stokes equation in Ω_f and by the Darcy's law in Ω_p.

In order to describe the motion of the fluid in Ω_f, we introduce the Navier-Stokes equations: $\forall t > 0$,

$$\begin{array}{rcll} \partial_t \mathbf{u}_f - \boldsymbol{\nabla} \cdot \mathsf{T}(\mathbf{u}_f, p_f) + (\mathbf{u}_f \cdot \nabla)\mathbf{u}_f & = & \mathbf{f} & \text{in } \Omega_f \ , \\ \nabla \cdot \mathbf{u}_f & = & 0 & \text{in } \Omega_f \ , \end{array} \tag{211}$$

where $\mathsf{T}(\mathbf{u}_f, p_f) = \nu(\nabla \mathbf{u}_f + \nabla^T \mathbf{u}_f) - p_f \mathsf{I}$ is the Cauchy stress tensor, $\nu > 0$ is the kinematic viscosity of the fluid, while $\mathbf{u}_f$ and p_f are the fluid velocity and pressure, respectively; ∇ is the gradient operator with respect to the space coordinates.

In the domain Ω_p we define the piezometric head $\varphi = z + p_p/(\rho_f g)$, where z is the elevation from a reference level, p_p is the pressure of the fluid in Ω_p, ρ_f its density and g is the gravity acceleration.

The fluid motion in Ω_p is described by the equations:

$$\begin{array}{rcll} n\mathbf{u}_p & = & -K\nabla\varphi & \text{in } \Omega_p \ , \\ \nabla \cdot \mathbf{u}_p & = & 0 & \text{in } \Omega_p \ , \end{array} \tag{212}$$

where $\mathbf{u}_p$ is the fluid velocity, n is the volumetric porosity and K is the hydraulic conductivity tensor $K = diag(K_1, \dots, K_d)$ with $K_i \in L^\infty(\Omega_p)$, $i = 1, \dots, d$. The first equation is Darcy's law. In the following we shall denote $\mathsf{K} = K/n = diag(K_i/n)$ $(i = 1, \dots, d)$.

For the sake of clarity, in our analysis we shall adopt homogeneous boundary conditions. The treatment of non-homogeneous conditions involves some additional technicalities, but neither the guidelines of the theory nor the final results are affected. In particular, for the Navier-Stokes problem we impose the no-slip condition $\mathbf{u}_f = \mathbf{0}$ on $\partial\Omega_f \setminus \Gamma$, while for the Darcy problem, we set the piezometric head $\varphi = 0$ on Γ_p^D and we require the normal velocity to be null on Γ_p^N, $\mathbf{u}_p \cdot \mathbf{n}_p = 0$ on Γ_p^N, where $\partial\Omega_p = \Gamma \cup \Gamma_p^D \cup \Gamma_p^N$. $\mathbf{n}_p$ and $\mathbf{n}_f$ denote the unit outward normal vectors to the surfaces $\partial\Omega_p$ and $\partial\Omega_f$, respectively, and we have $\mathbf{n}_f = -\mathbf{n}_p$ on Γ. We suppose $\mathbf{n}_f$ and $\mathbf{n}_p$ to be regular enough. In the following we shall indicate $\mathbf{n} = \mathbf{n}_f$ for simplicity of notation.

We supplement the Navier-Stokes and Darcy problems with the following conditions on Γ:

$$
\begin{aligned}
\mathbf{u}_p \cdot \mathbf{n} &= \mathbf{u}_f \cdot \mathbf{n} \,, && (213)\\
-\mathbf{n} \cdot (\mathsf{T}(\mathbf{u}_f, p_f) \cdot \mathbf{n}) &= g\varphi \,, && (214)\\
-\varepsilon \boldsymbol{\tau}_i \cdot (\mathsf{T}(\mathbf{u}_f, p_f) \cdot \mathbf{n}) &= \nu \mathbf{u}_f \cdot \boldsymbol{\tau}_i \,, \qquad i = 1, \dots, d-1 \,, && (215)
\end{aligned}
$$

where $\boldsymbol{\tau}_i$ $(i = 1, \dots, d-1)$ are linear independent unit tangential vectors to the boundary Γ, and ε is the characteristic length of the pores of the porous medium.

Conditions (213) and (214) impose the continuity of the normal velocity on Γ, as well as that of the normal component of the normal stress, however they allow pressure to be discontinuous across the interface. The so-called Beavers-Joseph-Saffman condition (215) does not yield any coupling. Indeed, it provides a boundary condition for the Navier-Stokes problem since it involves only quantities in the domain Ω_f.

From now on, we focus on the steady problem obtained by dropping the time derivative in the momentum equation (211). This can be motivated by, e.g., the use of an implicit time-advancing scheme on the time-dependent problem (211). Moreover, instead of (212), we consider the following equivalent formulation for Darcy problem:

$$
\text{find } \varphi \; : \quad -\nabla \cdot (\mathsf{K} \nabla \varphi) = 0 \quad \text{in } \Omega_p \,. \qquad (216)
$$

We denote by $|\cdot|_1$ and $\|\cdot\|_1$ the H^1–seminorm and norm, respectively, and by $\|\cdot\|_0$ the L^2–norm; it will always be clear form the context whether we are referring to spaces on Ω_f or Ω_p.

We define the following functional spaces:

$$H_f = \{\mathbf{v} \in (H^1(\Omega_f))^d : \mathbf{v} = \mathbf{0} \text{ on } \partial\Omega_f \setminus \Gamma\}, \tag{217}$$

$$H_f^0 = \{\mathbf{v} \in H_f : \mathbf{v} \cdot \mathbf{n} = 0 \text{ on } \Gamma\}, \tag{218}$$

$$V_f = \{\mathbf{v} \in H_f : \nabla \cdot \mathbf{v} = 0 \text{ in } \Omega_f\},\ V_f^0 = \{\mathbf{v} \in H_f^0 : \nabla \cdot \mathbf{v} = 0 \text{ in } \Omega_f\}, \tag{219}$$

$$H_p = \{\psi \in H^1(\Omega_p) : \psi = 0 \text{ on } \Gamma_p^D\},\ H_p^0 = \{\psi \in H_p : \psi = 0 \text{ on } \Gamma\}, \tag{220}$$

$$Q = L^2(\Omega_f),\ Q_0 = \{q \in Q : \textstyle\int_{\Omega_f} q = 0\}. \tag{221}$$

By imposing the continuity of the normal velocity as well as that of the normal component of the normal stress on the interface Γ, and using the Beavers-Joseph-Saffman condition, the weak formulation of the the coupling of the Navier-Stokes and Darcy equations is written as: find $\mathbf{u}_f \in H_f$, $p_f \in Q$, $\varphi \in H_p$ such that

$$\begin{aligned} & a_f(\mathbf{u}_f, \mathbf{v}) + c_f(\mathbf{u}_f; \mathbf{u}_f, \mathbf{v}) + b_f(\mathbf{v}, p_f) \\ & \quad + \int_\Gamma g\,\varphi(\mathbf{v} \cdot \mathbf{n}) + \int_\Gamma \sum_{j=1}^{d-1} \frac{\nu}{\varepsilon}(\mathbf{u}_f \cdot \boldsymbol{\tau}_j)(\mathbf{v} \cdot \boldsymbol{\tau}_j) = \int_{\Omega_f} \mathbf{f} \cdot \mathbf{v}\,, \end{aligned} \tag{222}$$

$$b_f(\mathbf{u}_f, q) = 0\,, \tag{223}$$

$$a_p(\varphi, \psi) = \int_\Gamma \psi(\mathbf{u}_f \cdot \mathbf{n})\,, \tag{224}$$

for all $\mathbf{v} \in H_f$, $q \in Q$, $\psi \in H_p$, where

$$a_f(\mathbf{v}, \mathbf{w}) = \int_{\Omega_f} \frac{\nu}{2}(\nabla\mathbf{v} + \nabla^T\mathbf{v}) \cdot (\nabla\mathbf{w} + \nabla^T\mathbf{w}) \qquad \forall \mathbf{v}, \mathbf{w} \in (H^1(\Omega_f))^d\,, \tag{225}$$

$$b_f(\mathbf{v}, q) = -\int_{\Omega_f} q\,\nabla \cdot \mathbf{v} \qquad \forall \mathbf{v} \in (H^1(\Omega_f))^d, \quad \forall q \in Q\,, \tag{226}$$

$$a_p(\varphi, \psi) = \int_{\Omega_p} \nabla\psi \cdot \mathsf{K}\nabla\varphi \qquad \forall \varphi, \psi \in H^1(\Omega_p)\,, \tag{227}$$

and, for all $\mathbf{v}, \mathbf{w}, \mathbf{z} \in (H^1(\Omega_f))^d$, the trilinear form

$$c_f(\mathbf{w}; \mathbf{z}, \mathbf{v}) = \int_{\Omega_f} [(\mathbf{w} \cdot \nabla)\mathbf{z}] \cdot \mathbf{v} = \sum_{i,j=1}^{d} \int_{\Omega_f} w_j \frac{\partial z_i}{\partial x_j} v_i\,. \tag{228}$$

Now, we consider the trace space $\Lambda = H_{00}^{1/2}(\Gamma)$ and its subspace $\Lambda_0 = \{\mu \in \Lambda : \int_\Gamma \mu = 0\}$, and define the *linear* extension operator:

$$R_f \colon \Lambda_0 \to H_f \times Q_0, \quad \eta \to R_f\eta = (\mathbf{R}_f^1\eta, R_f^2\eta), \tag{229}$$

satisfying $\mathbf{R}_f^1\eta \cdot \mathbf{n} = \eta$ on Γ, and, for all $\mathbf{v} \in H_f^0$, $q \in Q_0$,

$$a_f(\mathbf{R}_f^1\eta, \mathbf{v}) + b_f(\mathbf{v}, R_f^2\eta) + \int_\Gamma \sum_{j=1}^{d-1} \frac{\nu}{\varepsilon}(\mathbf{R}_f^1\eta \cdot \boldsymbol{\tau}_j)(\mathbf{v} \cdot \boldsymbol{\tau}_j) = 0, \tag{230}$$

$$b_f(\mathbf{R}_f^1\eta, q) = 0. \tag{231}$$

Moreover, we consider the *linear* extension operator

$$R_p \colon \Lambda_0 \to H_p, \qquad \eta \to R_p\eta \tag{232}$$

such that

$$a_p(R_p\eta, \psi) = \int_\Gamma \eta\psi \qquad \forall \psi \in H_p\,. \tag{233}$$

It is easy to see that problems (230)–(231) and (233) both have a unique solution.

Finally, let us introduce the following *nonlinear* extension operator:

$$\mathcal{R}_f \colon \Lambda_0 \to H_f \times Q_0, \quad \eta \to \mathcal{R}_f(\eta) = (\mathcal{R}_f^1(\eta), \mathcal{R}_f^2(\eta))$$

such that $\mathcal{R}_f^1(\eta) \cdot \mathbf{n} = \eta$ on Γ, and, for all $\mathbf{v} \in H_f^0$, $q \in Q_0$,

$$\begin{aligned} a_f(\mathcal{R}_f^1(\eta), \mathbf{v}) + c_f(\mathcal{R}_f^1(\eta); \mathcal{R}_f^1(\eta), \mathbf{v}) + b_f(\mathbf{v}, \mathcal{R}_f^2(\eta)) & \\ + \int_\Gamma \sum_{j=1}^{d-1} \frac{\nu}{\varepsilon}(\mathcal{R}_f^1(\eta) \cdot \boldsymbol{\tau}_j)(\mathbf{v} \cdot \boldsymbol{\tau}_j) &= \int_{\Omega_f} \mathbf{f} \cdot \mathbf{v}, \quad (234) \\ b_f(\mathcal{R}_f^1(\eta), q) &= 0. \quad (235) \end{aligned}$$

In order to prove the existence and uniqueness of $\mathcal{R}_f$, we define the auxiliary nonlinear operator

$$\begin{aligned} &\mathcal{R}_0 : \Lambda_0 \to H_f^0 \times Q_0, \qquad \eta \to \mathcal{R}_0(\eta) = (\mathcal{R}_0^1(\eta), \mathcal{R}_0^2(\eta)), \\ &\text{with } \mathcal{R}_0^i(\eta) = \mathcal{R}_f^i(\eta) - R_f^i\eta, \quad i = 1, 2. \end{aligned} \tag{236}$$

Clearly, $\mathcal{R}_0^1(\eta)\cdot\mathbf{n}=0$ on Γ, and it satisfies:

$$\begin{aligned}
a_f(\mathcal{R}_0^1(\eta),\mathbf{v}) + c_f(\mathbf{R}_f^1\eta + \mathcal{R}_0^1(\eta);\mathbf{R}_f^1\eta+\mathcal{R}_0^1(\eta),\mathbf{v})& \\
+b_f(\mathbf{v},\mathcal{R}_0^2(\eta)) + \int_\Gamma \sum_{j=1}^{d-1}\frac{\nu}{\varepsilon}(\mathcal{R}_0^1(\eta)\cdot\boldsymbol{\tau}_j)(\mathbf{v}\cdot\boldsymbol{\tau}_j) &= \int_{\Omega_f}\mathbf{f}\cdot\mathbf{v}\,, \quad (237)\\
b_f(\mathcal{R}_0^1(\eta),q) &= 0\,, \quad (238)
\end{aligned}$$

for all $\mathbf{v}\in H_f^0$, $q\in Q_0$. Remark that problem (237)–(238) is analogous to (234)–(235), but here $\mathcal{R}_0^1(\eta)\in H_f^0$, while $\mathcal{R}_f^1(\eta)\in H_f$.

Moreover, given $\eta\in\Lambda_0$, we define the form

$$\begin{aligned}
&a(\mathbf{w};\mathbf{z},\mathbf{v}) = a_f(\mathbf{z},\mathbf{v}) + c_f(\mathbf{w};\mathbf{z},\mathbf{v}) + c_f(\mathbf{R}_f^1\eta;\mathbf{z},\mathbf{v})\\
&+ c_f(\mathbf{z};\mathbf{R}_f^1\eta,\mathbf{v}) + \int_\Gamma\sum_{j=1}^{d-1}\frac{\nu}{\varepsilon}(\mathbf{z}\cdot\boldsymbol{\tau}_j)(\mathbf{v}\cdot\boldsymbol{\tau}_j) \qquad \forall\mathbf{w},\mathbf{z},\mathbf{v}\in(H^1(\Omega_f))^d\,, \quad (239)
\end{aligned}$$

and the functional

$$\langle\ell,\mathbf{v}\rangle = -c_f(\mathbf{R}_f^1\eta;\mathbf{R}_f^1\eta,\mathbf{v}) + \int_{\Omega_f}\mathbf{f}\cdot\mathbf{v} \qquad \forall\mathbf{v}\in(H^1(\Omega_f))^d\,. \quad (240)$$

Thus, we can rewrite (237)–(238) as: given $\eta\in\Lambda_0$,

$$\text{find } \mathcal{R}_0^1(\eta)\in V_f^0 \ :\quad a(\mathcal{R}_0^1(\eta);\mathcal{R}_0^1(\eta),\mathbf{v}) = \langle\ell,\mathbf{v}\rangle \qquad \forall\mathbf{v}\in V_f^0\,. \quad (241)$$

Finally, let us recall some useful inequalities: the Poincaré inequality

$$\exists C_{\Omega_f}>0:\quad \|\mathbf{v}\|_0\le C_{\Omega_f}|\mathbf{v}|_1 \qquad \forall\mathbf{v}\in H_f\,, \quad (242)$$

the Korn inequality

$$\exists C_\kappa>0:\ \int_{\Omega_f}\sum_{i,j=1}^d\left(\frac{\partial v_j}{\partial x_i}+\frac{\partial v_i}{\partial x_j}\right)^2 \ge C_\kappa\|\mathbf{v}\|_1^2 \quad \forall\mathbf{v}=(v_1,\dots,v_d)\in H_f, \quad (243)$$

and the following inequality

$$\exists C_\mathcal{N}>0:\quad |c_f(\mathbf{w};\mathbf{z},\mathbf{v})|\le C_\mathcal{N}|\mathbf{w}|_1\,|\mathbf{z}|_1\,|\mathbf{v}|_1 \qquad \forall\mathbf{w},\mathbf{z},\mathbf{v}\in H_f, \quad (244)$$

which follows from the Poincaré inequality (242) and the inclusion $(H^1(\Omega_f))^d\subset(L^4(\Omega_f))^d$ (for $d=2,3$) due to the Sobolev embedding theorem.

We can now state the following result,

Proposition 3.12 *Let* $\mathbf{f} \in L^2(\Omega_f)$ *be such that*

$$C_{\mathcal{N}} C_{\Omega_f} ||\mathbf{f}||_0 < \left(\frac{C_\kappa \nu}{2}\right)^2 , \qquad (245)$$

where C_κ *and* $C_{\mathcal{N}}$ *are the constants introduced in (243) and (244), respectively. If*

$$\eta \in \left\{ \mu \in \Lambda_0 \,:\, |\mathbf{R}_f^1 \mu|_1 < \frac{C_\kappa \nu - \sqrt{\left(\frac{C_\kappa \nu}{2}\right)^2 + 3C_{\mathcal{N}} C_\Omega ||\mathbf{f}||_0}}{3C_{\mathcal{N}}} \right\} , \qquad (246)$$

then there exists a unique nonlinear extension $\mathcal{R}_f(\eta) = (\mathcal{R}_f^1(\eta), \mathcal{R}_f^2(\eta)) \in H_f \times Q_0$.

We can reformulate the global coupled problem (222)–(224) as an interface equation depending solely on $\lambda = (\mathbf{u}_f \cdot \mathbf{n})_{|\Gamma}$. We define the *nonlinear* pseudo-differential operator $\mathcal{S} : \Lambda_0 \to \Lambda_0'$,

$$\begin{aligned} \langle \mathcal{S}(\eta), \mu \rangle \;=\;& a_f(\mathcal{R}_f^1(\eta), \mathbf{R}^1\mu) + c_f(\mathcal{R}_f^1(\eta); \mathcal{R}_f^1(\eta), \mathbf{R}^1\mu) + b_f(\mathbf{R}^1\mu, \mathcal{R}_f^2(\eta)) \\ & + \int_\Gamma \sum_{j=1}^{d-1} \frac{\nu}{\varepsilon} (\mathcal{R}_f^1(\eta) \cdot \boldsymbol{\tau}_j)(\mathbf{R}^1\mu \cdot \boldsymbol{\tau}_j) - \int_{\Omega_f} \mathbf{f} \cdot (\mathbf{R}^1\mu) \\ & + \int_\Gamma g(R_p\eta)\mu \qquad \forall \eta \in \Lambda_0, \forall \mu \in \Lambda \,. \end{aligned} \qquad (247)$$

The operator $\mathcal{S}$ is composed of two parts: a non-linear component associated to the fluid problem in Ω_f (the terms in the first two lines), and a linear part related to the problem in the porous media (corresponding to the last integral). The fluid part plays the role of a non-linear Dirichlet-to-Neumann map that associates at any given normal velocity η on Γ the normal component of the corresponding Cauchy stress tensor on Γ. On the other hand, the linear porous-media part is a Neumann-to-Dirichlet map that associates the trace on Γ of the piezometric head whose conormal derivative on Γ is equal to η. We have the following equivalence result,

Theorem 3.15 *The solution of (222)–(224) can be characterized as follows:*

$$\mathbf{u}_f = \mathcal{R}_f^1(\lambda), \quad p_f = \mathcal{R}_f^2(\lambda) + \hat{p}_f, \quad \varphi = R_p\lambda \,, \qquad (248)$$

where $\hat{p}_f = (meas(\Omega_f))^{-1} \int_{\Omega_f} p_f$*, and* $\lambda \in \Lambda_0$ *is the solution of the nonlinear interface problem:*

$$\langle \mathcal{S}(\lambda), \mu \rangle = 0 \qquad \forall \mu \in \Lambda_0 \,. \tag{249}$$

Moreover, $\hat{p}_f$ *can be obtained from* λ *by solving the algebraic equation*

$$\hat{p}_f = (meas(\Gamma))^{-1} \langle \mathcal{S}(\lambda), \varepsilon \rangle,$$

where $\varepsilon \in \Lambda$ *is a fixed function such that*

$$\frac{1}{meas(\Gamma)} \int_\Gamma \varepsilon = 1 \,. \tag{250}$$

The operator $\mathcal{S}$ can be characterized as

$$\begin{aligned} \langle \mathcal{S}(\eta), \mu \rangle \;&=\; a_f(\mathbf{R}_f^1 \eta, \mathbf{R}_f^1 \mu) + c_f(\mathcal{R}_0^1(\eta) + \mathbf{R}_f^1 \eta; \mathcal{R}_0^1(\eta) + \mathbf{R}_f^1 \eta, \mathbf{R}_f^1 \mu) \\ &\quad + \int_\Gamma \sum_{j=1}^{d-1} \frac{\nu}{\varepsilon} (\mathbf{R}_f^1 \eta \cdot \boldsymbol{\tau}_j)(\mathbf{R}_f^1 \mu \cdot \boldsymbol{\tau}_j) \\ &\quad - \int_{\Omega_f} \mathbf{f} \cdot (\mathbf{R}_f^1 \mu) + \int_\Gamma g(R_p \eta) \mu \,. \end{aligned} \tag{251}$$

Note that in view of (251), $\mathcal{S}(\lambda)$ is defined in terms of the operator $\mathcal{R}_0^1(\lambda)$, which, thanks to (237)–(238), satisfies in its turn the following problem:

$$\begin{aligned} &a_f(\mathcal{R}_0^1(\lambda), \mathbf{v}) + c_f(\mathcal{R}_0^1(\lambda) + \mathbf{R}_f^1 \lambda; \mathcal{R}_0^1(\lambda) + \mathbf{R}_f^1 \lambda, \mathbf{v}) \\ &\qquad + \int_\Gamma \sum_{j=1}^{d-1} \frac{\nu}{\varepsilon} (\mathcal{R}_0^1(\lambda) \cdot \boldsymbol{\tau}_j)(\mathbf{v} \cdot \boldsymbol{\tau}_j) = \int_{\Omega_f} \mathbf{f} \cdot \mathbf{v} \qquad \forall \mathbf{v} \in V_f^0 \,. \end{aligned} \tag{252}$$

Therefore, in order to prove the existence and uniqueness of the solution of the interface problem, we have to consider (249), with the characterization of $\mathcal{S}$ given in (251), coupled with (252), i.e., we have to guarantee at once the existence and uniqueness of $\lambda \in \Lambda_0$ and $\mathcal{R}_0^1(\lambda) \in V_f^0$. To this aim we consider the product space $W = \Lambda_0 \times V_f^0$ endowed with the norm

$$\|\bar{v}\|_W = (|\mathbf{R}_f^1 \mu|_1^2 + |\mathbf{v}|_1^2)^{1/2} \qquad \forall \bar{v} = (\mu, \mathbf{v}) \in W \,. \tag{253}$$

We introduce the trilinear form and the linear functional associated with our problem in the space W. For any fixed $(\eta, \mathbf{w}) \in W$, we define the following operator depending on $\bar{w}$:

$$\begin{aligned}\tilde{\mathcal{A}}(\eta, \mathbf{w}) : W &\rightarrow W', \\ \langle(\tilde{\mathcal{A}}(\eta, \mathbf{w}))(\xi, \mathbf{u}), (\mu, \mathbf{v})\rangle &= \langle(\mathcal{A}_f(\eta, \mathbf{w}))(\xi, \mathbf{u}), \mu\rangle + \langle(\mathcal{A}_0(\eta, \mathbf{w}))(\xi, \mathbf{u}), \mathbf{v}\rangle\end{aligned}$$

where, for every test function $\mu \in \Lambda_0$,

$$\begin{aligned}\langle(\mathcal{A}_f(\eta, \mathbf{w}))(\xi, \mathbf{u}), \mu\rangle &= a_f(\mathbf{R}_f^1\xi, \mathbf{R}_f^1\mu) + c_f(\mathbf{w} + \mathbf{R}_f^1\eta; \mathbf{u} + \mathbf{R}_f^1\xi, \mathbf{R}_f^1\mu) \\ &\quad + \int_\Gamma \sum_{j=1}^{d-1} \frac{\nu}{\varepsilon}(\mathbf{R}_f^1\xi \cdot \boldsymbol{\tau}_j)(\boldsymbol{R}_f^1\mu \cdot \boldsymbol{\tau}_j) + \int_\Gamma g(R_p\xi)\mu\,,\end{aligned}$$

whereas for any test function $\mathbf{v} \in V_f^0$,

$$\begin{aligned}\langle(\mathcal{A}_0(\eta, \mathbf{w}))(\xi, \mathbf{u}), \mathbf{v}\rangle &= a_f(\mathbf{u}, \mathbf{v}) + c_f(\mathbf{w} + \mathbf{R}_f^1\eta; \mathbf{u} + \mathbf{R}_f^1\xi, \mathbf{v}) \\ &\quad + \int_\Gamma \sum_{j=1}^{d-1} \frac{\nu}{\varepsilon}(\mathbf{u} \cdot \boldsymbol{\tau}_j)(\mathbf{v} \cdot \boldsymbol{\tau}_j)\,.\end{aligned}$$

We indicate by $\tilde{a}$ the form associated to the operator $\tilde{\mathcal{A}}$:

$$\tilde{a}(\bar{w}; \bar{u}, \bar{v}) = \langle(\tilde{\mathcal{A}}(\eta, \mathbf{w}))(\xi, \mathbf{u}), (\mu, \mathbf{v})\rangle \tag{254}$$

for all $\bar{w} = (\eta, \mathbf{w}), \bar{u} = (\xi, \mathbf{u}), \bar{v} = (\mu, \mathbf{v}) \in W$.

Next, we define two functionals $\ell_f : \Lambda_0 \rightarrow \mathbb{R}$ and $\ell_0 : V_f^0 \rightarrow \mathbb{R}$ as:

$$\begin{aligned}\langle \ell_f, \mu\rangle &= \int_{\Omega_f} \mathbf{f} \cdot (\mathbf{R}_f^1\mu) \qquad \forall \mu \in \Lambda_0\,, \\ \langle \ell_0, \mathbf{v}\rangle &= \int_{\Omega_f} \mathbf{f} \cdot \mathbf{v} \qquad \forall \mathbf{v} \in V_f^0\,,\end{aligned}$$

and denote

$$\langle \tilde{\ell}, \bar{v}\rangle = \langle \ell_f, \mu\rangle + \langle \ell_0, \mathbf{v}\rangle \qquad \forall \bar{v} = (\mu, \mathbf{v}) \in W\,. \tag{255}$$

Thus, the problem defined by (249) and (252) can be reformulated as:

$$\text{find } \bar{u} = (\lambda, u) \in W\ : \quad \tilde{a}(\bar{u}; \bar{u}, \bar{v}) = \langle \tilde{\ell}, \bar{v}\rangle \qquad \forall \bar{v} = (\mu, \mathbf{v}) \in W\,. \tag{256}$$

We shall prove the existence and uniqueness of the solution only in a closed convex subset of W.

Lemma 3.9 *Let* $\mathbf{f} \in L^2(\Omega_f)$ *be such that*

$$2(1+\sqrt{2})\sqrt{2C_{\mathcal{N}}C_{\Omega_f}\|\mathbf{f}\|_0} \leq C_\kappa \nu \,, \tag{257}$$

and consider two constants

$$r_m = \frac{C_1 - \sqrt{C_1^2 - 4C_2}}{2} \quad \text{and} \quad r_M = C_1 - \sqrt{\sqrt{2}C_2}\,, \tag{258}$$

where

$$C_1 = \frac{C_\kappa \nu}{4C_{\mathcal{N}}}\,, \quad C_2 = \frac{\sqrt{2}C_{\Omega_f}\|\mathbf{f}\|_0}{2C_{\mathcal{N}}}\,. \tag{259}$$

Notice that, thanks to (257), there holds

$$0 \leq r_m < r_M\,. \tag{260}$$

If we consider

$$\overline{B}_r = \{\bar{w} = (\eta, \mathbf{w}) \in W \,:\, |\mathbf{R}_f^1\eta|_1 \leq r\}\,, \tag{261}$$

with

$$r_m < r < r_M\,, \tag{262}$$

then, there exists a unique solution $\bar{u} = (\lambda, u) \in \overline{B}_r$ *of (256) with* $u = \mathcal{R}_0^1(\lambda)$.

The following theorem is a direct consequence of the previous lemma.

Theorem 3.16 *If (257) holds, then problem (256) has a unique solution* $\bar{u} = (\lambda, \mathcal{R}_0^1(\lambda))$ *in the set*

$$B_{r_M} = \{\bar{w} = (\eta, \mathbf{w}) \in W \,:\, |\mathbf{R}_f^1\eta|_1 < r_M\}\,,$$

and it satisfies $|\mathbf{R}_f^1\lambda|_1 \leq r_m$, *where* r_m *and* r_M *are defined in (258). In particular, it follows that (249) has a unique solution* λ *in the set* $S_{r_M} = \{\eta \in \Lambda_0 \,:\, |\mathbf{R}_f^1\eta|_1 < r_M\} \subset \Lambda_0$ *which indeed belongs to* $S_{r_m} = \{\eta \in \Lambda_0 \,:\, |\mathbf{R}_f^1\eta|_1 \leq r_m\}$.

Remark 3.7 *Notice that condition (257) is analogous to that usually required to prove existence and uniqueness of the solution of the Navier-Stokes equations. Moreover, we have proved that the solution is unique in* S_{r_M}. *Thus, Theorem 3.7 states that the solution is unique only for sufficiently small normal velocities* λ *across the interface* Γ. *Finally, notice that (257) implies (245) and that* S_{r_m} *is included in the set (246), so that the existence and uniqueness of the nonlinear extension* $\mathcal{R}_0^1(\lambda)$ *is ensured as well.*

Numerical methods. We prove the convergence of the fixed-point iteration, Newton method and preconditioned Richardson method for the interface problem (249). Then, writing the corresponding form, we get these of these methods are convergent for problem (222)–(224).

1. Fixed-point iteration to solve the coupled problem (222)–(224) can be written as follows. Given $\mathbf{u}_f^0 \in H_f$, for $n \geq 1$, find $\mathbf{u}_f^n \in H_f$, $p_f^n \in Q$, $\varphi^n \in H_p$ such that

$$a_f(\mathbf{u}_f^n, \mathbf{v}) + c_f(\mathbf{u}_f^{n-1}; \mathbf{u}_f^n, \mathbf{v}) + b_f(\mathbf{v}, p_f^n)$$

$$+ \int_\Gamma g\, \varphi^n (\mathbf{v}\cdot\mathbf{n}) + \int_\Gamma \sum_{j=1}^{d-1} \frac{\nu}{\varepsilon}(\mathbf{u}_f^n \cdot \boldsymbol{\tau}_j)(\mathbf{v}\cdot\boldsymbol{\tau}_j) = \int_{\Omega_f} \mathbf{f}\cdot\mathbf{v}\,, \tag{263}$$

$$b_f(\mathbf{u}_f^n, q) = 0\,, \tag{264}$$

$$a_p(\varphi^n, \psi) = \int_\Gamma \psi(\mathbf{u}_f^n \cdot \mathbf{n})\,, \tag{265}$$

for all $\mathbf{v} \in H_f$, $q \in Q$, $\psi \in H_p$.

Algorithm (263)–(265) requires to solve at each iteration a linear coupled problem, and it can be reinterpreted as a fixed-point method to solve the interface problem (249). Then, in view of this equivalence, the convergence of (263)–(265) is a direct consequence of Lemma 3.9. We can state the following result which is a straightforward corollary of Theorem 3.7.

Proposition 3.13 *If (257) holds and if $\mathbf{u}_f^0$ is such that $|\mathbf{R}_f^1(\mathbf{u}_f^0 \cdot \mathbf{n})|_1 < r_M$ with r_M given in (258), then the sequence $(\mathbf{u}_f^n, p_f^n, \varphi^n)$ converges for $n \to \infty$ to the unique solution $(\mathbf{u}_f, p_f, \varphi)$ of problem (222)–(224), and $|\mathbf{R}_f^1(\mathbf{u}_f \cdot \mathbf{n})|_1 \leq r_m$.*

2. Newton method to solve (the discrete form of) (222)–(224) can be written as follows. Let $\mathbf{u}_f^0 \in H_f$ be given. Then, for $n \geq 1$, the Newton method reads: find $\mathbf{u}_f^n \in H_f$, $p_f^n \in Q$, $\varphi^n \in H_p$ such that

$$a_f(\mathbf{u}_f^n, \mathbf{v}) + c_f(\mathbf{u}_f^n; \mathbf{u}_f^{n-1}, \mathbf{v}) + c_f(\mathbf{u}_f^{n-1}; \mathbf{u}_f^n, \mathbf{v}) + b_f(\mathbf{v}, p_f^n) + \int_\Gamma g\varphi^n(\mathbf{v}\cdot\mathbf{n})$$
$$+ \int_\Gamma \sum_{j=1}^{d-1} \frac{\nu}{\varepsilon}(\mathbf{u}_f^n \cdot \boldsymbol{\tau}_j)(\mathbf{v}\cdot\boldsymbol{\tau}_j) \ = \ c_f(\mathbf{u}_f^{n-1}; \mathbf{u}_f^{n-1}, \mathbf{v}) + \int_{\Omega_f} \mathbf{f}\cdot\mathbf{v}\,, \tag{266}$$
$$b_f(\mathbf{u}_f^n, q) = 0\,, \tag{267}$$
$$a_p(\varphi^n, \psi) = \int_\Gamma \psi(\mathbf{u}_f^n \cdot \mathbf{n})\,, \tag{268}$$

for all $\mathbf{v} \in H_f$, $q \in Q$, $\psi \in H_p$.

In order to reduce the computational cost, we might consider the modified Newton method: find $\mathbf{u}_f^n \in H_f$, $p_f^n \in Q$, $\varphi^n \in H_p$ such that

$$a_f(\mathbf{u}_f^n, \mathbf{v}) + c_f(\mathbf{u}_f^n; \mathbf{u}_f^0, \mathbf{v}) + c_f(\mathbf{u}_f^0; \mathbf{u}_f^n, \mathbf{v}) + b_f(\mathbf{v}, p_f^n) + \int_\Gamma g\varphi^n(\mathbf{v}\cdot\mathbf{n})$$
$$+ \int_\Gamma \sum_{j=1}^{d-1} \frac{\nu}{\varepsilon}(\mathbf{u}_f^n \cdot \boldsymbol{\tau}_j)(\mathbf{v}\cdot\boldsymbol{\tau}_j) \ = c_f(\mathbf{u}_f^{n-1}; \mathbf{u}_f^0, \mathbf{v})$$
$$+ c_f(\mathbf{u}_f^0 - \mathbf{u}_f^{n-1}; \mathbf{u}_f^{n-1}, \mathbf{v}) + \int_{\Omega_f} \mathbf{f}\cdot\mathbf{v}\,, \tag{269}$$
$$b_f(\mathbf{u}_f^n, q) = 0\,, \tag{270}$$
$$a_p(\varphi^n, \psi) = \int_\Gamma \psi(\mathbf{u}_f^n \cdot \mathbf{n})\,, \tag{271}$$

for all $\mathbf{v} \in H_f$, $q \in Q$, $\psi \in H_p$.

Like for fixed-point iterations, we have to solve a linearized coupled problem at each iteration of the Newton algorithms.

Rewriting the Newton methods (266)–(268) and (269)–(271) as iterative schemes for the interface equation (249), we can prove

Proposition 3.14 *Let* $\mathbf{f} \in L^2(\Omega_f)$ *and let*

$$\tilde{C}_1 = \frac{32 C_{\mathcal{N}} C_{\Omega_f} \|\mathbf{f}\|_0}{(C_\kappa \nu)^2}, \qquad \tilde{C}_2 = \frac{2\sqrt{2} C_{\Omega_f} ||\mathbf{f}||_0}{C_\kappa \nu}\,. \tag{272}$$

If

$$\tilde{C}_1 \leq \frac{1}{2}, \tag{273}$$

then, there exists a unique solution $\bar{u} = (\lambda, \mathcal{R}_0^1(\lambda)) \in \overline{B}_{r_0}$ *of (249), with*

$$\overline{B}_{r_0} = \{\bar{w} = (\eta, \mathbf{w}) \in W \,:\, \|\bar{w}\|_W \leq r_0\} \tag{274}$$

and

$$r_0 = \frac{1 - \sqrt{1 - 2\tilde{C}_1}}{\tilde{C}_1}\tilde{C}_2\,. \tag{275}$$

Moreover, the sequence $\bar{u}^n = (\lambda^n, \mathbf{u}^n)$, $n \geq 1$, *obtained by the Newton algorithms applied to the interface equation (249), taking* $\bar{u}^0 = (0, \mathbf{0})$, *converges to this solution.*

The following error estimate hold for the Newton method:

$$||\bar{u} - \bar{u}^n||_W \leq \frac{1}{2^n}(2\tilde{C}_1)^{2^n}\frac{\tilde{C}_2}{\tilde{C}_1}, \quad n \geq 0, \tag{276}$$

while for the modified Newton method we have (if $\tilde{C}_1 < 1/2$*):*

$$||\bar{u} - \bar{u}^n||_W \leq \frac{\tilde{C}_2}{\tilde{C}_1}\left(1 - \sqrt{1 - 2\tilde{C}_1}\right)^{n+1}, \quad n \geq 0. \tag{277}$$

Remark 3.8 *By a simple calculus, we can see that* $\tilde{C}_1$ *and* $\tilde{C}_2$ *are related to the constants* C_1 *and* C_2 *in (259) as:* $C_1 = 2\sqrt{2}\tilde{C}_2/\tilde{C}_1$ *and* $C_2 = 2\sqrt{2}\tilde{C}_2^2/\tilde{C}_1$. *Thus, condition (257) can be reformulated as* $\tilde{C}_1 \leq (3 + 2\sqrt{2})/8$. *If we compare it with (273), we can see that the condition required for the convergence of the Newton method is more restrictive than condition (257)*

Finally, notice that r_m *becomes*

$$r_m = \frac{1 - \sqrt{1 - \sqrt{2}\tilde{C}_1}}{\tilde{C}_1}\sqrt{2}\tilde{C}_2\,.$$

Thus, r_m *has a form similar to* r_0 *in (275) and* $r_0 \geq r_m$. *Notice however that in the definition of* $\overline{B}_{r_m}$ *(see (261)) we control only* $|\mathbf{R}_f^1\lambda|_1$, *while in* $\overline{B}_{r_0}$ *in (274) we take the whole norm* $\|\bar{u}\|_W$. *We can conclude that the well-posedness results of Lemma 3.9 and Proposition 3.14 are consistent.*

3. Preconditioned Richardson method to solve (222)–(224) can be written as: given $\mathbf{u}_f^0 \in H_f$, $\varphi^0 \in H_p$, for $n \geq 1$, find $\mathbf{u}_f^n \in H_f$, $q_f^n \in Q$, $\varphi^n \in H_p$ such that

$$\begin{aligned}
&a_f(\mathbf{u}_f^n - \mathbf{u}_f^{n-1}, \mathbf{v}) + b_f(\mathbf{v}, p_f^n - p_f^{n-1}) + \int_\Gamma \sum_{j=1}^{d-1} \frac{\nu}{\varepsilon}((\mathbf{u}_f^n - \mathbf{u}_f^{n-1}) \cdot \boldsymbol{\tau}_j)(\mathbf{v} \cdot \boldsymbol{\tau}_j) \\
&\quad = \theta \Bigg[\int_{\Omega_f} \mathbf{f} \cdot \mathbf{v} - a_f(\mathbf{u}_f^{n-1}, \mathbf{v}) - c_f(\mathbf{u}_f^{n-1}; \mathbf{u}_f^{n-1}, \mathbf{v}) - b_f(\mathbf{v}, p_f^{n-1}) \\
&\qquad - \int_\Gamma \sum_{j=1}^{d-1} \frac{\nu}{\varepsilon}(\mathbf{u}_f^{n-1} \cdot \boldsymbol{\tau}_j)(\mathbf{v} \cdot \boldsymbol{\tau}_j) - \int_\Gamma g\, \varphi^{n-1}(\mathbf{v} \cdot \mathbf{n}) \Bigg], \qquad (278)
\end{aligned}$$

$$b_f(\mathbf{u}_f^n - \mathbf{u}_f^{n-1}, q) = 0\,, \qquad (279)$$

$$a_p(\varphi^n, \psi) = \int_\Gamma \psi(\mathbf{u}_f^n \cdot \mathbf{n})\,, \qquad (280)$$

for all $\mathbf{v} \in H_f$, $q \in Q$, $\psi \in H_p$. $\theta > 0$ is a suitably chosen relaxation parameter. Unlike the fixed-point and the Newton methods, this algorithm requires to solve at each iteration two decoupled linear equations at each iteration: one in the fluid domain and one in the porous media subdomain.

Proceeding as for the previous methods, we can interpret (278)–(279) as an iterative method for the interface problem (249) and we can prove its convergence for θ chosen in a suitable interval $(0, \theta_{max})$ with θ_{max} depending on ν, g and $\|\mathbf{f}\|_0$.

Numerical experiments are given in the paper to illustrate and compare these methods.

3.1.11 Schwarz-Neumann method (paper [17])

In this paper, a generalization of the Schwarz-Neumann method to more than two domains is propsed. We prove the convergence and the numerical stability of the algorithm. The results apply to both bounded and unbounded domains, and are given for the weak solution of an elliptic problem with mixed boundary conditions. Numerical results are given for both bounded and unbounded domains.

Neumann proposed in [72] an iterative method in which the solution of a Dirichlet problem in a domain $\Omega \subset \mathbf{R}^2$ is found by alternately solving two problems in two domains Ω_1 and Ω_2 whose intersection is the domain Ω,

$$\Omega = \Omega_1 \cap \Omega_2.$$

The two problems have the same equation as the initial one. The sum of the restrictions to Ω of the solutions in the two sequences converges to the solution of the problem in Ω. To be more explicit, we briefly summarize an example in [52], where a proof of the convergence of the process is also given.

Let us consider the problem

$$\begin{aligned} &\Delta u = 0 \text{ in } \Omega \\ &u = g \text{ on } \partial\Omega. \end{aligned} \tag{281}$$

We consider two functions g_1 and g_2 on the boundaries $\partial\Omega_1$ and $\partial\Omega_2$, respectively, such that $g_1 + g_2 = g$ on $\partial\Omega \cap \partial\Omega_1 \cap \partial\Omega_2$. At iteration $n \geq 1$ of Neumann's algorithm we solve the problems

$$\begin{aligned} &\Delta u_1^n = 0 \text{ in } \Omega_1 \\ &u_1^n = g_1 \text{ on } \partial\Omega_1 \backslash \Omega_2 \\ &u_1^n = g - u_2^{n-1} \text{ on } \partial\Omega_1 \cap \Omega_2, \end{aligned} \tag{282}$$

and

$$\begin{aligned} &\Delta u_2^n = 0 \text{ in } \Omega_2 \\ &u_2^n = g_2 \text{ on } \partial\Omega_2 \backslash \Omega_1 \\ &u_2^n = g - u_1^n \text{ on } \partial\Omega_2 \cap \Omega_1, \end{aligned} \tag{283}$$

where u_2^0, used in (282) for $n = 1$, is an arbitrary continuous function on Ω_2. It has been proved that, under some assumptions, the sequence $(u_1^n + u_2^n)_n$ converges in Ω to u, the solution of problem (281).

As we can see, this method is closely related to both fictitious domain methods and domain decomposition methods.

We see that the value g of the boundary condition in (281) is kept in the boundary conditions of problems (282) and (283) on $\partial\Omega_1 \cap \partial\Omega$ and $\partial\Omega_2 \cap \partial\Omega$, respectively. This idea of taking the boundary conditions of the problems in the algorithm such that the sum of their solutions satisfies the boundary condition of problem (281) is used in the generalization to more than two domains of the method we propose in this paper. To be more explicit, we now rewrite problems (282) and (283) in another form which can be directly generalized to more than two domains.

We consider that the value g from the boundary conditions of problem (281) is the trace of a function g defined on $D = \Omega_1 \cup \Omega_2$. Also, g_1 and g_2 in (282) and (283) can be considered as the traces of some functions g_1, g_2 defined on D such that $g_1 + g_2 = g$ in D. We start the algorithm with $\bar{u}_2^0 = g_2$ defined on D. It is easy to see that, for $n \geq 1$, the algorithm given by problems (282) and (283) can be written as: find u_1^n as the solution of the problem

$$\begin{aligned} &\Delta u_1^n = 0 \text{ in } \Omega_1 \\ &u_1^n = g - \bar{u}_2^{n-1} \text{ on } \partial\Omega_1, \end{aligned} \tag{284}$$

and extend u_1^n to $\bar{u}_1^n$ on all of D such that $\bar{u}_1^n = g - \bar{u}_2^{n-1}$ in $D \backslash \Omega_1$. Then, find u_2^n as the solution of the problem

$$\begin{aligned} &\Delta u_2^n = 0 \text{ in } \Omega_2 \\ &u_2^n = g - \bar{u}_1^n \text{ on } \partial\Omega_2, \end{aligned} \tag{285}$$

and extend u_2^n to $\bar{u}_2^n$ on all of D such that $\bar{u}_2^n = g - \bar{u}_1^n$ in $D \backslash \Omega_2$. In the case of the method with two domains we have $\partial\Omega_1$, $\partial\Omega_2 \subset \partial\Omega \cup \partial D$. This fact is not true when Ω is the intersection of more than two domains, but the generalization of the algorithm written as (284)-(285) defines the boundary conditions on the parts of the boundaries which do not lie in $\partial\Omega \cup \partial D$, and, moreover, it will be useful in the convergence proof. At the same time, for practical implementations, we will also give the equivalent variant of the algorithm generalizing (282)–(283), which does not use the function extensions.

We think that the Schwarz-Neumann method can be very efficient, especially for exterior problems, when we are able to give direct solutions on the domains which contain the domain of problem (281). In this case the method consists in the iterative calculation of the boundary data of a problem from the values of the solutions on the other domains.

Schwarz-Neumann method for bounded domains. Let us consider in $\mathbf{R}^N$ the bounded domains Ω_i, $i = 1, \cdots, m$, and write

$$\Omega = \bigcap_{i=1}^{m} \Omega_i, \quad D = \bigcup_{i=1}^{m} \Omega_i \tag{286}$$

We assume that the boundaries ∂D, $\partial\Omega$ and $\partial\Omega_i$ of the domains D, Ω and Ω_i, $i = 1, \cdots, m$, respectively, are Lipschitz continuous. In the following, the notations L^2, H^1 and $H^{1/2}$ are used with the usual sense of Sobolev spaces.

Let $u \in H^1(\Omega)$ be the weak solution of the elliptic problem

$$\begin{array}{ll} Au = F & \text{in } \Omega \\ u = g & \text{on } \Gamma_d \\ \frac{\partial u}{\partial n_A} = h & \text{on } \Gamma_n, \end{array} \tag{287}$$

where $\Gamma_d \cap \Gamma_n = \phi$ and $\bar{\Gamma}_d \cup \bar{\Gamma}_n = \partial\Omega$. The differential operator A is considered in the form

$$Au = -\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}(a_{ij}\frac{\partial u}{\partial x_j}) + a_0 u, \tag{288}$$

where a_{ij}, $a_0 \in L^\infty(D)$, $a_{ij} = a_{ji}$, $i, j = 1, \cdots, n$, $a_0 \geq 0$, and we assume that there exists a constant $c > 0$ such that

$$\sum_{i,j=1}^{N} a_{ij}\xi_i\xi_j \geq c \mid \xi \mid^2 \text{ for any } \xi \in \mathbf{R}^N \text{ and } x \in D.$$

Above, we have denoted by $\dfrac{\partial}{\partial n_A} = \displaystyle\sum_{i,j=1}^{N} a_{ij}\frac{\partial}{\partial x_i} n_i$ the conormal derivative operator associated with the operator A. We assume that the measure of Γ_d is positive, $F \in L^2(D)$, $h \in L^2(\Gamma_n)$, and $g \in H^{1/2}(\Gamma_d)$ is the restriction to Γ_d of the trace on $\partial\Omega$ of a function $g \in H^1(D)$. Also, we assume that if $\Gamma_n \neq \phi$ then $\Gamma_n \subset \partial\Omega_i$, for all $i = 1, \cdots, m$.

Now, let us consider, for $i = 1, \cdots, m$, $g_i \in H^1(D)$, $F_i \in L^2(D)$ and $h_i \in L^2(\Gamma_n)$ such that $g = g_1 + \cdots g_m$, $F = F_1 + \cdots F_m$ and $h = h_1 + \cdots h_m$. The following algorithm is a direct extension to more than two domains of that given in (284)-(285).

Algorithm 3.19 *Firstly, for $i = 1, \cdots, m$, we write*

$$\bar{u}_i^0 = g_i \text{ in } D. \tag{289}$$

Assuming that at iteration $n \geq 1$ and domain $1 \leq i \leq m$ we have $\bar{u}_1^n, \cdots, \bar{u}_{i-1}^n$, $\bar{u}_i^{n-1}$, $\bar{u}_{i+1}^{n-1}, \cdots, \bar{u}_m^{n-1} \in H^1(D)$, we find $u_i^n \in H^1(\Omega_i)$ which satisfies

$$\begin{array}{ll} Au_i^n = F_i & \text{in } \Omega_i \\ u_i^n = g - \bar{u}_1^n - \cdots - \bar{u}_{i-1}^n - \bar{u}_{i+1}^{n-1} - \cdots - \bar{u}_m^{n-1} & \text{on } \partial\Omega_i \backslash \bar{\Gamma}_n \\ \frac{\partial u_i^n}{\partial \nu_A} = h_i & \text{on } \Gamma_n \end{array} \tag{290}$$

Then, we consider the extension of u_i^n on D given by

$$\bar{u}_i^n = \begin{cases} u_i^n \text{ on } \Omega_i \\ g - \bar{u}_1^n - \cdots - \bar{u}_{i-1}^n - \bar{u}_{i+1}^{n-1} - \cdots - \bar{u}_m^{n-1} \text{ on } D \backslash \Omega_i \end{cases} \tag{291}$$

To prove the convergence of this algorithm, we introduce

$$\tilde{u}_i^n = \bar{u}_1^n + \cdots + \bar{u}_i^n + \bar{u}_{i+1}^{n-1} + \cdots + \bar{u}_m^{n-1} \text{ and } \tilde{u}_0^n = \tilde{u}_m^{n-1}, \tag{292}$$

and we see that $\tilde{u}_i^n - \tilde{u}_{i-1}^n = \bar{u}_i^n - \bar{u}_i^{n-1} = u_i^n - u_i^{n-1}$ in Ω_i. Therefore, from (290), for $n \geq 2$ and $i = 1, \cdots, m$, we get that

$$A(\tilde{u}_i^n - \tilde{u}_{i-1}^n) = 0 \text{ in } \Omega_i \tag{293}$$

and

$$\tilde{u}_i^n = g \text{ on } \partial\Omega_i \tag{294}$$

As usual, we associate with the operator A the bilinear form

$$a(u, v) = \sum_{i,j=1}^{N} \int_D a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} + \int_D a_0 uv, \quad u, v \in H^1(D).$$

Also, we introduce the spaces

$$\begin{aligned} V(\Omega) &= \{v \in H^1(\Omega) : v = 0 \text{ on } \Gamma_d\}, \\ V(D) &= \{v \in H^1(D) : v = 0 \text{ on } \Gamma_d\}, \end{aligned} \tag{295}$$

and, for $i = 1, \cdots, m$,

$$V_i = \{v_i \in H^1(\Omega_i) : v_i = 0 \text{ on } \partial\Omega_i \backslash \bar{\Gamma}_n\}. \tag{296}$$

Following the classical way, we obtain from (287) that $w = u - g \in H^1(\Omega)$ is the weak solution of the problem

$$w \in V(\Omega) : a(w, v) = f(v) \text{ for any } v \in V(\Omega) \tag{297}$$

in which $f(v) = -a(g, v) + \int_\Omega Fv + \int_{\Gamma_n} hv$.

If we write $w_i^n = u_i^n - g_i$, and denote by $\bar{w}_i^n$ its extension with $-\bar{w}_1^n - \cdots - \bar{w}_{i-1}^n - w_i^n - \bar{w}_{i+1}^{n-1} - \cdots - \bar{w}_m^{n-1}$ in $D \backslash \Omega_i$, then problems (290) can be written as,

$$\begin{aligned} &w_i^n \in H^1(\Omega_i) : a(w_i^n, v_i) = f_i(v_i) \text{ for any } v_i \in V_i, \\ &\bar{w}_1^n + \cdots + \bar{w}_{i-1}^n + w_i^n + \bar{w}_{i+1}^{n-1} + \cdots + \bar{w}_m^{n-1} = 0 \text{ on } \partial\Omega_i \backslash \bar{\Gamma}_n, \end{aligned} \tag{298}$$

where $f_i(v_i) = -a(g_i, v_i) + \int_{\Omega_i} F_i v_i + \int_{\Gamma_n} h_i v_i$, for $i = 1, \cdots, m$.

Writing

$$\tilde{w}_i^n = \tilde{u}_i^n - g, \tag{299}$$

then we get from (293) and (294) that

$$\tilde{w}_i^n \in V_i : a(\tilde{w}_i^n - \tilde{w}_{i-1}^n, v_i) = 0, \text{ for any } v_i \in V_i, \tag{300}$$

and

$$\tilde{w}_i^n = 0 \text{ on } D \backslash \Omega_i. \tag{301}$$

Evidently, problems (297), (298) and (300) have unique solutions.

In what follows we need to specify the domain on which the functions in the bilinear form a are defined. For this reason we write, $a_\Omega(u, w)$, $a_D(u, w)$, $a_{\Omega_i}(u, w)$ or $a_{\Omega_i \backslash \bar{\Omega}}(u, w)$, when u and w are defined on Ω, D, Ω_i or $\Omega_i \backslash \bar{\Omega}$, respectively. We prove the convergence of the algorithm making the following assumption:

Assumption 3.10 *For each $i = 1, \cdots, m$, there exists an open set $O_i \subset \Omega_i$ such that $\partial\Omega \cap \Omega_i \subset O_i$ and $\overline{\partial\Omega \cap \Omega_i} \cap \overline{\Omega_i \backslash O_i} = \emptyset$, and there exists a one-to-one continuous mapping from $\overline{O_i \cap \Omega}$ onto $\overline{O_i \backslash \bar{\Omega}}$, $T_i : \overline{O_i \cap \Omega} \to \overline{O_i \backslash \bar{\Omega}}$, such that $T_i(\partial(O_i \cap \Omega)) = \partial(O_i \backslash \bar{\Omega})$, $T_i(x) = x$ for any $x \in \partial\Omega \cap \Omega_i$, and the partial derivatives of T_i and T_i^{-1} lie in $(L^\infty(O_i \cap \Omega))^N$ and $\left(L^\infty(O_i \backslash \bar{\Omega})\right)^N$, respectively.*

This is an easy enough constraint on the domains Ω_i and Ω. In fact, O_i can be seen as a strip in Ω_i which covers $\partial\Omega \cap \Omega_i$ and is split by it into two parts, $O_i \cap \Omega$ and $O_i \cap \backslash \bar{\Omega}$; the mappings T_i and T_i^{-1} transform one-to-one a part of O_i onto the other one. Using the fact that the partial derivatives of T_i and T_i^{-1} are in L^∞, we see that $H^1(O_i \cap \Omega) \ni v \to v \circ T_i^{-1} \in H^1(O_i \backslash \bar{\Omega})$ is a bijective correspondence, and there exists a constant C such that

$$\begin{aligned} &||v||_{H^1(O_i \cap \Omega)} \leq C ||v \circ T_i^{-1}||_{H^1(O_i \backslash \bar{\Omega})} \text{ for } v \in H^1(O_i \cap \Omega) \\ &\text{and} \\ &||v||_{H^1(O_i \backslash \bar{\Omega})} \leq C ||v \circ T_i||_{H^1(O_i \cap \Omega)} \text{ for } v \in H^1(O_i \backslash \bar{\Omega}). \end{aligned} \tag{302}$$

Moreover, v and $v \circ T_i^{-1}$ coincide on $\partial\Omega \cap \Omega_i$ and, evidently, $v(x) = v \circ T_i^{-1}(T_i(x))$ for the other points x of the boundary $\partial(O_i \cap \Omega)$.

The following result proves the geometrical convergence of our algorithm.

Theorem 3.17 *If Assumption 3.10 holds, and if u is the weak solution of problem (287), $\tilde{u}$ is its extension with g in $D\backslash\Omega$, and $\tilde{u}_i^n$, $i = 1, \cdots, m$, $n \geq 1$ are obtained from Algorithm 3.19 and (292), then $\tilde{u}_i^n \to \tilde{u}$ strongly in $H^1(D)$ as $n \to \infty$, for any $i = 1, \cdots, m$. Moreover, we have the following error estimate*

$$| \tilde{u} - \tilde{u}_i^{n+1} |_{H^1(D)}^2 \leq C[\frac{C(m-1)}{C(m-1)+1}]^n | \tilde{u} - \tilde{u}_i^1 |_{H^1(D)}^2 \tag{303}$$

where C depends only on the bilinear form a and the domains Ω and Ω_i.

We have seen in Algorithm 3.19 that, at each iteration, we have to evaluate the solution in a domain on parts of the boundaries of other domains. Even if we are able to find the exact solution on each domain Ω_i, its trace on an interior curve is usually calculated at some points, and then we use interpolation to approximate the value of the solution on that curve. Consequently, we assume that the boundary values might be transmitted with errors from one domain to another when we apply the algorithm, and we are interested in its stability.

Let us assume that instead of Algorithm 3.19 we have

Algorithm 3.20 *We choose, for $i = 1, \cdots, m$, $\bar{\vartheta}_i^0 \in H^1(D)$ such that $\bar{\vartheta}_i^0 = g_i$ in D. Assuming that at iteration $n \geq 1$ and domain $1 \leq i \leq m$ we have $\bar{\vartheta}_1^n, \cdots, \bar{\vartheta}_{i-1}^n, \bar{\vartheta}_i^{n-1}, \bar{\vartheta}_{i+1}^{n-1}, \cdots, \bar{\vartheta}_m^{n-1} \in H^1(D)$, we approximate the exact solution $u_i^n \in H^1(\Omega_i)$ of the problem*

$$\begin{array}{ll} Au_i^n = F_i & in\ \Omega_i \\ u_i^n = g - \bar{\vartheta}_1^n - \cdots - \bar{\vartheta}_{i-1}^n - \bar{\vartheta}_{i+1}^{n-1} - \cdots - \bar{\vartheta}_m^{n-1} & on\ \partial\Omega_i\backslash\bar{\Gamma}_n \\ \frac{\partial u_i^n}{\partial n_A} = h_i & on\ \Gamma_n \end{array} \tag{304}$$

with $\vartheta_i^n \in H^1(\Omega_i)$ such that $\vartheta_i^n = u_i^n$ on $\partial\Omega_i\backslash\bar{\Gamma}_n$. Then we consider $\bar{\vartheta}_i^n \in H^1(D)$, the extension of ϑ_i^n on $D\backslash\Omega_i$ by $g - \bar{\vartheta}_1^n - \cdots - \bar{\vartheta}_{i-1}^n - \bar{\vartheta}_{i+1}^{n-1} - \cdots - \bar{\vartheta}_m^{n-1}$, and we approximately solve (304) for the next domain.

First, we define $\bar{u}_i^n \in H^1(D)$ by writing $\bar{u}_i^n = u_i^n$ on Ω_i and $\bar{u}_i^n = \bar{\vartheta}_i^n$ on $D\backslash\Omega_i$, where u_i^n and $\bar{\vartheta}_i^n$ are defined in Algorithm 3.20. Using these $\bar{u}_i^n$, we define $\tilde{u}_i^n$ as in (292) and, using $\bar{\vartheta}_i^n$, we similarly write

$$\tilde{\vartheta}_i^n = \bar{\vartheta}_1^n + \cdots + \bar{\vartheta}_i^n + \bar{\vartheta}_{i+1}^{n-1} + \cdots + \bar{\vartheta}_m^{n-1},\ \tilde{\vartheta}_0^n = \tilde{\vartheta}_m^{n-1}. \tag{305}$$

Now, since $u_i^n = \vartheta_i^n$ on $\partial\Omega_i \backslash \Gamma_n$, from (304), for $n \geq 2$, $\tilde{u}_i^n$ is the solution of equation

$$A(\tilde{u}_i^n - \tilde{u}_{i-1}^n) = 0 \text{ in } \Omega_i \tag{306}$$

with the boundary conditions

$$\tilde{u}_i^n - \tilde{u}_{i-1}^n = g - \tilde{\vartheta}_{i-1}^n \text{ on } \partial\Omega_i \backslash \bar{\Gamma}_n. \tag{307}$$

To write the variational form of the above problems, we use the same spaces as in the previous subsection, $V(\Omega)$, $V(D)$ and V_i, $i = 1, \cdots, m$. We also write $w_i^n = u_i^n - g_i$, $v_i^n = \vartheta_i^n - g_i$, $\bar{w}_i^n = \bar{u}_i^n - g_i$, $\bar{v}_i^n = \bar{\vartheta}_i^n - g_i$, $\tilde{w}_i^n = \tilde{u}_i^n - g$ and $\tilde{v}_i^n = \tilde{\vartheta}_i^n - g$. Using these notations, problem (304) can be written as

$$\begin{aligned} &w_i^n \in H^1(\Omega_i) : a(w_i^n, v_i) = f_i(v_i) \text{ for any } v_i \in V_i, \\ &\bar{v}_1^n + \cdots + \bar{v}_{i-1}^n + w_i^n + \bar{v}_{i+1}^{n-1} + \cdots + \bar{v}_m^{n-1} = 0 \text{ on } \partial\Omega_i \backslash \bar{\Gamma}_n. \end{aligned} \tag{308}$$

Problem (306), (307) is written as

$$\begin{aligned} &\tilde{w}_i^n \in H^1(\Omega_i) : a(\tilde{w}_i^n - \tilde{w}_{i-1}^n, v_i) = 0 \text{ for any } v_i \in V_i, \\ &\tilde{w}_i^n - \tilde{w}_{i-1}^n = -\tilde{v}_{i-1}^n \text{ on } \partial\Omega_i \backslash \bar{\Gamma}_n, \end{aligned} \tag{309}$$

and we also have

$$\tilde{w}_i^n = \tilde{w}_{i-1}^n - \tilde{v}_{i-1}^n \text{ and } \tilde{v}_i^n = 0 \text{ in } D \backslash \Omega_i. \tag{310}$$

Starting from the choice of $\bar{\vartheta}_0^i$, $i = 1, \cdots, m$, in Algorithm 3.20, we can inductively prove that

$$\tilde{w}_i^n = \tilde{v}_i^n = 0 \text{ on } \partial D \backslash \Gamma_n. \tag{311}$$

The following theorem, whose proof is similar to that of Theorem 3.25, proves the numerical stability of Algorithm 3.19.

Theorem 3.18 *Suppose that Assumption 3.10 holds, u is the weak solution of problem (287), $\tilde{u}$ is its extension with g in $D \backslash \Omega$, and $\tilde{u}_i^n$ and $\tilde{\vartheta}_i^n$, $i = 1, \cdots, m$, $n \geq 1$, are obtained from Algorithm 3.20 by (292) and (305), respectively. Then, if*

$$|\tilde{u}_i^n - \tilde{\vartheta}_i^n|_{H^1(\Omega_i)} \leq \varepsilon, \text{ for any } i = 1, \cdots, m \text{ and } n \geq 1, \tag{312}$$

we have

$$\begin{aligned} &|\, \tilde{u} - \tilde{u}_i^{n+1} \,|_{H^1(D)}^2 \leq C[1 + \tfrac{1}{C(m-1)}]^{-n} \,|\, \tilde{u} - \tilde{u}_i^1 \,|_{H^1(D)}^2 + \\ &Cm[C(m-1) + 1]\varepsilon^2, \end{aligned} \tag{313}$$

where the constant C depends only on the bilinear form a and the domains Ω and Ω_i.

Schwarz-Neumann method for unbounded domains. We consider that the domain $\Omega \subset \mathbf{R}^N$ of problem (287) is the intersection of domains Ω_i, $i = 1, \cdots, m$, which are unbounded and locally lie on only one side of their boundaries. In order to follow as in the earlier section and prove that Theorems 3.17 and 3.18 hold for Algorithms 3.19 and 3.20 with unbounded domains, we have to specify:

i) the spaces in which the problems have solutions, proving that their norms are equivalent to that generated by the bilinear form a, and also,

ii) the trace spaces corresponding to these spaces of the solutions.

In the following we answer these two questions.

i) As in the case of the bounded domains, if there exists a constant $c_0 > 0$ such that $a_0(x) \geq c_0$ for any $x \in D$, then the bilinear form a generates in $H^1(D)$ a norm equivalent to the usual norm.

If a_0 does not have the above property and the domain is unbounded, then the closure of $\mathcal{D}(D)$ for the norm generated by a might not be a space of generalized functions (see [35]), and we have to introduce the weighted spaces which take into account the behavior of the functions at infinity. The type of weight for these spaces depends on the dimension N of the space $\mathbf{R}^N$ and we discuss, in the following, the cases $N = 2$ and $N \geq 3$.

If the domains lie in $\mathbf{R}^2$ we use the weighted spaces introduced in [61] or [62],

$$W^1(D) = \{v \in \mathcal{D}'(D) : (1+r^2)^{-1/2}(1+\log\sqrt{1+r^2})^{-1} v \in L^2(D),$$
$$\nabla v \in (L^2(D))^2\},$$

where r denotes the distance from the origin. The norm on $W^1(D)$ is given by

$$| v |_{W^1(D)} = [| (1+r^2)^{-1/2}(1+\log\sqrt{1+r^2})^{-1} v |^2_{L^2(D)} + | \nabla v |^2_{(L^2(D))^2}]^{1/2},$$

and we assume that the coefficient a_0 of the operator A satisfies

$$(1+r^2)(1+\log\sqrt{1+r^2})^2 a_0 \in L^\infty(D). \tag{314}$$

We notice that the space $H^1(D)$ is continuously embedded in $W^1(D)$ and, if D is bounded, then the two spaces coincide. We denote by $W^1_0(D)$ the closure of $\mathcal{D}(D)$ in $W^1(D)$. If $N = 2$ and D is the complement of the closure of a bounded domain, then the bilinear form a generates on $W_0^1(D)$ an equivalent norm to that induced by $W^1(D)$ (see [61]). This fact holds also if D is the complement of an unbounded domain. We notice that the bilinear form a might generate only a seminorm on $W^1(\mathbf{R}^2)$, but not a norm.

If the domains lie in $\mathbf{R}^N$, $N \geq 3$, suitable spaces introduced in [46] are

$$W^1(D) = \{v \in \mathcal{D}'(D) : (1+r^2)^{-1/2}v \in L^2(D),\ \nabla v \in (L^2(D))^N\},$$

with the norm

$$\mid v \mid_{W^1(D)} = [\mid (1+r^2)^{-1/2}v \mid^2_{L^2(D)} + \mid \nabla v \mid^2_{(L^2(D))^N}]^{1/2}.$$

In this case we assume that

$$(1+r^2)a_0 \in L^\infty(D). \tag{315}$$

The above norm on $W^1(\mathbf{R}^N)$, $N \geq 3$, is equivalent to that generated by a (see [46]). Now, if D is a domain in $\mathbf{R}^N$, extending the functions in $W_0^1(D)$ with zero in $\mathbf{R}^N \backslash D$, the bilinear form a generates on $W_0^1(D)$ an equivalent norm to that induced by $W^1(D)$. Evidently, as for $N = 2$, $H^1(D)$ and $W^1(D)$ coincide when the domain D is bounded.

Since the problems we have considered in this paper have mixed boundary conditions we introduce the space

$$W(D) = \{v \in W^1(D) : v = 0 \text{ on } \partial D \backslash \Gamma_n\},$$

where $\Gamma_n \subset \partial D$ is a bounded set such that $\text{meas}(\partial D \backslash \Gamma_n) > 0$. The following lemma proves that the norm generated by a and the norm of $W^1(D)$ are equivalent on $W(D)$. The proof we give is similar to that in [60] where it is proved that if D is the complement of the closure of a bounded domain in $\mathbf{R}^3$, then the norm generated by the Laplace operator and that of $W^1(D)$ are equivalent.

Lemma 3.10 *If* $\text{meas}(\partial D \backslash \Gamma_n) > 0$, *then*

$$\mid v \mid_{W(D)} = \mid \nabla v \mid_{(L^2(D))^N}$$

is a norm on the above space $W(D)$, *for* $N \geq 2$, *and it is equivalent to the norm induced by* $W^1(D)$.

The above lemma results directly from Friedrichs' inequality for bounded domains.

ii) Concerning the spaces of traces of a function v in $W^1(D)$, we first notice that if the domain has a bounded boundary, then the trace of v lies in $H^{1/2}(\partial D)$. We can easily prove it by considering a bounded domain which contains the boundary of D, and taking into account that W^1 and H^1 coincide on this domain. In this case, there is an isomorphism and homeomorphism of $W^1(D)/W_0^1(D)$ onto $H^{1/2}(\partial D)$.

To investigate the behavior at infinity of the traces, in [46], the traces of functions in $W^1(\mathbf{R}_+^N)$, $N \geq 3$, on a hyperplane are studied. It is proved there that $W^1(\mathbf{R}_+^N)/W_0^1(\mathbf{R}_+^N)$ is isomorphic and homeomorphic with

$$W^{1/2}(\mathbf{R}^{N-1}) = \{v \in \mathcal{D}'(\mathbf{R}^{N-1}) : (1+r^2)^{-1/2}v \in L^2(\mathbf{R}^{N-1}) \text{ and } \int_0^\infty t^{-2}dt \int_{\mathbf{R}^{N-1}} \mid v(x+te_i) - v(x) \mid^2 dx < \infty,\ i = 1, \cdots, N-1\},$$

where e_i, $i = 1, \cdots, N-1$, are the unit vectors of $\mathbf{R}^{N-1}$. The same techniques can be used to find the trace space of the functions in W^1 for other types of unbounded domains (see remarks in Sections 4.29, p. 89, and 7.45, p. 213, in [1]). Concerning the domain Ω of problem (397) we make the following assumption.

Assumption 3.11 *The space of traces $W^{1/2}(\partial\Omega)$ of the functions in $W^1(\Omega)$ is isomorphic and homeomorphic with $W^1(\Omega)/W_0^1(\Omega)$, i.e. there are two constants $k_1,\ k_2 > 0$ such that we have*

1. For any $v \in W^1(\Omega)$ there exists $w \in W^{1/2}(\partial\Omega)$ such that $v = w$ on $\partial\Omega$ and
$\mid w \mid_{W^{1/2}(\partial\Omega)} \leq k_1 \mid v \mid_{W^1(\Omega)}$.

2. For any $w \in W^{1/2}(\partial\Omega)$ there exists $v \in W^1(\Omega)$ such that $v = w$ on $\partial\Omega$ and
$\mid v \mid_{W^1(\Omega)} \leq k_2 \mid w \mid_{W^{1/2}(\partial\Omega)}$.

As we have already remarked, this assumption holds for domains with bounded boundaries.

Going back to problem (287) we assume that $g \in W^{1/2}(\Gamma_d)$, and also,

$$\begin{aligned} &(1+r^2)^{1/2}(1+\log\sqrt{1+r^2})F \in L^2(D) \text{ if } N = 2, \\ &(1+r^2)^{1/2}F \in L^2(D) \text{ if } N \geq 3. \end{aligned} \tag{316}$$

Consequently, if Γ_n is bounded and we assume (314)–(316), from Lemma 3.10 we get that all the problems defined in the previous section have unique solutions in the spaces W^1. Moreover, using Assumption 3.11, all the results in the case of the bounded domains hold, replacing the spaces H^1 and $H^{1/2}$ by W^1 and $W^{1/2}$, respectively. In this way we have proved

Theorem 3.19 *If Γ_n is bounded, Assumptions 3.10 and 3.11 hold, and $D \neq \mathbf{R}^2$ if $N = 2$ and there is no constant $c_0 > 0$ such that $a_0(x) \geq c_0$ for any $x \in D$, then*

1. Using the notations in Theorem 3.17 we have $\tilde{u}_i^n \to \tilde{u}$ strongly in $W^1(D)$ as $n \to \infty$, for any $i = 1, \cdots, m$, and the following error estimate

$$| \tilde{u} - \tilde{u}_i^{n+1} |_{W^1(D)}^2 \leq C[\frac{C(m-1)}{C(m-1)+1}]^n | \tilde{u} - \tilde{u}_i^1 |_{W^1(D)}^2 \tag{317}$$

holds.

2. Using the notations in Theorem 3.18 we have

$$\begin{aligned} &| \tilde{u} - \tilde{u}_i^{n+1} |_{W^1(D)}^2 \leq C[1 + \tfrac{1}{C(m-1)}]^{-n} | \tilde{u} - \tilde{u}_i^1 |_{W^1(D)}^2 + \\ &Cm[C(m-1)+1]\varepsilon^2. \end{aligned} \tag{318}$$

The above constant C depends only on the bilinear form a and the domains Ω and Ω_i.

Remark 3.1. a) Since the spaces H^1 and W^1 coincide on bounded domains, the result of the above theorem still holds even if the set of domains $\{\Omega_i, i = 1, \cdots, m\}$, contains both bounded and unbounded domains.

b) If $D = \mathbf{R}^2$ and the bilinear form a generates in $W^1(\mathbf{R}^2)$ only a semi-norm but not a norm, then Algorithm 3.19 might not converge, as in the following counter-example. At the same time, we notice that if $N > 2$ the algorithm converges even if $D = \mathbf{R}^N$; in this case the norm generated by a is equivalent to that in $W^1(\mathbf{R}^N)$.

Counter-example. Let us consider the Dirichlet problem (281) in which Ω is the annulus bounded by the circles C_1 and C_2 with the centers at the origin and having radii of 1 and 2, respectively. We consider $g = 1$ on the circle C_1 and $g = 2$ on C_2 in the boundary conditions of problem (281). The domain Ω_1 is the exterior domain of the circle C_1 and Ω_2 is the open disc bounded by the circle C_2. Let $g_1 \in H^1(\mathbf{R}^2)$ be a function vanishing in the exterior of the circle C_2, $g_1 = 1$ on the disc bounded by C_1. Also, we take a function $g_2 \in W^1(\mathbf{R}^2)$ such that $g_2 = 2$ in the exterior of C_2, and $g_2 = 0$ on the disc bounded by C_1. We consider $g = g_1 + g_2 \in W^1(\mathbf{R}^2)$, $u_1^0 = g_1$ and $u_2^0 = g_2$. Applying Algorithm 3.19, the solutions of problems (284) and (285) with constant boundary conditions being constant functions, we get $u_1^1 = 1$, $u_2^1 = 1$, $u_1^2 = 0$, $u_2^2 = 2$, $u_1^3 = -1$, $u_2^3 = 3$, $u_1^4 = -2$, $u_2^4 = 4$, and so on. Consequently, we obtain $\tilde{u}_1^n = u_1^n + u_2^{n-1} = 1$ and $\tilde{u}_2^n = u_1^n + u_2^n = 2$, for any $n \geq 2$. Evidently, neither $\tilde{u}_1^n$ nor $\tilde{u}_2^n$ converges to the solution of problem (281).

3.2 Optimal control, domain embedding methods and fast algorithms

3.2.1 Boundary control approach to domain embedding methods (paper [12])

In [12], a domain embedding method associated with an optimal boundary control problem with boundary observations to solve elliptic problems is proposed. We prove that the optimal boundary control problem has a unique solution if the controls are taken in a finite dimensional subspace of the space of the boundary conditions on the auxiliary domain. Using a controllability theorem due to J. L. Lions, we prove that the solutions of Dirichlet (or Neumann) problems can be approximated within any prescribed error, however small, by solutions of Dirichlet (or Neumann) problems in the auxiliary domain taking an appropriate subspace for such an optimal control problem. We also prove that the results obtained for the interior problems hold for the exterior problems. Some numerical examples are given for both the interior and the exterior Dirichlet problems.

Controllability. Let $\omega,\ \Omega \in \mathcal{N}^{(1),1}$ (i.e., the maps defining the boundaries of the domains and their derivatives are Lipschitz continuous) be two bounded domains in $\mathbf{R}^N$ such that $\bar{\omega} \subset \Omega$. Their boundaries are denoted by γ and Γ, respectively.

In this paper, we use domain embedding and the optimal boundary control approach to solve the elliptic equation

$$Ay = f \quad \text{in } \omega, \tag{319}$$

subject to either Dirichlet boundary conditions

$$y = g_\gamma \quad \text{on } \gamma \tag{320}$$

or Neumann boundary conditions

$$\tfrac{\partial y}{\partial n_A(\omega)} = h_\gamma \quad \text{on } \gamma, \tag{321}$$

where $\frac{\partial}{\partial n_A(\omega)}$ is the outward conormal derivative associated with A.

We assume that the operator A is of the form

$$A = -\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial}{\partial x_j}\right) + a_0$$

with $a_{ij} \in C^{(1),1}(\bar{\Omega})$, $a_0 \in C^{(0),1}(\bar{\Omega})$, $a_0 \geq 0$ in Ω, and there exists a constant $c > 0$ such that $\sum_{i,j=1}^{N} a_{ij}\xi_i\xi_j \geq c(\xi_1^2 + \cdots + \xi_N^2)$ in Ω for any $(\xi_1, \ldots, \xi_N) \in \mathbf{R}^N$. Also, we assume that $f \in L^2(\Omega)$, $g_\gamma \in L^2(\gamma)$, and $h_\gamma \in H^{-1}(\gamma)$.

Regarding the solutions of the above problems, we give the following definition (see [67, Chap. 2, section 7]).

Definition 3.1 *A function $y \in H^{1/2}(\omega)$ is called a solution of the Dirichlet problem (319)–(320) if it satisfies (319) in the sense of distributions and the boundary conditions (320) in the sense of traces in $L^2(\gamma)$. A function $y \in H^{1/2}(\omega)$ is called a solution of the Neumann problem (319), (321) if it satisfies (319) in the sense of distributions and the boundary conditions (321) in the sense of traces in $H^{-1}(\gamma)$.*

The Dirichlet problem (319)–(320) has a unique solution which depends continuously on the data

$$|y|_{H^{1/2}(\omega)} \leq C\{|f|_{L^2(\omega)} + |g_\gamma|_{L^2(\gamma)}\}. \tag{322}$$

If there exists a constant $c_0 > 0$ such that $a_0 \geq c_0$ in ω, then the Neumann problem (319), (321) has a unique solution which depends continuously on the data

$$|y|_{H^{1/2}(\omega)} \leq C\{|f|_{L^2(\omega)} + |h_\gamma|_{H^{-1}(\gamma)}\}. \tag{323}$$

If $a_0 = 0$ in ω, then the Neumann problem (319), (321) has a solution if

$$\int_\omega f + \int_\gamma h_\gamma = 0. \tag{324}$$

In this case, the problem has a unique solution in $H^{1/2}(\omega)/\mathbf{R}$ and

$$\inf_{r \in \mathbf{R}} |y + r|_{H^{1/2}(\omega)} \leq C\{|f|_{L^2(\omega)} + |h_\gamma|_{H^{-1}(\gamma)}\}. \tag{325}$$

We also remark that the solution of problem (319)–(320) can be viewed (see [67, Chap. 2, section 6]) as the solution of the problem

$$\begin{array}{l} y \in H^{1/2}(\omega): \ \int_\omega y A^* \psi = \int_\omega f \psi - \int_\gamma g_\gamma \frac{\partial \psi}{\partial n_{A^*(\omega)}} \\ \text{for any } \psi \in H^2(\omega),\ \psi = 0 \text{ on } \gamma, \end{array} \tag{326}$$

and that a solution of problem (319), (321) is also solution of the problem

$$\begin{array}{l} y \in H^{1/2}(\omega): \ \int_\omega y A^* \psi = \int_\omega f \psi + \int_\gamma h_\gamma \psi \\ \text{for any } \psi \in H^2(\omega),\ \frac{\partial \psi}{\partial n_{A^*(\omega)}} = 0 \text{ on } \gamma, \end{array} \tag{327}$$

where A^* is the adjoint operator of A given by

$$A^* = -\sum_{i,j=1}^N \frac{\partial}{\partial x_i}\left(a_{ji}\frac{\partial}{\partial x_j}\right) + a_0.$$

Evidently, the above results also hold for problems in the domain Ω.

We consider in the following only the cases in which the above problems have unique solutions, i.e., the Dirichlet problems, and we assume in the case of the Neumann problems that there exists a constant $c_0 > 0$ such that $a_0 \geq c_0$ in Ω.

Below we use the notations and the notions of optimal control from Lions [63]. First, we study the controllability of the solutions of the above two problems (defined by (319)–(321)) in ω with the solutions of a Dirichlet problem in Ω. Let

$$\mathcal{U} = L^2(\Gamma) \tag{328}$$

be the space of controls. The state of the system for a control $v \in L^2(\Gamma)$ is given by the solution $y(v) \in H^{1/2}(\Omega)$ of the following Dirichlet problem:

$$\begin{array}{lll} Ay(v) = f & \text{in} & \Omega, \\ y(v) = v & \text{on} & \Gamma. \end{array} \tag{329}$$

In the case of the Dirichlet problem (319)–(320), the space of observations is taken to be

$$\mathcal{H} = L^2(\gamma), \tag{330}$$

and the cost function is given by

$$J(v) = \frac{1}{2}|y(v) - g_\gamma|^2_{L^2(\gamma)}, \tag{331}$$

where $v \in L^2(\Gamma)$ and $y(v)$ is the solution of problem (329). For the Neumann problem given by (319) and (321), the space of observations is taken to be

$$\mathcal{H} = H^{-1}(\gamma), \tag{332}$$

and the cost function is given by

$$J(v) = \frac{1}{2}\left|\frac{\partial y(v)}{\partial n_A(\omega)} - h_\gamma\right|^2_{H^{-1}(\gamma)}. \tag{333}$$

Remark 3.9 *Since $y(v) \in H^{1/2}(\Omega)$ and $Ay(u) = f \in L^2(\Omega)$, we have $y(v) \in H^2(D)$ for any domain D which satisfies $\bar{\omega} \subset D \subset \bar{D} \subset \Omega$ (see [71, Chap. 4, section 1.2, Theorem 1.3], for instance). Therefore, $y(v) \in H^{3/2}(\gamma)$ with the same values on both the sides of γ. Also, $\frac{\partial y(v)}{\partial n_A(\omega)} \in H^{1/2}(\gamma)$, $\frac{\partial y(v)}{\partial n_A(\Omega-\bar{\omega})} \in H^{1/2}(\gamma)$, and $\frac{\partial y(v)}{\partial n_A(\omega)} + \frac{\partial y(v)}{\partial n_A(\Omega-\bar{\omega})} = 0$. Consequently, the above two cost functions make sense.*

We have the following

Proposition 3.15 *A control $u \in L^2(\Gamma)$ satisfies $J(u) = 0$, where the control function is given by (331), if and only if the solution of (329) for $v = u$, $y(u) \in H^{1/2}(\Omega)$ satisfies*

$$\begin{array}{ll} Ay(u) = f & \text{in } \Omega - \bar{\omega}, \\ y(u) = y & \text{on } \gamma, \\ \frac{\partial y(u)}{\partial n_A(\Omega-\bar{\omega})} + \frac{\partial y}{\partial n_A(\omega)} = 0 & \text{on } \gamma, \end{array} \tag{334}$$

and

$$y(u) = y \quad \text{in } \omega, \tag{335}$$

where y is the solution of the Dirichlet problem defined by (319) and (320) in the domain ω. The same result holds if the control function is given by (333) and y is the solution of the Neumann problem (319) and (321).

Since (334) is not a properly posed problem, it follows from the above proposition that the optimal control might not exist. However, J. L. Lions proves in [63, Chap. 2, section 5.3, Theorem 5.1] a controllability theorem which can be directly applied to problem (329). We mention this theorem below.

LIONS'S CONTROLLABILITY THEOREM. *The set* $\{\frac{\partial z_0(v)}{\partial n_A(\Omega-\bar{\omega})} \in H^{-1}(\gamma) : v \in L^2(\Gamma)\}$ *is dense in* $H^{-1}(\gamma)$, *where* $z_0(v) \in H^{1/2}(\Omega - \bar{\omega})$ *is the solution of the problem*

$$\begin{aligned} &Az_0(v) = 0 && \text{in } \Omega - \bar{\omega}, \\ &z_0(v) = v && \text{on } \Gamma, \\ &z_0(v) = 0 && \text{on } \gamma. \end{aligned}$$

Using this theorem, we get

Lemma 3.11 *For any* $g \in L^2(\gamma)$, *the set* $\{\frac{\partial z(v)}{\partial n_A(\Omega-\bar{\omega})} \in H^{-1}(\gamma) : v \in L^2(\Gamma)\}$ *is dense in* $H^{-1}(\gamma)$, *where* $z(v) \in H^{1/2}(\Omega - \bar{\omega})$ *is the solution of the problem*

$$\begin{aligned} &Az(v) = f && \text{in } \Omega - \bar{\omega}, \\ &z(v) = v && \text{on } \Gamma, \\ &z(v) = g && \text{on } \gamma. \end{aligned} \tag{336}$$

The following theorem proves controllability of the solutions of problems in ω by the solutions of Dirichlet problems in Ω.

Theorem 3.20 *The set* $\{y(v)_{|\omega} : v \in L^2(\Gamma)\}$ *is dense, using the norm of* $H^{1/2}(\omega)$, *in* $\{y \in H^{1/2}(\omega) : Ay = f \text{ in } \omega\}$, *where* $y(v) \in H^{1/2}(\Omega)$ *is the solution of the Dirichlet problem* (329) *for a given* $v \in L^2(\Gamma)$.

For the controllability of the solutions of the Dirichlet and the Neumann problems (given by (319), (320), and (319), (321), respectively) in ω by Neumann problems in Ω, we take the space of controls

$$\mathcal{U} = H^{-1}(\Gamma), \tag{337}$$

and for a $v \in H^{-1}(\Gamma)$, the state of the system is the solution $y(v) \in H^{1/2}(\Omega)$ of the problem

$$\begin{aligned} &Ay(v) = f && \text{in} \quad \Omega, \\ &\frac{\partial y(v)}{\partial n_A(\Omega)} = v && \text{on} \quad \Gamma. \end{aligned} \tag{338}$$

We remark that

$$\begin{aligned}&i : \{y(v) \in H^{1/2}(\Omega) : v \in L^2(\Gamma),\ y(v) \text{ solution of problem (329)}\} \to \\ &\quad \{y(w) \in H^{1/2}(\Omega) : w \in H^{-1}(\Gamma),\ y(w) \text{ solution of problem (338)}\}, \\ &i(y(v)) = y(w) \Leftrightarrow y(v) = y(w) \text{ in } \Omega\end{aligned} \tag{339}$$

establish a bijective correspondence. Consequently, Proposition 3.15 also holds if the space of controls there is changed to $H^{-1}(\Gamma)$ and the states $y(v)$ of the system are solutions of problem (338). Theorem 3.20 in this case becomes the following theorem.

Theorem 3.21 *The set $\{y(v)_{|\omega} : v \in H^{-1}(\Gamma)\}$ is dense, using the norm of $H^{1/2}(\omega)$, in $\{y \in H^{1/2}(\omega) : Ay = f \text{ in } \omega\}$, where $y(v) \in H^{1/2}(\Omega)$ is a solution of the Neumann problem* (338) *for a given $v \in H^{-1}(\Gamma)$.*

Controllability with finite dimensional spaces. Let $\{U_\lambda\}_\lambda$ be a family of finite dimensional subspaces of the space $L^2(\Gamma)$ such that, given (328) as a space of controls with the Dirichlet problems, we have

$$\bigcup_\lambda U_\lambda \text{ is dense in } \mathcal{U} = L^2(\Gamma). \tag{340}$$

For a $v \in L^2(\Gamma)$ we consider the solution $y'(v) \in H^{1/2}(\Omega)$ of the problem

$$\begin{aligned} Ay'(v) &= 0 \quad \text{in } \Omega, \\ y'(v) &= v \quad \text{on } \Gamma. \end{aligned} \tag{341}$$

We fix a U_λ. The cost functions J defined by (331) and (333) are differentiable and convex. Consequently, an optimal control

$$u_\lambda \in U_\lambda : J(u_\lambda) = \inf_{v \in U_\lambda} J(v) \tag{342}$$

exists if and only if it is a solution of the equation

$$u_\lambda \in U_\lambda : (y(u_\lambda), y'(v))_{L^2(\gamma)} = (g_\gamma, y'(v))_{L^2(\gamma)} \text{ for any } v \in U_\lambda, \tag{343}$$

when the control function is (331), and

$$u_\lambda \in U_\lambda : \left(\frac{\partial y(u_\lambda)}{\partial n_A(\omega)}, \frac{\partial y'(v)}{\partial n_A(\omega)}\right)_{H^{-1}(\gamma)} = \left(h_\gamma, \frac{\partial y'(v)}{\partial n_A(\omega)}\right)_{H^{-1}(\gamma)} \text{ for any } v \in U_\lambda, \tag{344}$$

when the control function is (333). Above, $y(u_\lambda)$ is the solution of problem (329) corresponding to u_λ, and $y'(v)$ is the solution of problem (341) corresponding to v. If $y_f \in H^2(\Omega)$ is the solution of the problem

$$\begin{aligned} Ay_f &= f \quad \text{in } \Omega, \\ y_f &= 0 \quad \text{on } \Gamma, \end{aligned} \tag{345}$$

then, for a $v \in L^2(\Gamma)$, we have

$$y(v) = y'(v) + y_f, \tag{346}$$

where $y(v)$ and $y'(v)$ are the solutions of problems (329) and (341), respectively. Therefore, we can rewrite problems (343) and (344) as

$$u_\lambda \in U_\lambda : \ (y'(u_\lambda), y'(v))_{L^2(\gamma)} = (g_\gamma - y_f, y'(v))_{L^2(\gamma)} \tag{347}$$

for any $v \in U_\lambda$, and

$$u_\lambda \in U_\lambda : \ \left(\frac{\partial y'(u_\lambda)}{\partial n_A(\omega)}, \frac{\partial y'(v)}{\partial n_A(\omega)}\right)_{H^{-1}(\gamma)} = \left(h_\gamma - \frac{\partial y_f}{\partial n_A(\omega)}, \frac{\partial y'(v)}{\partial n_A(\omega)}\right)_{H^{-1}(\gamma)} \tag{348}$$

for any $v \in U_\lambda$, respectively. We have the following lemma.

Lemma 3.12 *For a fixed λ, let $\varphi_1, \ldots, \varphi_{n_\lambda}$, $n_\lambda \in \mathbf{N}$, be a basis of U_λ, and let $y'(\varphi_i)$ be the solution of problem* (341) *for $v = \varphi_i$, $i = 1, \ldots, n_\lambda$. Then $\{y'(\varphi_1)_{|\gamma}, \ldots, y'(\varphi_{n_\lambda})_{|\gamma}\}$ and $\{\frac{\partial y'(\varphi_1)}{\partial n_A(\omega)}|\gamma, \ldots, \frac{\partial y'(\varphi_{n_\lambda})}{\partial n_A(\omega)}|\gamma\}$ are linearly independent sets.*

The following proposition proves the existence and uniqueness of the optimal control when the states of the system are the solutions of the Dirichlet problems.

Proposition 3.16 *Let us consider a fixed U_λ. Then problems* (347) *and* (348) *have unique solutions. Consequently, if the boundary conditions of Dirichlet problems* (329) *lie in the finite dimensional space U_λ, then there exists a unique optimal control of problem* (342) *corresponding to either the Dirichlet problem* (319), (320) *or the Neumann problem* (319), (321).

The following theorem proves the controllability of the solutions of the Dirichlet and Neumann problems in ω by the solutions of the Dirichlet problems in Ω.

Theorem 3.22 *Let $\{U_\lambda\}_\lambda$ be a family of finite dimensional spaces satisfying* (340)*. We associate the solution y of the Dirichlet problem* (319), (320) *in ω with problem* (342), *in which the cost function is given by* (331)*. Also, the solution y of the Neumann problem* (319), (321) *is associated with problem* (342), *in which the cost function is given by* (333)*. In both cases, there exists a positive constant C, and for any given $\varepsilon > 0$ there exists U_{λ_ε} such that*

$$|y(u_{\lambda_\varepsilon})_{|\omega} - y|_{H^{1/2}(\omega)} < C\varepsilon,$$

where $u_{\lambda_\varepsilon} \in U_{\lambda_\varepsilon}$ is the optimal control of the corresponding problem (342) *with $\lambda = \lambda_\varepsilon$, and $y(u_{\lambda_\varepsilon})$ is the solution of problem* (329) *with $v = u_{\lambda_\varepsilon}$.*

Using the basis $\varphi_1, \dots, \varphi_{n_\lambda}$ of the space U_λ, we define the matrix

$$\Pi_\lambda = ((y'(\varphi_i), y'(\varphi_j))_{L^2(\gamma)})_{1\le i,j\le n_\lambda} \tag{349}$$

and the vector

$$l_\lambda = ((g_\gamma - y_f, y'(\varphi_i))_{L^2(\gamma)})_{1\le i\le n_\lambda}. \tag{350}$$

Then problem (347) can be written as

$$\xi_\lambda = (\xi_{\lambda,1}, \dots, \xi_{\lambda,n_\lambda}) \in \mathbf{R}^{n_\lambda} : \Pi_\lambda \xi_\lambda = l_\lambda. \tag{351}$$

Consequently, using Theorem 3.22, the solution y of problem (123), (320) can be obtained within any prescribed error by setting the restriction to ω of

$$y(u_\lambda) = \xi_{\lambda,1} y'(\varphi_1) + \cdots + \xi_{\lambda,n_\lambda} y'(\varphi_{n_\lambda}) + y_f, \tag{352}$$

where $\xi_\lambda = (\xi_{\lambda,1}, \dots, \xi_{\lambda,n_\lambda})$ is the solution of algebraic system (351). Above, y_f is the solution of problem (345), and $y'(\varphi_i)$ are the solutions of problems (341) with $v = \varphi_i$, $i = 1, \dots, n_\lambda$.

An algebraic system (351) is also obtained in the case of problem (348). This time the matrix of the system is given by

$$\Pi_\lambda = \left(\left(\frac{\partial y'(\varphi_i)}{\partial n_A(\omega)}, \frac{\partial y'(\varphi_j)}{\partial n_A(\omega)} \right)_{H^{-1}(\gamma)} \right)_{1\le i,j\le n_\lambda}, \tag{353}$$

and the free term is

$$l_\lambda = \left(\left(h_\gamma - \frac{\partial y_f}{\partial n_A(\omega)}, \frac{\partial y'(\varphi_i)}{\partial n_A(\omega)} \right)_{H^{-1}(\gamma)} \right)_{1 \le i \le n_\lambda}. \tag{354}$$

Therefore, using Theorem 3.22, the solution y of problem (123), (321) can be estimated by (352). Also, y_f is the solution of problem (345), and $y'(\varphi_i)$ are the solutions of problems (341) with $v = \varphi_i$, $i = 1, \ldots, n_\lambda$.

The case of the controllability with finite dimensional optimal controls for states of the system given by the solution of a Neumann problem is treated in a similar way. As in the previous section, the space of the controls is $\mathcal{U}$, given in (337), and the state of the system $y(v) \in H^{1/2}(\Omega)$ is given by the solution of Neumann problem (338) for a $v \in H^{-1}(\Gamma)$.

Let $\{U_\lambda\}_\lambda$ be a family of finite dimensional subspaces of the space $H^{-1}(\Gamma)$ such that

$$\bigcup_\lambda U_\lambda \text{ is dense in } \mathcal{U} = H^{-1}(\Gamma). \tag{355}$$

This time, the function $y'(v) \in H^{1/2}(\Omega)$ appearing in (343), (344), (347), and (348) is the solution of the problem

$$\begin{aligned} Ay'(v) &= 0 \quad \text{in } \Omega, \\ \tfrac{\partial y'(v)}{\partial n_A(\Omega)} &= v \quad \text{on } \Gamma \end{aligned} \tag{356}$$

for a $v \in H^{-1}(\Gamma)$. Also, $y_f \in H^2(\Omega)$ appearing in (346), (347), and (348) is the solution of the problem

$$\begin{aligned} Ay_f &= f \quad \text{in } \Omega, \\ \tfrac{\partial y_f}{\partial n_A(\Omega)} &= 0 \quad \text{on } \Gamma. \end{aligned} \tag{357}$$

With these changes, Lemma 3.12 also holds in this case, and the proof of the following proposition is similar to that of Proposition 3.16.

Proposition 3.17 *For a given U_λ, the problems* (347) *and* (348) *have unique solutions. Consequently, if the boundary conditions of Neumann problems* (338) *lie in the finite dimensional space U_λ, then there exists a unique optimal control of problem* (342)*, corresponding to either Dirichlet problem* (319), (320) *or Neumann problem* (319), (321).

A proof similar to that given for Theorem 3.22 can also be given for the following theorem.

Theorem 3.23 *Let* $\{U_\lambda\}_\lambda$ *be a family of finite dimensional spaces satisfying* (355). *We associate the solution* $y \in H^{1/2}(\omega)$ *of problem* (319), (320) *with problem* (342), *in which the cost function is given by* (331). *Also, the solution* y *of problem* (319), (321) *is associated with problem* (342), *in which the cost function is given by* (333). *In both cases, there exists a positive constant* C, *and for any given* $\varepsilon > 0$ *there exists* λ_ε *such that*

$$|y(u_{\lambda_\varepsilon})_{|\omega} - y|_{H^{1/2}(\omega)} < C\varepsilon,$$

where $u_{\lambda_\varepsilon} \in U_{\lambda_\varepsilon}$ *is the optimal control of the corresponding problem* (342) *with* $\lambda = \lambda_\varepsilon$, *and* $y(u_{\lambda_\varepsilon})$ *is the solution of problem* (338) *with* $v = u_{\lambda_\varepsilon}$.

Evidently, in the case of the controllability with solutions of Neumann problem (338) we can also write algebraic systems (351) using a basis $\varphi_1, \ldots, \varphi_{n_\lambda}$ of a given subspace U_λ of the space $\mathcal{U} = H^{-1}(\Gamma)$. As in the case of the controllability with solutions of the Dirichlet problem (329), these algebraic systems have unique solutions.

Theorems 3.22 and 3.23 prove the convergence of the embedding method associated with the optimal boundary control. An error analysis would be desirable, but it would go beyond the scope of this paper.

Remark 3.10 *We have defined* y_f *as a solution of problems (345) or (357) in order to have* $y(v) = y'(v) + y_f$ *or* $\frac{\partial y(v)}{\partial n_A(\Omega)} = \frac{\partial y'(v)}{\partial n_A(\Omega)} + \frac{\partial y_f}{\partial n_A(\Omega)}$, *respectively, on the boundary* Γ. *In fact, we can replace* $y(v)$ *by* $y'(v) + y_f$ *in the cost functions (331) and (333) with* $y_f \in H^2(\Omega)$ *satisfying only*

$$Ay_f = f \text{ in } \Omega, \tag{358}$$

and the results obtained in this section still hold.

Approximate observations in finite dimensional spaces. In solving problems (347), (348), we require an appropriate interpolation which makes use of the values of $y'(v)$ computed only at some points on the boundary γ. We show below that using these interpolations, i.e., observations in finite dimensional subspaces, we can obtain the approximate solutions of problems (319), (320) and (319), (321).

As in the previous sections, we first deal with the case when the states of the system are given by the Dirichlet problem (329). Let U_λ be a fixed finite dimensional subspace of $\mathcal{U} = L^2(\Gamma)$ with the basis $\varphi_1, \ldots, \varphi_{n_\lambda}$.

Let us assume that for problem (319), (320), we choose a family of finite dimensional spaces $\{H_\mu\}_\mu$ such that

$$\bigcup_\mu H_\mu \text{ is dense in } \mathcal{H} = L^2(\gamma). \tag{359}$$

Similarly, for problem (319), (321) we choose the finite dimensional spaces $\{H_\mu\}_\mu$ such that

$$\bigcup_\mu H_\mu \text{ is dense in } \mathcal{H} = H^{-1}(\gamma). \tag{360}$$

The subspace H_μ given in (359) and (360) is a subspace of $\mathcal{H}$ given in (330) and (332), respectively.

An appropriate choice of H_μ is made based on the problem to be solved as discussed above. For a given φ_i, $i = 1, \ldots, n_\lambda$, we consider below the solution $y'(\varphi_i)$ of problem (341) corresponding to $v = \varphi_i$, and we approximate its trace on γ by $y'_{\mu,i}$. Also, the approximation of $\frac{\partial y'(\varphi_i)}{\partial n_A(\omega)}$ on γ is denoted by $\frac{\partial y'_{\mu,i}}{\partial n_A(\omega)}$.

Since the system (351) has a unique solution, the determinants of the matrices Π_λ given in (349) and (353) are nonzero. Consequently, if $|y'(\varphi_i) - y'_{\mu,i}|_{L^2(\gamma)}$ or $|\frac{\partial y'(\varphi_i)}{\partial n_A(\omega)} - \frac{\partial y'_{\mu,i}}{\partial n_A(\omega)}|_{H^{-1}(\gamma)}$ are small enough, then the matrices

$$\Pi_{\lambda\mu} = ((y'_{\mu,i}, y'_{\mu,j})_{L^2(\gamma)})_{1\le i,j\le n_\lambda} \tag{361}$$

and

$$\Pi_{\lambda\mu} = \left(\left(\frac{\partial y'_{\mu,i}}{\partial n_A(\omega)}, \frac{\partial y'_{\mu,j}}{\partial n_A(\omega)} \right)_{H^{-1}(\gamma)} \right)_{1\le i,j\le n_\lambda} \tag{362}$$

have nonzero determinants. In this case, each of the algebraic systems

$$\xi_{\lambda\mu} = (\xi_{\lambda\mu,1}, \ldots, \xi_{\lambda\mu,n_\lambda}) \in \mathbf{R}^{n_\lambda} : \Pi_{\lambda\mu}\xi_{\lambda\mu} = l_{\lambda\mu} \tag{363}$$

has a unique solution. In this system, the free term is

$$l_{\lambda\mu} = ((g_{\gamma\mu} - y_{f\mu}, y'_{\mu,i})_{L^2(\gamma)})_{1\le i\le n_\lambda} \tag{364}$$

if the matrix $\Pi_{\lambda\mu}$ is given by (361) and

$$l_{\lambda\mu} = \left(\left(h_{\gamma\mu} - \frac{\partial y_{f\mu}}{\partial n_A(\omega)}, \frac{\partial y'_{\mu,i}}{\partial n_A(\omega)} \right)_{H^{-1}(\gamma)} \right)_{1 \le i \le n_\lambda} \tag{365}$$

if the matrix $\Pi_{\lambda\mu}$ is given by (362). Above, we have denoted by $g_{\gamma\mu}$ and $h_{\gamma\mu}$ some approximations in H_μ of g_γ and h_γ, respectively. Also, $y_{f\mu}$ and $\frac{\partial y_{f\mu}}{\partial n_A(\omega)}$ are some approximations of y_f and $\frac{\partial y_f}{\partial n_A(\omega)}$ in the corresponding H_μ of $L^2(\gamma)$ and $H^{-1}(\gamma)$, respectively, with $y_f \in H^2(\Omega)$ satisfying (419).

The solution y of problems (123), (320) and (123), (321) can be approximated with the restriction to ω of

$$y(u_{\lambda\mu}) = \xi_{\lambda\mu,1} y'(\varphi_1) + \cdots + \xi_{\lambda\mu,n_\lambda} y'(\varphi_{n_\lambda}) + y_f, \tag{366}$$

where $\xi_\lambda = (\xi_{\lambda\mu,1}, \ldots, \xi_{\lambda\mu,n_\lambda})$ is the solution of appropriate algebraic system (363).

In both cases (i.e., when the control is affected via Dirichlet and Neumann problems), we obtain the following theorem.

Theorem 3.24 *Let $\{U_\lambda\}_\lambda$ be a family of finite dimensional spaces, either satisfying* (340) *if we consider problem* (329)*, or satisfying* (355) *if we consider problem* (338)*. Also, we associate problem* (319), (320) *or* (319), (321) *with a family of spaces $\{H_\mu\}_\mu$ satisfying* (359) *or* (360)*, respectively. Then, for any $\varepsilon > 0$, there exists λ_ε such that the following hold.*

(i) *If the space H_μ is taken such that $|y'(\varphi_i) - y'_{\mu,i}|_{L^2(\gamma)}$, $i = 1, \ldots, n_{\lambda_\varepsilon}$, are small enough, y is the solution of problem* (319)–(320)*, then*

$$\begin{aligned} &|y(u_{\lambda_\varepsilon \mu})_{|\omega} - y|_{H^{1/2}(\omega)} < C\varepsilon \\ &+ C_{\lambda_\varepsilon} \left(|g_\gamma - g_{\gamma\mu}|_{L^2(\gamma)} + |y_f - y_{f\mu}|_{L^2(\gamma)} + \max_{1 \le i \le n_\lambda} |y'(\varphi_i) - y'_{\mu,i}|_{L^2(\gamma)} \right). \end{aligned}$$

(ii) *If the space H_μ is taken such that $|\frac{\partial y'(\varphi_i)}{\partial n_A(\omega)} - \frac{\partial y'_{\mu,i}}{\partial n_A(\omega)}|_{H^{-1}(\gamma)}$, $i = 1, \ldots, n_{\lambda_\varepsilon}$, are small enough, y is the solution of problem* (319), (321)*, then*

$$\begin{aligned} &|y(u_{\lambda_\varepsilon \mu})_{|\omega} - y|_{H^{1/2}(\omega)} < C\varepsilon \\ &+ C_{\lambda_\varepsilon} \left(|h_\gamma - h_{\gamma\mu}|_{H^{-1}(\gamma)} + \left| \frac{\partial y_f}{\partial n_A(\omega)} - \frac{\partial y_{f\mu}}{\partial n_A(\omega)} \right|_{H^{-1}(\gamma)} \right. \\ &\left. + \max_{1 \le i \le n_\lambda} \left| \frac{\partial y'(\varphi_i)}{\partial n_A(\omega)} - \frac{\partial y'_{\mu,i}}{\partial n_A(\omega)} \right|_{H^{-1}(\gamma)} \right), \end{aligned}$$

where C is a constant and C_{λ_ε} depends on the basis of U_{λ_ε}.

Remark 3.11 *Since the matrices $\Pi_{\lambda\mu}$ given by (361) and (362) are assumed to be nonsingular, it follows that $\{y'_{\mu,i}\}_{i=1,\ldots,n_\lambda}$ and $\{\frac{\partial y'_{\mu,i}}{\partial n_A(\omega)}\}_{i=1,\ldots,n_\lambda}$ are some linearly independent sets in $L^2(\gamma)$ and $H^{-1}(\gamma)$, respectively. Consequently, if m_μ is the dimension of the corresponding subspace H_μ, then $n_\lambda \leq m_\mu$.*

Exterior problems. We consider the domain $\omega \subset \mathbf{R}^N$ of problems (319), (320) and (319), (321) as the complement of the closure of a bounded domain, and it lies on only one side of its boundary. The same assumptions are made on the domain Ω of problems (329) and (338), and, evidently, $\omega \subset \Omega$. In order to retain continuity and to prove that the solutions of the problems in ω can be approximated by the solutions of problems in Ω, we have to specify the spaces in which the problems have solutions and also their correspondence with the trace spaces.

Since the domain $\Omega - \bar{\omega}$ is bounded, Lions's controllability theorem does not need to be extended to unbounded domains. Moreover, we see that the boundaries γ and Γ of the domains ω and Ω are bounded, and, consequently, we can use finite open covers of them (as for the bounded domains) to define the traces.

In order to avoid the use of the fractional spaces of the spaces in ω and Ω, we simply remark that if the controls in the Lions controllability theorem are taken in $H^{1/2}(\Gamma)$ instead of $L^2(\Gamma)$, then a similar proof of it gives the following.

The set $\{\frac{\partial z_0(v)}{\partial n_A(\Omega-\bar{\omega})} \in H^{-1/2}(\gamma) : v \in H^{1/2}(\Gamma)\}$ is dense in $H^{-1/2}(\gamma)$, where $z_0(v) \in H^1(\Omega - \bar{\omega})$ is the solution of the problem

$$\begin{aligned}
&Az_0(v) = 0 && \text{in } \Omega - \bar{\omega},\\
&z_0(v) = v && \text{on } \Gamma,\\
&z_0(v) = 0 && \text{on } \gamma.
\end{aligned}$$

Now we associate to the operator A the symmetric bilinear form

$$a(y,z) = \sum_{i,j=1}^{N} \int_\Omega a_{ij} \frac{\partial y}{\partial x_i} \frac{\partial z}{\partial x_j} + \int_\Omega a_0 yz \quad \text{for} \quad y,\ z \in H^1(\Omega),$$

which is continuous on $H^1(\Omega) \times H^1(\Omega)$. Evidently, a is also continuous on $H^1(\omega) \times H^1(\omega)$. Now if $f \in L^2(\omega)$, taking the boundary data $g_\gamma \in H^{1/2}(\gamma)$ and $h_\gamma \in H^{-1/2}(\gamma)$, then problems (319), (320) and (319), (321) can be written in the variational form

$$\begin{aligned} &y \in H^1(\omega) : \ a(y,z) = \textstyle\int_\omega fz \quad \text{for any} \quad z \in H_0^1(\omega), \\ &y = g_\gamma \text{ on } \gamma, \end{aligned} \tag{367}$$

and

$$y \in H^1(\omega) : \ a(y,z) = \int_\omega fz + \int_\gamma h_\gamma z \quad \text{for any} \quad z \in H^1(\omega), \tag{368}$$

respectively. Similar equations can also be written for problems (329) and (338).

Therefore, *if there exists a constant* $c_0 > 0$ *such that* $a_0 \geq c_0$ *in* Ω, then the bilinear form a is $H^1(\Omega)$-elliptic, i.e., there exists a constant $\alpha > 0$ such that $\alpha|y|^2_{H^1(\Omega)} \leq a(y,y)$ for any $y \in H^1(\Omega)$. It follows from the Lax–Milgram lemma that problems (329) and (338) have unique weak solutions in $H^1(\Omega)$. Naturally, problems (319), (320) and (319), (321) in ω also have unique weak solutions given by the solutions of problems (367) and (368), respectively. In this case, we have the following continuous dependence of the solutions on data for problems (319), (320) and (319), (321) we have

$$|y|_{H^1(\omega)} \leq C\{|f|_{L^2(\omega)} + |g_\gamma|_{H^{1/2}(\gamma)}\}$$

and

$$|y|_{H^1(\omega)} \leq C\{|f|_{L^2(\omega)} + |h_\gamma|_{H^{-1/2}(\gamma)}\},$$

respectively.

Therefore, if there exists a constant $c_0 > 0$ such that $a_0 \geq c_0$ in Ω, then we can proceed in the same manner and obtain similar results for the exterior problems to those obtained in the previous sections for the interior problems. Evidently, in this case we take

$$\mathcal{U} = H^{1/2}(\Gamma) \tag{369}$$

as a space of the controls for problem (329), in place of that given in (328), and the space of controls for problem (338) is taken as

$$\mathcal{U} = H^{-1/2}(\Gamma), \tag{370}$$

in place of the space given in (337).

If $a_0 = 0$ *in* Ω, the domain being unbounded, then the problems might not have solutions in the classical Sobolev spaces, and we have to introduce the weighted spaces which take into account the particular behavior of the solutions at infinity.

For domains in $\mathbf{R}^2$, we use the weighted spaces

$$W^1(\Omega) = \{v \in \mathcal{D}'(\Omega) : (1+r^2)^{-1/2}(1+\log\sqrt{1+r^2})^{-1}v \in L^2(\Omega), \ \nabla v \in (L^2(\Omega))^2\},$$

where $\mathcal{D}'(\Omega)$ is the space of the distributions on Ω, and r denotes the distance from the origin. The norm on $W^1(\Omega)$ is given by

$$\mid v \mid_{W^1(\Omega)} = \left(\mid (1+r^2)^{-1/2}(1+\log\sqrt{1+r^2})^{-1}v \mid^2_{L^2(\Omega)} + \mid \nabla v \mid^2_{(L^2(\Omega))^2} \right)^{1/2}.$$

For domains in $\mathbf{R}^N$, $N \geq 3$, appropriate spaces, are

$$W^1(\Omega) = \{v \in \mathcal{D}'(\Omega) : (1+r^2)^{-1/2}v \in L^2(\Omega), \ \nabla v \in (L^2(\Omega))^N\}$$

with the norm

$$\mid v \mid_{W^1(\Omega)} = \left(\mid (1+r^2)^{-1/2}v \mid^2_{L^2(\Omega)} + \mid \nabla v \mid^2_{(L^2(\Omega))^N} \right)^{1/2}.$$

We remark that the space $H^1(\Omega)$ is continuously embedded in $W^1(\Omega)$, and the two spaces coincide for the bounded domains. We use $W^1_0(\Omega)$ to denote the closure of $\mathcal{D}(\Omega)$ in $W^1(\Omega)$.

Concerning the space of the traces of the functions in $W^1(\Omega)$, we notice that, the boundary Γ being bounded, these traces lie in $H^{1/2}(\Gamma)$.

Assuming that

$$(1+r^2)^{1/2}(1+\log\sqrt{1+r^2})f \in L^2(\Omega) \text{ if } N = 2,$$

$$(1+r^2)^{1/2}f \in L^2(\Omega) \text{ if } N \geq 3,$$

and using the spaces W^1 in place of the spaces H^1, we can rewrite the problems (367) and (368) and also similar equations for problems (329) and (338).

We can prove that, in the case of $a_0 = 0$ on Ω, the exterior problems have unique solutions in the spaces W^1 if $N \geq 3$. If $N = 2$, the Dirichlet problems have unique solutions in W^1, and the Neumann problems have unique solutions in $W^1/\mathbf{R}$. Also, we get the following continuous dependence on the data of the solution y of problem (319), (320)

$$|y|_{W^1(\omega)} \leq C\{|(1+r^2)^{1/2}(1+\log\sqrt{1+r^2})f|_{L^2(\omega)} + |g_\gamma|_{H^{1/2}(\gamma)}\} \quad \text{if} \;\; N=2,$$

and

$$|y|_{W^1(\omega)} \leq C\{|(1+r^2)^{1/2}f|_{L^2(\omega)} + |g_\gamma|_{H^{1/2}(\gamma)}\} \quad \text{if} \;\; N \geq 3.$$

For the problem (319), (321), we have

$$\inf_{s\in\mathbf{R}} |y+s|_{W^1(\omega)} \leq C\{|(1+r^2)^{1/2}(1+\log\sqrt{1+r^2})f|_{L^2(\omega)} + |h_\gamma|_{H^{-1/2}(\gamma)}\} \quad \text{if} \;\; N=2,$$

and

$$|y|_{W^1(\omega)} \leq C\{|(1+r^2)^{1/2}f|_{L^2(\omega)} + |h_\gamma|_{H^{-1/2}(\gamma)}\} \quad \text{if} \;\; N \geq 3.$$

Therefore, we can prove in a manner similar to the previous sections that when $a_0 = 0$ on Ω and $N \geq 3$, the solutions of the Dirichlet and Neumann problems in ω can be approximated with solutions of both the Dirichlet and the Neumann problems in Ω. Naturally, the controls are taken in the appropriate space (369) or (370). If $a_0 = 0$ on Ω and $N = 2$, the solutions of the Dirichlet problems in ω can be approximated with solutions of the Dirichlet problem in Ω. The Neumann problems do not have unique solutions.

Since $y(v)$ and g_γ lie in $H^{1/2}(\gamma)$ in the case of problem (319), (320), and $\frac{\partial y(v)}{\partial n_{A(\omega)}}$ and h_γ lie in $H^{-1/2}(\gamma)$ when we solve (319), (321), the natural choices for the space of observations are

$$\mathcal{H} = H^{1/2}(\gamma) \tag{371}$$

and

$$\mathcal{H} = H^{-1/2}(\gamma), \tag{372}$$

respectively. Even if the convergence is assured for these spaces, their norms are numerically estimated with much difficulty. However, noticing that the inclusions $H^{1/2}(\gamma) \subset L^2(\gamma) \subset H^{-1/2}(\gamma) \subset H^{-1}(\gamma)$ are continuous, we can take the spaces of observations, as in the case of the bounded domains, given in (330) and (332).

Numerical results. In the following, we succinctly describe the numerical examples we have performed. Details and the obtained numerical results are given in the paper [12].

Interior problems. *Example 1.* The first numerical test refers to the Dirichlet problem Numerical experiments were performed to find the solution of the Dirichlet problem

$$\begin{aligned} &-\Delta y = f \text{ in } \omega, \\ &y = g_\gamma \text{ on } \gamma, \end{aligned} \tag{373}$$

where $\omega \subset \mathbf{R}^2$ is either the interior of a square centered at the origin. The approximate solution of this problem is given by the solution of the Dirichlet problem

$$\begin{aligned} &-\Delta y(v) = f \text{ in } \Omega, \\ &y(v) = v \text{ on } \Gamma, \end{aligned} \tag{374}$$

in which the domain Ω is a disc centered at the origin which contains the square. The solutions of the homogeneous Dirichlet problems in Ω are found by the Poisson formula

$$y(v)(z) = \frac{1}{2\pi r} \int_{|\zeta|=r} v(\zeta) \frac{r^2 - |z|^2}{|z-\zeta|^2} dS_\zeta. \tag{375}$$

Example 2. This example concerns the Dirichlet problem

$$\begin{aligned} &\Delta y - \sigma^2 y = f \text{ in } \omega, \\ &y = g_\gamma \text{ on } \gamma, \end{aligned} \tag{376}$$

where $\omega \subset \mathbf{R}^2$ is bounded by the straight lines $x_1 = -\pi/2$, $x_1 = \pi/2$, and $x_2 = -1.5$ and the curve $y = 0.5 + \cos(x + \pi/2)$. We approximate the solution of this problem by a solution of the Dirichlet problem

$$\begin{aligned} &\Delta y(v) - \sigma^2 y(v) = f \text{ in } \Omega, \\ &y(v) = v \text{ on } \Gamma, \end{aligned} \tag{377}$$

in which the domain Ω is the disc centered at the origin which contains ω. The solution of (377) has been calculated by the discrete Fourier transform.

We approximate the functions f and v by the discrete Fourier transforms

$$
\begin{aligned}
f(r,\theta) &= \sum_{k=-n/2}^{n/2-1} f_k(r)e^{ik\theta}, \\
v(\theta) &= \sum_{k=-n/2}^{n/2-1} v_k e^{ik\theta}.
\end{aligned} \tag{378}
$$

Then the solution of problem (377),

$$
y(v) = y_f + y'(v), \tag{379}
$$

can also be written as a discrete Fourier transform

$$
\begin{aligned}
y_f(r,\theta) &= \sum_{k=-n/2}^{n/2-1} y_k(r)e^{ik\theta}, \\
y'(v)(r,\theta) &= \sum_{k=-n/2}^{n/2-1} y'_k(r)e^{ik\theta},
\end{aligned} \tag{380}
$$

where the Fourier coefficients $y_k(r)$ and $y'_k(r)$ are given by

$$
\begin{aligned}
y_k(r) &= -\int_0^r \rho K_k(\sigma r)I_k(\sigma\rho)f_k(\rho)d\rho - \int_r^R \rho I_k(\sigma r)K_k(\sigma\rho)f_k(\rho)d\rho \\
&\quad + \frac{I_k(\sigma r)}{I_k(\sigma R)}\int_0^R \rho K_k(\sigma R)I_k(\sigma\rho)f_k(\rho)d\rho, \\
y'_k(r) &= \frac{I_k(\sigma r)}{I_k(\sigma R)}v_k.
\end{aligned} \tag{381}
$$

Above, R is the radius of the disc, and I_k and K_k are the modified Bessel functions of the first and second kinds, respectively. We recall that $y'(v)$ and y_f in (379) are the solutions of problems (341) and (345), respectively. A fast algorithm is proposed in [13], which, using (381) and the fast Fourier transforms, evaluates y_f and $y'(v)$ in (380) at the nodes of a mesh on the disc Ω with n equidistant nodes in tangential direction and l equidistant nodes in the radial direction.

Exterior problems. *Example 3.* We solve the same problem as defined by (373) in Example 1 except that the domain ω is now the exterior of a square centered at the origin with sides parallel to the axes. For this problem, we consider exterior Dirichlet problem (374) with the embedding domain Ω as the exterior of a disc with its center at the origin included in ω.

Similar to Example 1, the solutions of the homogeneous Dirichlet problems in Ω are found by the Poisson formula

$$y(v)(z) = \frac{-1}{2\pi r} \int_{|\zeta|=r} v(\zeta) \frac{r^2 - |z|^2}{|z-\zeta|^2} dS_\zeta. \tag{382}$$

Example 4. We have solved the same problem as defined by (376) except that the domain ω now is the open complement of the domain bounded by the straight lines $x_1 = -\pi/2$, $x_1 = \pi/2$, and $x_2 = -1.5$ and the curve $y = 0.5 + \cos(x)$. For this problem, the embedding domain Ω is taken to be the exterior of a disc included in ω, and we consider the exterior Neumann problem

$$\begin{array}{l} \Delta y(v) - \sigma^2 y(v) = f \text{ in } \Omega, \\ \frac{\partial y(v)}{\partial n_A(\Omega)} = v \text{ on } \Gamma. \end{array} \tag{383}$$

where Γ is the inner boundary of the embedding domain Ω.

As before, functions f and v are approximated by the discrete Fourier transforms (378). Then the solution of problem (383) admits representation given by (379) and (380) except that the Fourier coefficients $y_k(r)$ and $y_k'(r)$ are now given by

$$\begin{aligned} y_k(r) = &- \int_R^r \rho K_k(\sigma r) I_k(\sigma\rho) f_k(\rho) d\rho - \int_r^\infty \rho I_k(\sigma r) K_k(\sigma\rho) f_k(\rho) d\rho \\ &- \frac{K_k(\sigma r)}{K_{k-1}(\sigma R) + K_{k+1}(\sigma R)} \int_R^\infty \rho [I_{k-1}(\sigma R) + I_{k+1}(\sigma R)] K_k(\sigma\rho) f_k(\rho) d\rho, \end{aligned} \tag{384}$$

$$y_k'(r) = \frac{K_k(\sigma r)}{K_{k-1}(\sigma R) + K_{k+1}(\sigma R)} \frac{2}{\sigma} v_k.$$

Above, R is the radius of the disc whose complement is the domain Ω, and I_k and K_k are the modified Bessel functions of first and second kinds, respectively. In order to compute the solution of problem (383) at mesh points of the domain Ω with n equidistant nodes in the tangential direction and l equidistant nodes in the radial direction, we use the algorithm proposed in

[13]. This algorithm uses (384) and the fast Fourier transforms to compute y_f and $y'(v)$ in (380). For numerical computations, the domain Ω is considered to be the annulus with the radii R and R_∞, where R_∞ is chosen very large so that its effect is minimal on the accuracy of the solutions.

3.2.2 Fast algorithm (paper [13])

In [13], analysis-based fast algorithms to solve inhomogeneous elliptic equations of three different types in three different two-dimensional domains are derived. Dirichlet, Neumann and mixed boundary value problems are treated in all these cases. Three different domains considered are: (i) interior of a circle, (ii) exterior of a circle, and (iii) circular annulus. Three different types of elliptic problems considered are: (i) Poisson equation, (ii) Helmholtz equation (oscillatory case), and (iii) Helmholtz equation (monotone case). These algorithms are derived from an exact formula for the solution of a large class of elliptic equations (the coefficients of the equation do not depend on the polar angle when we use the polar coordinates) based on Fourier series expansion and one-dimensional ordinary differential equation. The performance of these algorithms are illustrated on several of these problems. Numerical results are presented.

We consider domain $\Omega \subset \mathbf{R}^2$ which can be an open disc, or an open annulus or the complement of a closed disc, all centered at the origin. The boundary of the domain is denoted by $\partial\Omega$ and the two radii limiting Ω are denoted by $R_1 < R_2$ where R_1 can be zero and R_2 can be infinity.

Let L be an elliptic operator and its coefficients have sufficient regularity so that the coefficients of its adjoint L^* are continuous. For a Dirichlet problem associated with the equation

$$Lu = f \text{ in } \Omega, \tag{385}$$

and the boundary condition

$$u = g \text{ on } \partial\Omega, \tag{386}$$

we assume that f is continuous in $\bar{\Omega}$ and g is continuous on $\partial\Omega$. Therefore the problem has a solution in the classical sense, i.e. $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$, satisfies the equation (385) pointwise, by continuity the boundary conditions on $\partial\Omega$, and also the conditions at infinity for exterior problems.

The data f and h of a Neumann problem with equation (385) and boundary condition

$$\partial_n u = h \text{ on } \partial\Omega, \tag{387}$$

are assumed to be also continuous on $\bar{\Omega}$ and $\partial\Omega$, respectively, and to satisfy appropriate conditions for the existence of classical solutions. Evidently, for the Laplace operator the uniqueness is up to an additive constant and we must have

$$\int_\Omega f = \int_{\partial\Omega} h. \tag{388}$$

Also, for both problems, Dirichlet and Neumann, we assume that f has a compact support for the exterior problems.

We consider now the equation

$$Lu(r,\theta) = f(r,\theta),\ R_1 < r < R_2,\ 0 \le \theta \le 2\pi, \tag{389}$$

where L is written in polar coordinates. We assume in the following that the coefficients of the operator L using the polar coordinates are independent of θ. With this assumption, for each integer n we can define an ordinary differential operator of second order L_n whose coefficients do not depend on θ satisfying

$$L_n v(r) = e^{-in\theta} L(v(r)e^{in\theta}), \tag{390}$$

for any $v(r) \in C^2(R_1, R_2)$. Now, writing $L = L_r + a(r)\partial_\theta + b(r)\partial_r\partial_\theta + c(r)\partial_\theta^2$, where the operator L_r depends only on r, and using integration by parts we can verify that the operator L_n introduced in (390) satisfies the equation

$$L_n u_n(r) \equiv L_n \int_0^{2\pi} u(r,\theta)e^{-in\theta}d\theta = \int_0^{2\pi} (Lu(r,\theta))e^{-in\theta}d\theta \equiv (Lu(r,\theta))_n, \tag{391}$$

for any $u(r,\theta) \in C^2(\Omega)$, and for any integer n and $R_1 < r < R_2$.

Now, for each $r \in (R_1, R_2)$, we write $f(r,\theta)$ and $u(r,\theta)$ as Fourier series on $[0, 2\pi]$

$$f(r,\theta) = \sum_{n=-\infty}^{\infty} f_n(r)e^{in\theta}, \tag{392}$$

and

$$u(r,\theta) = \sum_{n=-\infty}^{\infty} u_n(r)e^{in\theta}, \tag{393}$$

respectively. Applying equation (391) to the solution $u(r,\theta)$ of equation (389) and using the above Fourier series expansions we get

$$L_n u_n(r) = f_n(r), \tag{394}$$

for any integer number n. Thus, the Fourier coefficients of u satisfy equation (394). We state this result in a theorem.

Theorem 3.25 *Assume that the coefficients of the operator L in its polar form do not depend on the angle θ. If a solution u of equation (389) is written as the Fourier expansion (393), then its coefficients $u_n(r)$ are solutions of the equations*

$$L_n u_n(r) = f_n(r), \quad R_1 < r < R_2, \tag{395}$$

where operator L_n is given by (390) and f_n are the Fourier coefficients of the function f given by (392).

Now, let us assume that the boundary data g and h are written as Fourier series

$$g(r,\theta) = \sum_{n=-\infty}^{\infty} g_n(r) e^{in\theta}, \quad (r,\theta) \in \partial\Omega, \tag{396}$$

and

$$h(r,\theta) = \sum_{n=-\infty}^{\infty} h_n(r) e^{in\theta}, \quad (r,\theta) \in \partial\Omega, \tag{397}$$

respectively. Since the boundaries of the domains are given by either one or two circles, we see that the conditions (398) and (399) below follow respectively from boundary conditions (386) and (387) for any integer number n.

$$u_n(R_1) = g_n(R_1), \quad u_n(R_2) = g_n(R_2), \tag{398}$$

and

$$d_r u_n(R_1) = -h_n(R_1) \equiv -h_n^{(1)}, \quad d_r u_n(R_2) = h_n(R_2) \equiv h_n^{(2)}. \tag{399}$$

We have formally included in (398) and (399) the conditions at origin and infinity that the solution when written in polar coordinates must satisfy. Therefore, when Ω is a disc, $u_n(R_1) = g_n(R_1)$ or $d_r u_n(R_1) = h_n(R_1)$ means "$u_n(r)$ has a finite limit when $r \to 0$ for each n". Also, when Ω is the complement of a closed disc, $u_n(R_2) = g_n(R_2)$ or $d_r u_n(R_2) = h_n(R_2)$ means "$u_n(r)$ and/or $d_r u_n(r)$ satisfy appropriate conditions at infinity which arise from the conditions at infinity of the problem in Ω". Thus, the Dirichlet problem defined by (385) and (386) is reduced to one-dimensional problems given by (395) and (398), while the Neumann problem given by (385) and (387) is reduced to one-dimensional problems given by (395) and (399).

We look for the solution of one-dimensional problems associated with (395), (398) and (399) in two steps. First, we look for a solution $v_n(r)$ satisfying only equation (395),

$$L_n v_n(r) = f_n(r), \quad R_1 < r < R_2, \tag{400}$$

of the form

$$v_n(r) = \int_{R_1}^{R_2} f_n(\rho) V_n(\rho, r) d\rho, \tag{401}$$

where $V_n(\rho, r)$ satisfies in the sense of distributions the equation

$$L_n^* V_n(\rho, r) = \delta(\rho - r), \quad R_1 < \rho < R_2, \tag{402}$$

where $\delta(\rho - r)$ is the Dirac delta function. Writing

$$L_n v_n(r) \equiv \alpha_n(r) d_r^2 v_n(r) + \beta_n(r) d_r v_n(r) + \gamma_n(r) v_n(r), \quad R_1 < r < R_2, \tag{403}$$

let us assume that the homogeneous equation

$$L_n^* v_n^*(r) \equiv d_r^2(\alpha_n(r) v_n^*(r)) - d_r(\beta_n(r) v_n^*(r)) + \gamma_n(r) v_n^*(r) = 0, \quad R_1 < r < R_2, \tag{404}$$

has two linearly independent solutions, $v_{n,1}^*(r)$ and $v_{n,2}^*(r)$. In the above, L_n^* is the adjoint of the operator L_n. We seek solutions of equation (402) in the form

$$V_n(\rho, r) = \begin{cases} a_n(r) v_{n,1}^*(\rho), & \text{for } R_1 < \rho < r, \\ b_n(r) v_{n,2}^*(\rho), & \text{for } r < \rho < R_2. \end{cases} \tag{405}$$

Now the functions $a_n(r)$ and $b_n(r)$ are to be found from the conditions that $V_n(\rho, r)$ is continuous at $\rho = r$,

$$a_n(r) v_{n,1}^*(r) = b_n(r) v_{n,2}^*(r), \tag{406}$$

and the jump of its first derivative $\partial_\rho V_n(\rho, r)$ at $\rho = r$ satisfies

$$\alpha_n(r)[b_n(r) d_r v_{n,2}^*(r) - a_n(r) d_r v_{n,1}^*(r)] = 1, \tag{407}$$

where it is assumed that $\alpha_n(r) \neq 0$ for any $R_1 < r < R_2$. In this way, we find

$$v_n(r) = \frac{v^*_{n,2}(r)}{\alpha_n(r)D^*_n(r)} \int_{R_1}^{r} f_n(\rho)v^*_{n,1}(\rho)d\rho + \frac{v^*_{n,1}(r)}{\alpha_n(r)D^*_n(r)} \int_{r}^{R_2} f_n(\rho)v^*_{n,2}(\rho)d\rho, \tag{408}$$

is a solution of equation (400),where

$$D^*_n(r) = v^*_{n,1}(r)d_r v^*_{n,2}(r) - d_r v^*_{n,1}(r)v^*_{n,2}(r). \tag{409}$$

Next, writing

$$D_n(r) = v_{n,1}(r)d_r v_{n,2}(r) - d_r v_{n,1}(r)v_{n,2}(r), \tag{410}$$

where $v_{n,1}(r)$ and $v_{n,2}(r)$ are two linearly independent solutions of the homogeneous form of equation (400), we can prove

Proposition 3.18 *If the coefficients of the operator L_n given by (403) satisfy $\alpha_n(r) \neq 0$, $r \in (R_1, R_2)$, $\alpha_n(r) \in C^2(R_1, R_2)$, $\beta_n(r) \in C^1(R_1, R_2)$ and $\gamma_n(r) \in C^0(R_1, R_2)$, then equations*

$$v_{n,1}(r) = \frac{v^*_{n,2}(r)}{\alpha_n(r)D^*_n(r)} \text{ and } v_{n,2}(r) = \frac{v^*_{n,1}(r)}{\alpha_n(r)D^*_n(r)}, \tag{411}$$

and

$$v^*_{n,1}(r) = \frac{-v_{n,2}(r)}{\alpha_n(r)D_n(r)} \text{ and } v^*_{n,2}(r) = \frac{-v_{n,1}(r)}{\alpha_n(r)D_n(r)}, \tag{412}$$

where $D^*_n(r)$ and $D_n(r)$ are given in (409) and (410), respectively, are reciprocal transformations and establish bijective correspondences between pairs of linearly independent solutions of the homogeneous form of equation (400) and of equation (404).

In view of this proposition and (408) we get

Theorem 3.26 *If the coefficients of the operator L_n given by (403) satisfy $\alpha_n(r) \neq 0$, $r \in (R_1, R_2)$, $\alpha_n(r) \in C^2(R_1, R_2)$, $\beta_n(r) \in C^1(R_1, R_2)$ and $\gamma_n(r) \in C^0(R_1, R_2)$, and if $v_{n,1}(r)$ and $v_{n,2}(r)$ are two linearly independent solutions of the homogeneous form of equation (400), then*

$$v_n(r) = -v_{n,1}(r) \int_{R_1}^{r} \frac{v_{n,2}(\rho)}{\alpha_n(\rho)D_n(\rho)} f_n(\rho)d\rho - v_{n,2}(r) \int_{r}^{R_2} \frac{v_{n,1}(\rho)}{\alpha_n(\rho)D_n(\rho)} f_n(\rho)d\rho, \tag{413}$$

is a solution of equation (400) provided that the integral is convergent.

In the following, three different domains are considered:

(i) interior of a circle,

(ii) exterior of a circle, and

(iii) circular annulus.

Also, three different types of elliptic problems are considered:

(i) Poisson equation, $Lu \equiv \Delta u = f$, with the two linearly independent solutions

$$\begin{aligned} &v_{0,1}(r) = 1, \quad v_{0,2}(r) = \log(r), \\ &v_{n,1}(r) = r^{|n|}, \quad v_{n,2}(r) = r^{-|n|} \qquad \text{for } n \neq 0. \end{aligned} \tag{414}$$

(ii) Helmholtz equation (oscillatory case), $Lu \equiv \Delta u + k^2 u = f$, with the linearly independent solutions

$$v_{n,1}(r) = J_n(kr) \text{ and } v_{n,2}(r) = Y_n(kr) \quad \text{for any } n, \tag{415}$$

where $J_n(r)$ and $Y_n(r)$ are Bessel functions of the first and the second kind of order n, respectively.

(iii) Helmholtz equation (monotone case), $Lu \equiv \Delta u - k^2 u = f$, with the two linearly independent solutions

$$v_{n,1}(r) = I_n(kr) \text{ and } v_{n,2}(r) = K_n(kr) \quad \text{for any } n, \tag{416}$$

where $I_n(r)$ and $K_n(r)$ are modified Bessel functions of the first and the second kind of order n respectively.

In all these nine cases, the solutions of Dirichlet and Neumann value problems are explicitly written. Based on radial and tangential discretizations of the domain, detailed fast algorithms are given in the paper. Finally, numerical results are presented.

3.2.3 Distributed optimal control associated domain embedding method (papers [14] and [18])

In papers [14] and [18] the domain embedding method is associated with a distributed control to solve boundary value problems.

Distributed optimal control for periodic solutions on a rectangle. In [14], the method is based on formulating the problem as an optimal distributed control problem inside a rectangle in which the arbitrary domain is embedded. A periodic solution of the equation under consideration is constructed by making use of Fourier series. Numerical examples are given for the solution of a Dirichlet problem in a hexagonal domain. The numerical tests show a high accuracy of the proposed algorithm and the computed solutions are in very good agreement with the exact solutions.

We look for the solution of problem

$$\begin{aligned} &\Delta u - \sigma u = f \quad && \text{in } \Omega, \\ &u = g && \text{on } \Gamma, \end{aligned} \tag{417}$$

where σ is a positive constant and Γ is the boundary of the domain $\Omega \subset \mathbf{R}^2$. The domain Ω is embedded in a rectangle $D = (0, a) \times (0, b)$ on which a regular rectangular mesh is considered. By considering extensions $\tilde{f}$ of f in D, $\tilde{f}_{|\Omega} = f$, we look for a periodic solution $\tilde{u}$ of the equation

$$\Delta \tilde{u} - \sigma \tilde{u} = \tilde{f} \quad \text{in } D \tag{418}$$

such that $\tilde{u}_{|\Omega} = u$.

If we know $\tilde{f}$ and write it as a double Fourier series

$$\tilde{f}(x, y) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \hat{f}_{mn} e^{i(mX+nY)}, \tag{419}$$

where $X = 2\pi x/a$, $Y = 2\pi y/b$, then a periodic solution $\tilde{u}$ of (418) can be found from its Fourier series,

$$\tilde{u}(x, y) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \hat{u}_{mn} e^{i(mX+nY)}, \tag{420}$$

with $\hat{u}_{mn}$ now given by

$$\hat{u}_{mn} = -\frac{\hat{f}_{mn}}{\left(\frac{4\pi^2}{a^2} m^2 + \frac{4\pi^2}{b^2} n^2 + \sigma\right)}. \tag{421}$$

If we consider a rectangular mesh on $[0, a] \times [0, b]$ with $2M + 1$ and $2N + 1$ segments on $[0, a]$ and $[0, b]$ respectively, then the discrete Fourier transforms approximating $\tilde{f}$ and $\tilde{u}$ are given by

$$\tilde{f}_{ij} = \sum_{m=-M}^{M} \sum_{n=-N}^{N} \hat{f}_{mn} e^{i(mX_i + nY_j)} \tag{422}$$

and

$$\tilde{u}_{ij} = \sum_{m=-M}^{M} \sum_{n=-N}^{N} \hat{u}_{mn} e^{i(mX_i + nY_j)}, \tag{423}$$

respectively, where $\tilde{f}_{ij} = \tilde{f}(x_i, y_j)$, $\tilde{u}_{ij} = \tilde{u}(x_i, y_j)$, $X_i = 2\pi x_i/a$, and $Y_j = 2\pi y_j/b$. It is well known that the inverse discrete Fourier transforms of (422) and (423) are written as

$$\hat{f}_{mn} = \frac{1}{(2M+1)(2N+1)} \sum_{i=0}^{2M} \sum_{j=0}^{2N} \tilde{f}_{ij} e^{-i(mX_i + nY_j)} \tag{424}$$

and

$$\hat{u}_{mn} = \frac{1}{(2M+1)(2N+1)} \sum_{i=0}^{2M} \sum_{j=0}^{2N} \tilde{u}_{ij} e^{-i(mX_i + nY_j)}, \tag{425}$$

respectively. Therefore, for arbitrary values of $\tilde{f}$ at the mesh nodes in $\bar{D}\backslash\bar{\Omega}$, we can find a Fourier approximation of the periodic solution of equation (418) using (391), (421) and (423).

If the trace on Γ of $\tilde{u}$ written in (423) is a good approximation for the boundary data g, then $\tilde{u}$ is also a good approximation in Ω for the solution u of problem (417). Construction of such a $\tilde{u}$ leads to an optimization problem: we solve for $\tilde{f}$ the following optimal distributed control problem

$$\min_{\tilde{h}|\Omega = f} J(\tilde{h}), \quad J(\tilde{h}) = \frac{1}{2} |\tilde{u}(\tilde{h}) - g|^2_{L^2(\Gamma)}, \tag{426}$$

where $\tilde{u}(\tilde{h})$ is the periodic solution of problem (418) corresponding to the right hand side $\tilde{h}$. In this way, the condition

$$\tilde{u} = g \quad \text{on } \Gamma,$$

is satisfied approximately by $\tilde{u}(\tilde{f})$. Since J depends only on the values of $\tilde{h}$ at the mesh nodes in $\bar{D}\backslash\bar{\Omega}$, and if we write

$$\tilde{h} = f + h$$

where f and h are assumed to be extended with zero in $\bar{D}\backslash\bar{\Omega}$ and $\bar{\Omega}$, respectively, then problem (426) can be written as

$$\min_h J(h), \quad J(h) = \frac{1}{2}|\tilde{u}(f+h) - g|^2_{L^2(\Gamma)}. \tag{427}$$

Since the Gâteaux derivative of $J(h)$ is

$$J'(h)(e) = \int_\Gamma [\tilde{u}(h) + \tilde{u}(f) - g]\tilde{u}(e),$$

the minimization problem (427) is equivalent with the linear algebraic system

$$\int_\Gamma \tilde{u}(h)\tilde{u}(e) = \int_\Gamma [g - \tilde{u}(f)]\tilde{u}(e), \tag{428}$$

for all e vanishing at the mesh nodes in $\bar{\Omega}$.

In order to write the matrix and the right hand side of the linear system (428), we denote by e_{ij} the discrete Fourier transform of the function which takes the value 1 at the mesh node (x_i, y_j) and vanishes at the other nodes. Then, the linear system (428) can be written as

$$\sum_{(x_k,y_l)\in\bar{D}\backslash\bar{\Omega}} h_{kl} \textstyle\int_\Gamma \tilde{u}(e_{kl})\tilde{u}(e_{ij}) = \int_\Gamma [g - \tilde{u}(f)]\tilde{u}(e_{ij}), \quad \text{for } (x_i, y_j) \in \bar{D}\backslash\bar{\Omega}, \tag{429}$$

where $h_{kl} = h(x_k, y_l)$. Denoting the Fourier coefficients of e_{ij} by $\hat{e}_{ij,mn}$, $-M \le m \le M$, $-N \le n \le N$, we get from (424),

$$\hat{e}_{ij,mn} = \frac{e^{-i(mX_i+nY_j)}}{(2M+1)(2N+1)}.$$

Then, from (421), the Fourier coefficients $\hat{u}_{ij,mn}$ of $\tilde{u}(e_{ij})$ are given by

$$\hat{u}_{ij,mn} = \frac{-e^{-i(mX_i+nY_j)}}{(2M+1)(2N+1)} \frac{1}{\left(\frac{4\pi^2}{a^2}m^2 + \frac{4\pi^2}{b^2}n^2 + \sigma\right)}, \tag{430}$$

and the value of $\tilde{u}(e_{ij})$ at the mesh node (x_k, y_l) can be approximated by

$$\tilde{u}(e_{ij})_{kl} = \sum_{m=-M}^{M} \sum_{n=-N}^{N} \hat{u}_{ij,mn} e^{i(mX_k+nY_l)}. \tag{431}$$

Since the mesh on D has been taken regular in the directions x and y, we can use a double fast Fourier transform for the calculation of the right hand side in (422)–(425).

Below, we outline the steps of the proposed algorithm in which the embedding method which uses the Fourier approximations on a rectangle is associated with an optimal distributed control.

Numerical algorithm

1. We extend f with zero at the mesh nodes in $\bar{D}\backslash\bar{\Omega}$ and calculate the Fourier coefficients $\hat{f}_{mn}$ given in (391).

2. Using (421) we get the Fourier coefficients $\hat{u}_{mn}$, and then using (423) we calculate $\tilde{u}(f)$ at the mesh nodes of $\bar{D}$.

3. Using (401), we calculate the coefficients $\hat{u}_{ij,mn}$ for the mesh nodes $(x_i, y_j) \in \bar{D}\backslash\bar{\Omega}$, and then using (402) we calculate the values of $\tilde{u}(e_{ij})$ at the mesh nodes of $\bar{D}$.

4. Using the computed values of $\tilde{u}(f)$ and $\tilde{u}(e_{ij})$ at the mesh nodes of $\bar{D}$, we calculate, by interpolation, the values of $\tilde{u}(f)$ and $\tilde{u}(e_{ij})$ at the mesh points of Γ, which are subsequently used in the numerical integration of the integrals appearing in the algebraic system (399). These boundary integrals have been calculated by the trapezoidal rule.

5. Using the solution of algebraic system (399), which gives the extension $\tilde{f}$ of f in $\bar{D}\backslash\bar{\Omega}$, and the values of f given in $\bar{\Omega}$, we get the values of $\tilde{u}$ at the mesh nodes of Ω from (391), (421) and (423) .

Distributed optimal control associated with a fast algorithm. In [18], a domain embedding method is proposed to solve second order elliptic problems in arbitrary two-dimensional domains. The method is based on formulating the problem as an optimal distributed control problem inside a disc in which the arbitrary domain is embedded. The optimal distributed control problem inside the disc is solved by the fast algorithm given in [13].

We consider the following Dirichlet problem associated with the Poisson equation

$$\begin{aligned} \Delta u = f \quad & \text{in } \Omega, \\ u = g \quad & \text{on } \partial\Omega, \end{aligned} \tag{432}$$

where Ω is a bounded domain, not necessarily simply connected, in $\mathbf{R}^2$. Embedding Ω within a disc D and using an optimal distributed control, we look for an extension $\tilde{f}$ of f from Ω to D, such that $\tilde{f} = f$ in Ω and the trace on $\partial\Omega$ of the solution $\tilde{u}$ of the Dirichlet problem in D,

$$\begin{array}{ll} \Delta\tilde{u} = \tilde{f} & \text{in } D, \\ \tilde{u} = 0 & \text{on } \partial D, \end{array} \tag{433}$$

optimally approximates the given function g on $\partial\Omega$. In this way, the restriction of $\tilde{u}$ to Ω optimally approximates the solution u of problem (432). The zero boundary condition (433) is a natural choice for this Dirichlet problem.

Using the polar coordinates, we assume that for $0 < r < R$ the function $\tilde{f}$ can be written as a Fourier series

$$\tilde{f}(re^{i\theta}) = \sum_{n=-\infty}^{\infty} f_n(r)e^{in\theta}. \tag{434}$$

Then the Fourier coefficients $u_n(re^{i\theta})$, $-\infty < n < \infty$, of the solution $\tilde{u}(re^{i\theta})$ of equation (392) can be written as (see [13])

$$\begin{array}{rl} u_0(r) = & \int_0^r \rho\log(r)f_0(\rho)d\rho + \int_r^R \rho\log(\rho)f_0(\rho)d\rho - \int_0^R \rho\log(R)f_0(\rho)d\rho, \\ & \text{for } n = 0, \\ u_n(r) = & -\int_0^r \frac{\rho}{2|n|}(\frac{\rho}{r})^{|n|}f_n(\rho)d\rho - \int_r^R \frac{\rho}{2|n|}(\frac{r}{\rho})^{|n|}f_n(\rho)d\rho + \\ & (\frac{r}{R})^{|n|}\int_0^R \frac{\rho}{2|n|}(\frac{\rho}{R})^{|n|}f_n(\rho)d\rho, \text{for } n \neq 0. \end{array} \tag{435}$$

The numerical computation of the integrals in (435) assumes the approximation of the Fourier coefficients $f_n(r)$ by some polynomial functions between two consecutive nodes of the discretization $0 = r_1 < r_2 < \cdots < r_M = R$. We consider $f_n(r)$ to be some continuous functions which are linear between two consecutive nodes. Consequently, denoting such a function by $\varphi_m(r)$, $m = 1, \cdots, M$, which takes the value 1 at r_m and 0 at the other points of the discretization according to the following formulas

$$\varphi_m(r) = \begin{cases} \frac{r-r_{m-1}}{r_m-r_{m-1}}, & r_{m-1} \leq r \leq r_m, \\ \frac{r_{m+1}-r}{r_{m+1}-r_m}, & r_m \leq r \leq r_{m+1}, \end{cases} \tag{436}$$

we can write

$$f_n(r) = \sum_{m=1}^{M} f_n(r_m)\varphi_m(r). \tag{437}$$

With these approximations of the Fourier coefficients and taking N terms in the Fourier series, the function $\tilde{f}$ is approximated by

$$\tilde{f}(re^{i\theta}) = \sum_{n=-N/2}^{N/2-1} \sum_{m=1}^{M} f_n(r_m)\varphi_m(r)e^{in\theta}. \tag{438}$$

We see that $\tilde{f}(re^{i\theta})$ is also linearly approximated between two consecutive radial nodes, and in the following we implicitly assume that $f_n(r)$ and $\tilde{f}(re^{i\theta})$ are of the form (437) and (438), respectively. For a given r_i, $i = 1, \cdots, M$, from (438) we have

$$\tilde{f}(r_i e^{i\theta}) = \sum_{n=-N/2}^{N/2-1} f_n(r_i)e^{in\theta}, \tag{439}$$

and therefore,

$$f_n(r_i) = \frac{1}{N}\sum_{j=0}^{N-1} \tilde{f}(r_i e^{i\theta_j})e^{-in\theta_j}, \tag{440}$$

where $\theta_j = \frac{2\pi j}{N}$. Substituting (440) into (438) we get

$$\tilde{f}(re^{i\theta}) = \frac{1}{N} \sum_{n=-N/2}^{N/2-1} \sum_{i=1}^{M} \sum_{j=0}^{N-1} \tilde{f}(r_i e^{i\theta_j})\varphi_i(r)e^{in(\theta-\theta_j)}, \tag{441}$$

and writing

$$\tilde{f}(r_i e^{i\theta_j}) = \begin{cases} f(r_i e^{i\theta_j}) & \text{if } r_i e^{i\theta_j} \in \bar{\Omega}, \\ h(r_i e^{i\theta_j}) & \text{if } r_i e^{i\theta_j} \in \bar{D}\backslash\bar{\Omega}, \end{cases} \tag{442}$$

we have from (441)

$$\tilde{f}(re^{i\theta}) = f(re^{i\theta}) + h(re^{i\theta}), \tag{443}$$

where

$$f(re^{i\theta}) = \frac{1}{N} \sum_{n=-N/2}^{N/2-1} \sum_{r_i e^{i\theta_j} \in \bar{\Omega}} f(r_i e^{i\theta_j})\varphi_i(r)e^{in(\theta-\theta_j)}, \tag{444}$$

and

$$h(re^{i\theta}) = \frac{1}{N} \sum_{n=-N/2}^{N/2-1} \sum_{r_i e^{i\theta_j} \in \bar{D}\backslash\bar{\Omega}} h(r_i e^{i\theta_j})\varphi_i(r)e^{in(\theta-\theta_j)}. \tag{445}$$

In the embedding method, using the optimal distributed control and the above approximation for $\tilde{f}$, we look for the extension $h(re^{i\theta})$ of $\tilde{f}$, in particular for the values $h(r_ie^{i\theta_j})$, $r_ie^{i\theta_j} \in \bar{D}\backslash\bar{\Omega}$, such that

$$J(h) = \min_{\chi} J(\chi),\ J(\chi) = \frac{1}{2}\int_{\partial\Omega} [\tilde{u}(f+\chi) - g]^2, \tag{446}$$

where the function χ is of the form

$$\chi(re^{i\theta}) = \frac{1}{N}\sum_{n=-N/2}^{N/2-1} \sum_{r_ie^{i\theta_j} \in \bar{D}\backslash\bar{\Omega}} \chi_{ij}\varphi_i(r)e^{in(\theta-\theta_j)}, \tag{447}$$

χ_{ij} being some real values, and $\tilde{u}(f+\chi)$ is the solution of the problem (433) corresponding to $\tilde{f} = f + \chi$. Since $J(\chi)$ is a differentiable convex function, its minimum is the solution of the following equation

$$J'(h)(\chi) \equiv \int_{\partial\Omega} [\tilde{u}(f+h) - g]\tilde{u}(\chi) = 0, \quad \text{for any } \chi, \tag{448}$$

where $J'(h)(\chi)$ is the Gâteaux derivative of J at h in χ direction. Since the solution $\tilde{u}$ of equation (433) depends linearly on the nonhomogeneous term $\tilde{f}$, and noticing that the functions belong to the finite linear space generated by the functions

$$\phi_{ij}(re^{i\theta}) = \frac{1}{N}\sum_{n=-N/2}^{N/2-1} \varphi_i(r)e^{in(\theta-\theta_j)},\ 1 \le i \le M,\ 0 \le j \le N-1, \tag{449}$$

the equation (448) can be written as

$$\begin{aligned} \sum_{r_ie^{i\theta_j} \in \bar{D}\backslash\bar{\Omega}} h_{ij} \int_{\partial\Omega} \tilde{u}(\phi_{ij}(re^{i\theta}))\tilde{u}(\phi_{kl}(re^{i\theta})) &= \int_{\partial\Omega} g\tilde{u}(\phi_{kl}(re^{i\theta})) \\ - \sum_{r_ie^{i\theta_j} \in \bar{\Omega}} f_{ij} \int_{\partial\Omega} \tilde{u}(\phi_{ij}(re^{i\theta}))\tilde{u}(\phi_{kl}(re^{i\theta})), &\quad \text{for any } r_ke^{i\theta_l} \in \bar{D}\backslash\bar{\Omega}, \end{aligned} \tag{450}$$

where we have used

$$h_{ij} = h(r_ie^{i\theta_j}) \text{ and } f_{ij} = f(r_ie^{i\theta_j}).$$

Taking into account the particular form of $\phi_{ij}(re^{i\theta})$ as a nonhomogeneous term of the equation (433), we can apply directly (435) to find $\tilde{u}(\phi_{ij})$. First, we see that in equations (435) the Fourier coefficients $u_n(r)$ of the solution $\tilde{u}(re^{i\theta})$ are some linear functions of the Fourier coefficients $f_n(r)$ of the nonhomogeneous term $\tilde{f}(re^{i\theta})$. In order to specify this linear dependence, we write $(u_n(f_n))(r)$ for $u_n(r)$ which is useful for our purposes below. The Fourier coefficients of the functions $\phi_{ij}(re^{i\theta})$ can be written as

$$\phi_{ij,n}(r) = \frac{1}{N}\varphi_i(r)e^{-in\theta_j},$$

and therefore, the Fourier coefficients of $(\tilde{u}(\phi_{ij}))(re^{i\theta})$ are given by

$$(u_n(\phi_{ij}))(re^{i\theta}) = \frac{1}{N}(u_n(\varphi_i))(r)e^{-in\theta_j}. \tag{451}$$

Therefore, we can write

$$(\tilde{u}(\phi_{ij}))(re^{i\theta}) = \frac{1}{N}\sum_{n=-N/2}^{N/2-1}(u_n(\varphi_i))(r)e^{in(\theta-\theta_j)}. \tag{452}$$

To conclude, we summarize the steps involved in the above described embedding method using the optimal distributed control as follows.

1. In order to numerically evaluate the curvilinear integrals in (450), we calculate the values of the function $(\tilde{u}(\phi_{ij}))(re^{i\theta})$ (given by (414)) at some points $re^{i\theta} \in \partial\Omega$ for each mesh point $r_ie^{i\theta_j}$.

2. Using the above calculated values, we evaluate the matrix and the right hand side of the algebraic linear system (450) by numerical integration. By solving this linear system, we find the values $h_{ij} = h(r_ie^{i\theta_j}) = \tilde{f}(r_ie^{i\theta_j})$ at the points $r_ie^{i\theta_j} \in \bar{D}\backslash\bar{\Omega}$.

3. Using the inverse fast Fourier transform we calculate the Fourier coefficients $f_n(r_i)$ of $\tilde{f}(r_ie^{i\theta})$, for all r_i, $i = 1, \cdots, M$.

4. Using Algorithm 1.1, we find the Fourier coefficients $u_n(r_i)$ of $\tilde{u}(r_ie^{i\theta})$, for all r_i, $i = 1, \cdots, M$.

5. Finally, we determine the values $\tilde{u}(r_ie^{i\theta_j})$ of the solution $\tilde{u}$ at the mesh points $r_ie^{i\theta_j}$, using the fast Fourier transform to calculate the values of $\tilde{u}(r_ie^{i\theta})$, $i = 1, \cdots, M$, at $\theta_1, \cdots, \theta_N$.

3.3 Financial problem: valuation of the American options (papers [9], [10] and [21])

Papers [9], [10] and [21] deal with theoretical study of existence and uniqueness as well as the numerical computation of the solution for the problem of the valuation of American options.

The value of the live American call is governed by the following parabolic equation

$$\frac{\partial w}{\partial t} - \frac{\sigma^2}{2} x^2 \frac{\partial^2 w}{\partial x^2} - (r-\delta) x \frac{\partial w}{\partial x} + rw = 0, \tag{453}$$

for $0 < x < s(t)$ and $0 < t \leq T$. The initial and boundary conditions for the equation (453) are given by

$$w(x,0) = w_0(x), \qquad 0 \leq x < s(0), \tag{454}$$

and

$$\begin{aligned} w(0,t) &= 0, \\ w(s(t),t) &= w_0(s(t)), \\ \frac{\partial w}{\partial x}(s(t),t) &= 1, \qquad\qquad t \in (0,T]. \end{aligned} \tag{455}$$

Above, x is the asset price, t is the time to expiration of the option, $w(x,t)$ denote the value of the live American call, r is the interest rate, δ is the rate of the dividends, σ is the volatility parameter, T is the time at which the option expires,

$$D = \{(x,t);\ 0 < x < s(t), 0 < t \leq T\}, \tag{456}$$

where $x = s(t)$ for $0 < t \leq T$ is the optimal exercise boundary representing the asset price above which American calls are exercised optimally (which is unknown), $w_0(x) = \max(x - Z, 0)$ is the "reward" function, where Z is the exercise price. The classical solution of (453)–(455) has the following properties:

(**P1**) *There exist positive numbers s_0 and S_0 such that*

$$s_0 \leq s(t) \leq S_0, \qquad \forall t \in (0,T],$$

(**P2**) *The solution w is non-decreasing in time; i.e., one has $\frac{\partial w}{\partial t} \geq 0$ for all $t \in (0,T]$, D given in (456).*

(**P3**) *The solution $w = w(x,t)$ of (453)–(455) is larger than w_0 in the continuation region D given in (456).*

In particular, it is proved in [9] that if $w(x,t)$ is a solution of (453)–(455), then $Z < s(t) < S_0$ for any $t \in (0,T]$, where

$$S_0 = \frac{\lambda Z}{\lambda - 1}, \tag{457}$$

where

$$\lambda = \frac{\sigma^2/2 - r + \delta + \sqrt{(\sigma^2/2 - r + \delta)^2 + 2\sigma^2 r}}{\sigma^2}. \tag{458}$$

Let $S > S_0$ and $Q_T = (0,S) \times (0,T)$ and

$$W(0,T) = \{v : v \in L^2(0,T; H_0^1(0,S)),\ x^{-1}\frac{\partial v}{\partial t} \in L^2(0,T; L^2(0,S))\}.$$

We extend w with w_0 in $Q_T - D$, and consider the difference

$$u(x,t) = w(x,t) - w_0(x),\ (x,t) \in Q_T. \tag{459}$$

Then, the weak form of problem (453)–(455) can be written as

Find $u \in W(0,T)$ satisfying almost for all $t \in (0,T)$

$$(x^{-1}\frac{\partial v}{\partial t}, x^{-1}v) + \mathcal{R}(u,v) + (qH(u),v) = \frac{\sigma^2}{2}v(Z), \tag{460}$$

for all $v \in H_0^1(0,S)$, and the following initial condition

$$u(x,0) = 0, \qquad x \in (0,S). \tag{461}$$

Above, $(\cdot,\cdot)$ is the L^2 inner product, $\mathcal{R} : H_0^1(0,S) \times H_0^1(0,S) \to \mathbf{R}$ is the bilinear

$$\mathcal{R}(u,v) = \frac{\sigma^2}{2}(\frac{\partial u}{\partial x}, \frac{\partial v}{\partial x}) - (r-\delta)(\frac{\partial u}{\partial x}, x^{-1}v) + r(x^{-1}u, x^{-1}v), \tag{462}$$

$$q(x) = (\delta x^{-1} - rZx^{-2})H(x-Z), \quad x \in (0,S), \tag{463}$$

and H is the Heaviside function.

3.3.1 Uniqueness and regularity of the solution

The results in [9] concerning the uniqueness and regularity of the weak solution can be synthesized as

Theorem 3.27 *The following results hold true for solutions of the problem (453)–(455) and its weak form (460)–(461):*

1. *Weak problem (460)–(461) has at most one solution.*
2. *If u is a weak solution of (460)–(461) such that $\frac{\partial u}{\partial t}$ a.e. in the rectangular region $Q_T = (0, S) \times (0, T)$, then one has the following results:*
   - *There exists a large, but fixed S_0, such that for any $S > S_0$ the weak solution u vanishes in $[S_0, S) \times (0, T)$ and is independent of S in $(S_0, S) \times (0, T)$;*
   - *The weak solution u provides a solution for the free boundary value problem (453)–(455) in the following sense: (a) the free boundary is determined by*

     $$s(t) = \inf\{x : x \in (0, S),\ u(x, t) = 0\},$$

     and it is a non-decreasing function; (b) the valuation function w is given by $w = u + w_0$ in D such that $w \in C^\infty(D)$ with $D = (0, s(t)) \times (0, T]$. It verifies the equation (453) in the classical sense in D and $w \in C([0, s(t)] \times (0, T])$; (c) for $t \in (0, T]$, the first and the second boundary conditions (455) are verified in the classical sense and the third one is verified, with the norm in C, on each horizontal segment; (d) the initial condition (454) is verified in $C((0, T); L^2(0, S))$, i.e. $\lim_{t\to 0} \int_0^{s(t)} [w(x, t) - w_0(x)]^2 dx = 0$.
3. *Let $w = w(x, t)$ and $x = s(t)$ solve the free-boundary value problem (453)–(455) in the classical sense. Then $u = w - w_0$ is a weak solution of the problem (460)–(461) for any $S > S_0$. Consequently the problem (453)–(455) has at most one classical solution.*

3.3.2 Existence and other properties of the solution

The existence and other properties of the weak solution are proved in [10],

Theorem 3.28 *The weak problem (460) and (461) has at least one solution $u = u(x,t) \in W(0,T)$ satisfying*

$$x^{-1}u \in L^\infty(0,T;L^2(0,S)), \tag{464}$$

$$\partial_x u \in L^\infty(0,T;L^2(0,S)), \tag{465}$$

Moreover, $u \geq 0$ and $\partial_t u \geq 0$ a.e. in $(0,S)\times(0,T)$.

3.3.3 Weak form of the problem as a variational inequality and numerical experiments using the Schwarz method

In [21], in is proved that problem given by (460) and (461) can be written as a variational inequality. We introduce the convex set

$$K = \{v \in H_0^1(0,S) \ : \ v \geq 0 \ in \ (0,S)\}$$

and we look for an $u(x,t)$ such that, for almost all $t \in (0,T)$, $u(x,t) \in K$ and satisfies the following inequality

$$\begin{array}{l}(x^{-2}\frac{\partial u}{\partial t}, v-u) + \mathcal{R}(u, v-u) + \int_Z^S q(x)(v-u) \geq \\ \frac{\sigma^2}{2}(v(Z) - u(Z)) \quad \forall v \in K.\end{array} \tag{466}$$

with the initial condition (461). We have the following equivalence theorem

Theorem 3.29 *Problem (460)–(461) is equivalent with the problem of finding $u \in L^2(0,T;K)$ with $x^{-1}\partial_t u \in L^2(0,T;L^2(0,S))$ which satisfies inequality (466), a.e. in $(0,T)$, and the initial condition (461).*

Writing $u_s(x,t) = x^p u(x,t)$, with $p = \frac{r-\delta}{\sigma^2}$, problem (466)–(461) can be written as

$$\begin{array}{l}u \in K : (x^{-2}\frac{\partial u_s}{\partial t}, v-u_s) + \mathcal{S}(u_s, v-u_s) + \int_Z^S x^p q(x)(v-u_s) \geq \\ \frac{\sigma^2}{2}Z^p(v(Z) - u_s(Z)) \quad \forall v \in K,\end{array} \tag{467}$$

$$u_s(x,0) = 0 \tag{468}$$

where

$$\mathcal{S}(u,v) = \frac{\sigma^2}{2}(\frac{\partial u}{\partial x}, \frac{\partial v}{\partial x}) + \frac{1}{2}(p^2\sigma^2 + r + \delta)(x^{-1}u, x^{-1}v).$$

The advantage of the above inequality in comparison with (466) is that the bilinear form $\mathcal{S}$ is symmetric.

Now, for an $n \in \mathbf{N}$, we consider a uniform partition $0 = t_0 < t_1 < \cdots < t_n = T$, $t_i = ik$ and $k = T/n$, of the interval $[0, T]$. Let V_h be the linear finite element space obtained from a uniform partition of the interval $[0, S]$, $0 = x_0 < x_1 < \cdots < x_m = S$, with $x_j = jh$ and $h = S/m$, $m \in \mathbf{N}$. Also, we define

$$K_h = \{v_h \in V_h : v_h(x_j) \geq 0,\ j = 0, \cdots, m\}$$

We consider the following discretization of problem (467)–(468).

Find $u_{hk}^i \in K_h,\ i = 1, \ldots, n$ satisfying

$$\begin{aligned}(x^{-2}\partial_k u_{hk}^i, v_h - u_{hk}^i) + \mathcal{S}(u_{hk}^i, v_h - u_{hk}^i) + \textstyle\int_Z^S x^p q(x)(v_h - u_{hk}^i) \geq \\ \tfrac{\sigma^2}{2} Z^p(v_h(Z) - u_{hk}^i(Z)) \quad \forall v_h \in K_h.\end{aligned} \tag{469}$$

with the initial condition

$$u_{hk}^0 = 0. \tag{470}$$

We have denoted above by $\partial_k u_{hk}^i$ the difference $(u_{hk}^i - u_{hk}^{i-1})/k$. Inequalities (469) are solved by the relaxation method, and several numerical examples have been given in [21].

4 Future directions of research

In this section, some possible extensions of the previous results are given. First, we present the results in two unpublished papers which are an attempt to extend the one- and two-level methods in the previous sections to multigrid methods. These results refer to the variational inequalities and to the inequalities with a contraction operator. We hope to extend these results to other types of non-linear problems like the variational inequalities of the second kind and quasi-variational inequalities. This would increase very much the number of problems in mechanics and engineering which could be solved by multigrid methods. We also discuss in this section on the application of the domain decomposition methods for Navier-Stokes problem and saddle point problems. These problems have convex sets of a type different from the ones considered in the previous sections.

4.1 Multigrid methods for nonlinear problems

As we have seen in the previous section, my main direction of research has been the study of the domain decomposition methods for variational inequalities, variational inequalities of the second kind, quasi-variational inequalities and inequalities which do not arise from a minimization problem. These methods are first introduced as subspace correction algorithms in a general reflexive Banach space. Under some assumptions on the decomposition of the elements in the closed convex set of the problem, we prove that the algorithms are globally convergent and estimate the global convergence rate. This convergence rate depends on the data of the problem, the number of the utilized subspaces and on a constant C_0 introduced in the assumption we made. This constant is introduced by a condition which, in the case of equations, has named stability condition of the decomposition of the space as a sum of subspaces (see [83] or [84]). These general algorithms become one- or multilevel methods if we use finite element spaces associated with the level meshes of the domain and with the domain decompositions on each level. In these cases, we prove that the assumptions we made in the general convergence theory hold, and we are able to write the convergence rate in terms of the parameters of the meshes and domain decompositions. We point out that the obtained results refer to problems in $W^{1,\sigma}$, $1 < \sigma < \infty$, not only to problems in H^1 as in the case of most papers in the literature.

Except the papers [22] (in Section 3.1.4) and [28] (in Section 3.1.9), the papers in in Sections 3.1.1 – 3.1.8 deal with the one- or two-level methods. The multilevel method proposed in [22] uses the convex set of the problem, which is defined on the finest mesh, for the constraints on the coarse levels. This lead to a sub-optimal computing complexity of the iterations. This drawback is avoided in paper [28] by writing the convex set of the problem as a sum of level convex sets. However, this procedure can be applied only for the convex sets of one-obstacle type. In the preprint [30], which will be presented in Section 4.1.1, we propose multigrid algorithms for two-obstacle problems. In these algorithms, the convex set of the problem is not decomposed as in [28], but we introduce convex sets for each mesh level where we look for the corrections.

The extension of these multigrid algorithms to inequalities which do not arise from the minimization of a differentiable functional, like inequalities of the second kind or quasi-variational inequalities, is not very evident because of the difficulties introduced by the non-differentiable term. A more simple case is that where the non-differentiable term is given by an operator $T : V \to V'$, as in [24], Section 3.1.6. In Section 4.1.2, we prove the convergence of the multigrid method corresponding to Algorithm 3.15, in Section 3.1.9 (paper [28]), for such problems. However, we find that the general algorithm is convergent only if a certain convergence condition is satisfied. This condition will limit the number of levels we can use in the corresponding multigrid method.

To conclude, we consider that the multigrid methods are very efficient and robust and consequently, they deserve a particular study when they are also applied for nonlinear problems. For this reason, one of my research direction in the future will be the extension of the one- and two-level methods in the previous section to multigrid methods. In the following, the results in the above mentioned two unpublished papers [30] and [29] are presented. They represent a first attempt in the study of the multigrid methods for nonlinear problems.

4.1.1 Multigrid methods for variational inequalities

The multigrid methods presented in Section 3.1.9 (paper [28]) have been given for variational inequalities whose convex set is of one-obstacle type. We shall introduce in the following other four algorithms for the variational inequalities with convex sets of two-obstacle type. As the algorithms introduced in [28], these algorithms are combinations of additive or multiplicative algorithms over the levels with additive or multiplicative algorithms on each level. The results concerning the convergence of these algorithms have been published only in the preprint [30].

The abstract framework is the same with that in [30]. We consider inequality (10) with F satisfying (38) with p and q as in (166), and (167). This time, the convex set of the problem is not decomposed as a sum of level convex sets as in (163), but we instead make an assumption on choice of the convex sets where we look for the level corrections. The chosen level convex sets depend on the current approximation in the algorithms.

Assumption 4.1 *For a given $w \in K$, we recursively introduce the convex sets K_j, $j = J,\ J-1,\ \ldots, 1$, as*

- at level J: we assume that $0 \in K_J$, $K_J \subset \{v_J \in V_J : w + v_J \in K\}$ and consider a $w_J \in K_J$

- at a level $J-1 \geq j \geq 1$: we assume that $0 \in K_j$, $K_j \subset \{v_j \in V_j : w + w_J + \ldots + w_{j+1} + v_j \in K\}$ and consider a $w_j \in K_j$

We can easily check that if we take, for $j = J-1, \ldots, 1$,

$$K_j \subset \{v_j \in V_j : w_{j+1} + v_j \in K_{j+1}\}. \tag{471}$$

then $K_j \subset \{v_j \in V_j : w + w_J + \ldots + w_{j+1} + v_j \in K\}$. Evidently, the optimal convergence of the algorithms depends on the effective choice of these level convex sets K_j.

We first introduce the algorithm which is of the multiplicative type over the levels as well as on each level.

Algorithm 4.1 *We start the algorithm with an arbitrary $u^0 \in K$. Assuming that at iteration $n \geq 0$ we have $u^n \in K$, we successively perform the following steps:*

- at the level J, as in Assumption 4.1, with $w = u^n$, we construct the convex set K_J. Then, we first write $w_J^n = 0$, and, for $i = 1, \ldots, I_J$, we successively calculate $w_{Ji}^{n+1} \in V_{Ji}$, $w_J^{n+\frac{i-1}{I_J}} + w_{Ji}^{n+1} \in K_J$, the solution of the inequalities

$$\langle F'(u^n + w_J^{n+\frac{i-1}{I_J}} + w_{Ji}^{n+1}), v_{Ji} - w_{Ji}^{n+1}\rangle \geq 0 \tag{472}$$

for any $v_{Ji} \in V_{Ji}$, $w_J^{n+\frac{i-1}{I_J}} + v_{Ji} \in K_J$, and write $w_J^{n+\frac{i}{I_J}} = w_J^{n+\frac{i-1}{I_J}} + w_{Ji}^{n+1}$.

- at a level $J-1 \geq j \geq 1$, as in Assumption 4.1, we construct the convex set K_j with $w = u^n$ and $w_J = w_J^{n+1}, \ldots, w_{j+1} = w_{j+1}^{n+1}$. Then, we write $w_j^n = 0$, and for $i = 1, \ldots, I_j$, we successively calculate $w_{ji}^{n+1} \in V_{ji}$, $w_j^{n+\frac{i-1}{I_j}} + w_{ji}^{n+1} \in K_j$, the solution of the inequalities

$$\langle F'(u^n + \sum_{k=j+1}^{J} w_k^{n+1} + w_j^{n+\frac{i-1}{I_j}} + w_{ji}^{n+1}), v_{ji} - w_{ji}^{n+1}\rangle \geq 0 \tag{473}$$

for any $v_{ji} \in V_{ji}$, $w_j^{n+\frac{i-1}{I_j}} + v_{ji} \in K_j$, and write $w_j^{n+\frac{i}{I_j}} = w_J^{n+\frac{i-1}{I_j}} + w_{ji}^{n+1}$.

- *we write* $u^{n+1} = u^n + \sum_{j=1}^{J} w_j^{n+1}$.

The algorithm which is of the multiplicative type over the levels and of the additive type on each level is written as

Algorithm 4.2 *We start the algorithm with an arbitrary* $u^0 \in K$. *Assuming that at iteration* $n \geq 0$ *we have* $u^n \in K$, *we successively perform the following steps:*

- *at the level J, as in Assumption 4.1, we construct the convex set* K_J *with* $w = u^n$. *Then, we simultaneously calculate* $w_{Ji}^{n+1} \in V_{Ji} \cap K_J$, $i = 1, \ldots, I_J$, *the solutions of the inequalities*

$$\langle F'(u^n + w_{Ji}^{n+1}), v_{Ji} - w_{Ji}^{n+1} \rangle \geq 0 \tag{474}$$

for any $v_{Ji} \in V_{Ji} \cap K_J$, *and write* $w_J^{n+1} = \frac{r}{I} \sum_{i=1}^{I_J} w_{Ji}^{n+1}$.

- *at a level* $J - 1 \geq j \geq 1$, *as in Assumption 4.1, we construct the convex set* K_j *with* $w = u^n$ *and* $w_J = w_J^{n+1}, \ldots, w_{j+1} = w_{j+1}^{n+1}$. *Then, we simultaneously calculate* $w_{ji}^{n+1} \in V_{ji} \cap K_j$, $i = 1, \ldots, I_j$, *the solutions of the inequalities*

$$\langle F'(u^n + \sum_{k=j+1}^{J} w_k^{n+1} + w_{ji}^{n+1}), v_{ji} - w_{ji}^{n+1} \rangle \geq 0 \tag{475}$$

for any $v_{ji} \in V_{ji} \cap K_j$, *and write* $w_j^{n+1} = \frac{r}{I} \sum_{i=1}^{I_j} w_{ji}^{n+1}$.

- *we write* $u^{n+1} = u^n + \sum_{j=1}^{J} w_j^{n+1}$.

Above, r is a constant in the interval $(0, 1]$.

The algorithm which is of the additive type over the levels and of the multiplicative type on each level is written as,

Algorithm 4.3 *We start the algorithm with an $u^0 \in K$. Assuming that at iteration $n \geq 0$ we have $u^n \in K$, for $j = 1, \ldots, J$, we simultaneously perform the following steps*

- we construct the convex set K_j as in Assumption 4.1 with $w = u^n$ and $w_J = \ldots = w_1 = 0$,

- we write $w_j^n = 0$, and for $i = 1, \ldots, I_j$, we successively calculate $w_{ji}^{n+1} \in V_{ji}$, $w_j^{n+\frac{i-1}{I_j}} + w_{ji}^{n+1} \in K_j$, the solution of the inequalities

$$\langle F'(u^n + w_j^{n+\frac{i-1}{I_j}} + w_{ji}^{n+1}), v_{ji} - w_{ji}^{n+1} \rangle \geq 0 \tag{476}$$

for any $v_{ji} \in V_{ji}$, $w_j^{n+\frac{i-1}{I_j}} + v_{ji} \in K_j$, and write $w_j^{n+\frac{i}{I_j}} = w_j^{n+\frac{i-1}{I_j}} + w_{ji}^{n+1}$,

Then, we write $u^{n+1} = u^n + \frac{s}{J} \sum_{j=1}^{J} w_j^{n+1}$, with a fixed $0 < s \leq 1$.

Finally, the algorithm which is of the additive type over the levels as well as on each level, is written as,

Algorithm 4.4 *We start the algorithm with an $u^0 \in K$. Assuming that at iteration $n \geq 0$ we have $u^n \in K$, we simultaneously perform, for $j = 1, \ldots, J$, the following steps*

- we construct the convex sets K_j as in Assumption 4.1 with $w = u^n$ and $w_J = \ldots = w_1 = 0$,

- we simultaneously calculate, for $i = 1 \ldots, I_j$, $w_{ji}^{n+1} \in V_{ji} \cap K_j$, the solutions of the inequalities

$$\langle F'(u^n + w_{ji}^{n+1}), v_{ji} - w_{ji}^{n+1} \rangle \geq 0 \tag{477}$$

for any $v_{ji} \in V_{ji} \cap K_j$, and write $w_j^{n+1} = \frac{r}{I} \sum_{i=1}^{I_j} w_{ji}^{n+1}$, with a fixed $0 < r \leq 1$.

Then, we write $u^{n+1} = u^n + \frac{s}{J} \sum_{j=1}^{J} w_j^{n+1}$, with a fixed $0 < s \leq 1$.

Evidently, inequalities (473), (475), (476) and (477) are equivalent, respectively, with the following minimization problems

– find $w_{ji}^{n+1} \in V_{ji}$, $w_j^{n+\frac{i-1}{I_j}} + w_{ji}^{n+1} \in K_j$,

$$
\begin{aligned}
&F(u^n + \sum_{k=j+1}^{J} w_k^{n+1} + w_j^{n+\frac{i-1}{I_j}} + w_{ji}^{n+1}) \leq \\
&F(u^n + \sum_{k=j+1}^{J} w_k^{n+1} + w_j^{n+\frac{i-1}{I_j}} + v_{ji})
\end{aligned}
\tag{478}
$$

for any $v_{ji} \in V_{ji}$, $w_j^{n+\frac{i-1}{I_j}} + v_{ji} \in K_j$,

– find $w_{ji}^{n+1} \in V_{ji} \cap K_j$,

$$
F(u^n + \sum_{k=j+1}^{J} w_k^{n+1} + w_{ji}^{n+1}) \leq F(u^n + \sum_{k=j+1}^{J} w_k^{n+1} + v_{ji}) \tag{479}
$$

for any $v_{ji} \in V_{ji} \cap K_j$,

– find $w_{ji}^{n+1} \in V_{ji}$, $w_j^{n+\frac{i-1}{I_j}} + w_{ji}^{n+1} \in K_j$,

$$
F(u^n + w_j^{n+\frac{i-1}{I_j}} + w_{ji}^{n+1}) \leq F(u^n + w_j^{n+\frac{i-1}{I_j}} + v_{ji}) \tag{480}
$$

for any $v_{ji} \in V_{ji}$, $w_j^{n+\frac{i-1}{I_j}} + v_{ji} \in K_j$,

– find $w_{ji}^{n+1} \in V_{ji} \cap K_j$,

$$
F(u^n + w_{ji}^{n+1}) \leq F(u^n + v_{ji}) \tag{481}
$$

for any $v_{ji} \in V_{ji} \cap K_j$.

In order to prove the convergence of the above algorithms, we shall introduce the constant C_1 as in (164) and make new assumptions. For Algorithms 4.1 and 4.3, we assume

Assumption 4.2 *There exists two constants C_2, $C_3 > 0$ such that for any $w \in K$, $w_{ji} \in V_{ji}$, $w_{j1} + \ldots + w_{ji} \in K_j$, $j = J, \ldots, 1$, $i = 1, \ldots, I_j$, and*

$u \in K$, there exist $u_{ji} \in V_{ji}$, $j = J, \dots, 1$, $i = 1, \dots, I_j$, which satisfy

$$u_{j1} \in K_j \text{ and } w_{j1} + \ldots + w_{ji-1} + u_{ji} \in K_j,\ i = 2, \ldots, I_j,\ j = J, \ldots, 1$$
$$u - w = \sum_{j=1}^{J} \sum_{i=1}^{I_j} u_{ji}$$
$$\sum_{j=1}^{J} \sum_{i=1}^{I_j} ||u_{ji}||^\sigma \le C_2^\sigma ||u - w||^\sigma + C_3^\sigma \sum_{j=1}^{J} \sum_{i=1}^{I_j} ||w_{ji}||^\sigma$$

The convex sets K_j, $j = J, \dots, 1$, are constructed as in Assumption 4.1 with the above w and $w_j = \sum_{i=1}^{I_j} w_{ji}$, $j = J, \dots, 1$, for Algorithm 4.1, and with w and $w_J = \ldots = w_1 = 0$, for Algorithm 4.3.

For Algorithms 4.2 and 4.4, we assume

Assumption 4.3 *There exists two constants C_2, $C_3 > 0$ such that for any $w \in K$, $w_{ji} \in V_{ji} \cap K_j$, $j = J, \dots, 1$, $i = 1, \dots, I_j$, and $u \in K$, there exist $u_{ji} \in V_{ji} \cap K_j$, $j = J, \dots, 1$, $i = 1, \dots, I_j$, which satisfy*

$$u - w = \sum_{j=1}^{J} \sum_{i=1}^{I_j} u_{ji}$$
$$\sum_{j=1}^{J} \sum_{i=1}^{I_j} ||u_{ji}||^\sigma \le C_2^\sigma ||u - w||^\sigma + C_3^\sigma \sum_{j=1}^{J} \sum_{i=1}^{I_j} ||w_{ji}||^\sigma$$

The convex sets K_j, $j = J, \dots, 1$, are constructed as in Assumption 4.1 with the above w and $w_j = \frac{r}{I} \sum_{i=1}^{I_j} w_{ji}$, $j = J, \dots, 1$, for Algorithm 4.2, and with $w \in K$ and $w_J = \ldots = w_1 = 0$, for Algorithm 4.4.

The convergence result is given by

Theorem 4.1 *We consider that V is a reflexive Banach space, V_j, $j = 1, \dots, J$, are closed subspaces of V, and V_{ji}, $i = 1, \dots, I_j$, are closed subspaces of V_j. Also, let K be a non empty closed convex subset of V, and K_j, $j = 1, \dots, J$, be non empty closed subsets of V_j given by Assumption 4.1. We* *consider a Gâteaux differentiable functional F on V which is supposed to be coercive if K is not bounded, and which satisfies (38). Also, we assume that Assumption 4.2 or 4.3 holds if we refer to Algorithms 4.1 and 4.3, or to Algorithms 4.2 and 4.4, respectively. On these conditions, if u is the solution of problem (10) and u^n, $n \geq 0$, are its approximations obtained from one of Algorithms 4.1–4.4, then there exists $M > 0$ such that $||u||$, $||u^n|| \leq M$, $n \geq 0$, and the following error estimations hold:*

(i) if $p = q = 2$ we have

$$F(u^n) - F(u) \leq (\frac{\tilde{C}_1}{\tilde{C}_1 + 1})^n [F(u^0) - F(u)], \tag{482}$$

$$||u^n - u||^2 \leq \frac{2}{\alpha_M} (\frac{\tilde{C}_1}{\tilde{C}_1 + 1})^n [F(u^0) - F(u)], \tag{483}$$

where

$$\tilde{C}_1 = \frac{1-t}{t} + \frac{1}{C_2 t \varepsilon} \left[\frac{C_2}{\varepsilon} + 1 + C_1 C_2 + C_3 \right] \tag{484}$$

and

(ii) if $p > q$ we have

$$F(u^n) - F(u) \leq \frac{F(u^0) - F(u)}{[1 + n\tilde{C}_2 (F(u^0) - F(u))^{\frac{p-q}{q-1}}]^{\frac{q-1}{p-q}}}, \tag{485}$$

$$||u - u^n||^p \leq \frac{p}{\alpha_M} \frac{F(u^0) - F(u)}{[1 + n\tilde{C}_2 (F(u^0) - F(u))^{\frac{p-q}{q-1}}]^{\frac{q-1}{p-q}}}, \tag{486}$$

where

$$\tilde{C}_2 = \frac{p - q}{(p-1)(F(u^0) - F(u))^{\frac{p-q}{q-1}} + (q-1)\tilde{C}_3^{\frac{p-1}{q-1}}}. \tag{487}$$

with t in (180),

$$\begin{aligned} \tilde{C}_3 = \frac{1-t}{t} (F(u^0) - F(u))^{\frac{p-q}{p-1}} + \frac{\frac{\alpha_M}{p}}{C_2 \varepsilon} \left[\frac{C_2}{\varepsilon^{\frac{1}{p-1}} (t \frac{\alpha_M}{p})^{\frac{q-1}{p-1}}} + \right. \\ \left. \frac{(1 + C_1 C_2 + C_3)(IJ)^{\frac{p-\sigma}{p\sigma}}}{(t \frac{\alpha_M}{p})^{\frac{q}{p}}} (F(u^0) - F(u))^{\frac{p-q}{p(p-1)}} \right] \end{aligned} \tag{488}$$

and

$$\varepsilon = \frac{\alpha_M}{p} \frac{1}{2C_2\beta_M I^{\frac{\sigma-1}{\sigma}+\frac{p-q+1}{p}} J^{\frac{\sigma-1}{\sigma}-\frac{q-1}{p}} (\max_{k=1,\cdots,J} \sum_{j=1}^{J} \beta_{kj})} \tag{489}$$

Multilevel methods. We consider a family of regular meshes $\mathcal{T}_{h_j}$ of mesh sizes h_j, $j = 1, \ldots, J$ over the domain $\Omega \subset \mathbf{R}^d$, and make the same assumptions on them as in Section 3.1.4 (paper [22]) or in Section 3.1.9 (paper [28]). Also, we introduce the same linear finite element spaces V_{h_j}, $j = 1, \ldots, J$, corresponding to the levels, and their subspaces $V_{h_j}^i$, $i = 1, \ldots, I_j$ associated with the domain decompositions $\{\Omega_j^i\}_{1 \le i \le I_j}$, at each level $j = 1, \ldots, J$. These finite element spaces will be considered as subspaces of $W^{1,\sigma}$, $1 \le \sigma \le \infty$.

In V_{h_J}, we consider the one-obstacle problem given by inequality (10) with the convex set

$$K = \{v \in V_{h_J} : \varphi \le v \le \psi\}, \tag{490}$$

with $\varphi,\ \psi \in V_{h_J}$, $\varphi \le \psi$. We shall prove that Assumptions 4.1–4.3 hold for this type of convex set, and, as in Section 3.1.9, we explicitly write the constants C_2 and C_3 in function of the mesh and overlapping parameters. We can then conclude from Theorem 4.1 that if the functional F has the asked properties, then Algorithms 4.1–4.4 are globally convergent.

Now, we define the level convex sets $K_j \subset V_{h_j}$, $j = J, \ldots, 1$, satisfying Assumption 4.1. Let K be the convex set defined in (490), and a $w \in K$. For the level J, we define

$$\begin{aligned} &\varphi_J = \varphi - w,\ \psi_J = \psi - w, \\ &K_J = [\varphi_J, \psi_J], \ \text{ and consider an arbitrary } w_J \in K_J \end{aligned} \tag{491}$$

At a level $j = J-1, \ldots, 1$, we define

$$\begin{aligned} &\varphi_j = I_{h_j}(\varphi_{j+1} - w_{j+1}),\ \psi_j = I_{h_j}(\psi_{j+1} - w_{j+1}), \\ &K_j = [\varphi_j, \psi_j], \ \text{ and consider an arbitrary } w_j \in K_j \end{aligned} \tag{492}$$

We have

Proposition 4.1 *Assumption 4.1 holds for the convex sets K_j, $j = J, \ldots, 1$, defined in (491) and (492), for any $w \in K$.*

In order to prove that Assumptions 4.2 and 4.3 hold for the convex sets defined in (491) and (492), we consider $u,\ w \in K$ and some $w_j \in K_j$, $j = J, \dots, 1$. First, we define

$$v_J = u - w \text{ and } v_j = I_{h_j}(v_{j+1} - w_{j+1}) \text{ for } j = J-1, \dots, 1 \tag{493}$$

and then,

$$\begin{aligned} u_j &= v_j - v_{j-1} = v_j - I_{h_{j-1}}(v_j - w_j) \text{ for } j = J, \dots, 2 \\ u_1 &= v_1 = I_{h_1}(v_2 - w_2) \end{aligned} \tag{494}$$

With these notations, we have

Lemma 4.1 *If K_j are defined in (491) and (492), and v_j and u_j are defined in (104) and (494), respectively, then $v_j,\ u_j \in K_j$, $j = J, \dots, 1$, and*

$$u - w = \sum_{j=1}^{J} u_j \tag{495}$$

The following result gives some properties of u_j in (494).

Lemma 4.2 *If u_j are defined in (494), then*

$$|u_j|_{1,\sigma}^{\sigma} \le C(J-1)^{\sigma-1} C_{d,\sigma}(h_{j-1}, h_J)^{\sigma} [\sum_{k=2}^{J} |w_k|_{1,\sigma}^{\sigma} + |u - w|_{1,\sigma}^{\sigma}] \tag{496}$$

for $j = J, \dots, 1$, where we take $h_0 = h_1$ for $j = 1$, and

$$\begin{aligned} &||u_j||_{0,\sigma}^{\sigma} \le ||w_j||_{0,\sigma}^{\sigma} + C(J-1)^{\sigma-1} h_{j-1}^{\sigma} C_{d,\sigma}(h_j, h_J)^{\sigma} \cdot \\ &[\sum_{k=2}^{J} |w_k|_{1,\sigma}^{\sigma} + |u - w|_{1,\sigma}^{\sigma}], \ \textit{for } j = J, \dots, 2, \ \textit{and} \\ &||u_1||_{0,\sigma}^{\sigma} \le C(J-1)^{\sigma-1} [||u - w||_{0,\sigma}^{\sigma} + \sum_{j=2}^{J} ||w_j||_{0,\sigma}^{\sigma}] \end{aligned} \tag{497}$$

To prove that Assumption 3.9 holds, we associate to the level domain decompositions the functions θ_j^i defined in (191) and (193). Using these functions, as in [22], we define

$$\begin{aligned} u_{j1} &= L_{h_j}(\theta_j^1 u_j + (1 - \theta_j^1) w_{j1}) \\ u_{ji} &= L_{h_j}(\theta_j^i (u_j - \sum_{l=1}^{i-1} u_{jl}) + (1 - \theta_j^i) w_{ji}), \quad i = 2, \dots, I_j \end{aligned}$$

which are decompositions of u_j, at each level $j = 1, \dots, J$, and we can prove

Proposition 4.2 *Assumption 4.2 holds for the convex sets K_j, $j = J, \dots, 1$, defined in (491) and (492). The constants C_2 and C_3 can be written as*

$$\begin{aligned} C_2 &= CI^{\frac{\sigma+1}{\sigma}}(I+1)^{\frac{\sigma-1}{\sigma}}(J-1)^{\frac{\sigma-1}{\sigma}}[\sum_{j=2}^{J} C_{d,\sigma}(h_{j-1}, h_J)^{\sigma}]^{\frac{1}{\sigma}} \\ C_3 &= CI^2(I+1)^{\frac{\sigma-1}{\sigma}}(J-1)^{\frac{\sigma-1}{\sigma}}[\sum_{j=2}^{J} C_{d,\sigma}(h_{j-1}, h_J)^{\sigma}]^{\frac{1}{\sigma}} \end{aligned} \tag{498}$$

for Algorithm 4.1, and as

$$\begin{aligned} C_2 &= CI^{\frac{\sigma+1}{\sigma}}(I+1)^{\frac{\sigma-1}{\sigma}}(J-1)^{\frac{\sigma-1}{\sigma}}[\sum_{j=2}^{J} C_{d,\sigma}(h_{j-1}, h_J)^{\sigma}]^{\frac{1}{\sigma}} \\ C_3 &= CI^{\frac{\sigma+1}{\sigma}}(I+1)^{\frac{\sigma-1}{\sigma}} \end{aligned} \tag{499}$$

for Algorithm 4.3.

Also, using the functions θ_j^i defined in (192) and (193) we define

$$u_{ji} = L_{h_j}(\theta_j^i u_j), \quad i = 1, \dots, I_j \text{ for } j = J, \dots, 2, \text{ and } u_{11} = u_1 \tag{500}$$

and we get

Proposition 4.3 *Assumption 4.3 holds for the convex sets K_j, $j = J, \dots, 1$, defined in (491) and (492). The constants C_2 and C_3 can be written as*

$$\begin{aligned} C_2 &= CI^{\frac{1}{\sigma}}(J-1)^{\frac{\sigma-1}{\sigma}}[\sum_{j=2}^{J} C_{d,\sigma}(h_{j-1}, h_J)^{\sigma}]^{\frac{1}{\sigma}} \\ C_3 &= C(J-1)^{\frac{\sigma-1}{\sigma}}[\sum_{j=2}^{J} C_{d,\sigma}(h_{j-1}, h_J)^{\sigma}]^{\frac{1}{\sigma}} \end{aligned} \tag{501}$$

for Algorithm 4.2, and as,

$$C_2 = CI^{\frac{1}{\sigma}}(J-1)^{\frac{\sigma-1}{\sigma}}[\sum_{j=2}^{J} C_{d,\sigma}(h_{j-1}, h_J)^{\sigma}]^{\frac{1}{\sigma}} \text{ and } C_3 = 0 \tag{502}$$

for Algorithm 4.4.

Now, we estimate the constants C_2 and C_3 as functions of J. Constant C_1 as a function of J is given in (198). Using $S_{d,\sigma}$ in (197), we have

$$C_2 = C(J-1)^{\frac{\sigma-1}{\sigma}} S_{d,\sigma}(J) \tag{503}$$

$$C_3 = \begin{cases} C(J-1)^{\frac{\sigma-1}{\sigma}} S_{d,\sigma}(J) & \text{for Algorithms 4.1 and 4.2} \\ C & \text{for Algorithm 4.3} \\ 0 & \text{for Algorithm 4.4} \end{cases} \tag{504}$$

Using these estimations of C_1–C_3 as functions of J, we write for example (202) the convergence rate of the multilevel Algorithms 4.1–4.4.

For $\sigma = 2$, $p = q = 2$ and $d = 1, 2, 3$, we get

$$\tilde{C}_1(J) = \begin{cases} CJ^3 S_{d,2}(J)^2 & \text{for Algorithms 4.1 and 4.2} \\ CJ^4 S_{d,2}(J)^2 & \text{for Algorithms 4.3 and 4.4} \end{cases} \tag{505}$$

and, from Theorem 4.1, we have

$$||u^n - u||_{1,2}^2 \le \tilde{C}_0 \left(1 - \frac{1}{1 + \tilde{C}_1(J)}\right)^n \tag{506}$$

where $\tilde{C}_0$ is a constant independent of J.

For $1 < q = \sigma < 2$, $p = 2$ and $d = 1$, we have

$$\tilde{C}_3(J) = \begin{cases} CJ^{\frac{4\sigma-1}{\sigma}} & \text{for Algorithms 4.1 and 4.2} \\ CJ^{\frac{7\sigma-2-\sigma^2}{\sigma}} & \text{for Algorithms 4.3 and 4.4} \end{cases} \tag{507}$$

Also, for $d = 2, 3$, we can take

$$\tilde{C}_3(J) = C^J \text{ for Algorithms } 4.1 - 4.4 \tag{508}$$

From Theorem 4.1, we get that

$$||u^n - u||_{1,\sigma}^2 \le \tilde{C}_0 \frac{1}{\left(1 + n\tilde{C}_2(J)\right)^{\frac{\sigma-1}{2-\sigma}}} \tag{509}$$

where, in view of (487), we can take

$$\tilde{C}_2(J) = \frac{1}{1 + \tilde{C}_3(J)^{\frac{1}{\sigma-1}}} \tag{510}$$

For $p = \sigma > 2$, $q = 2$, $d = 1, 2, 3$ and $\sigma \leq 3$, we get

$$\tilde{C}_3(J) = \begin{cases} CJ^3 S_{d,\sigma}(J) & \text{for Algorithms 4.1 and 4.2} \\ CJ^{\frac{3\sigma-1}{\sigma-1}} S_{d,\sigma}(J) & \text{for Algorithms 4.3 and 4.4} \end{cases} \tag{511}$$

Also, for $\sigma > 3$, we have

$$\tilde{C}_3(J) = \begin{cases} CJ^{\frac{2\sigma-1}{\sigma}} & \text{for Algorithms 4.1 and 4.2} \\ CJ^{\frac{2\sigma+1}{\sigma}} & \text{for Algorithms 4.3 and 4.4} \end{cases} \tag{512}$$

Finally, in this case, we have

$$||u^n - u||_{1,\sigma}^{\sigma} \leq \tilde{C}_0 \frac{1}{\left(1 + n\tilde{C}_2(J)\right)^{\frac{1}{\sigma-1}}} \tag{513}$$

where

$$\tilde{C}_2(J) = \frac{1}{1 + \tilde{C}_3(J)^{\sigma-1}} \tag{514}$$

Multigrid methods. The multigrid methods are obtained from the above multilevel methods by taking the subspaces $V_{h_j}^i$, $i = 1, \ldots, I_j$, as the one-dimensional spaces generated by the nodal basis functions associated with the nodes of $\mathcal{T}_{h_j}$, $j = J, \ldots, 1$.

In view of (200) and (201), we can consider $\max_{k=1,\ldots,J} \sum_{j=1}^{J} \beta_{kj}$ and the constant C_1 as being independent of J. Using this and the estimations of C_2 and C_3 in (503) and (504), respectively, we can write, as in the case of the multilevel methods, the convergence rate of the multigrid methods obtained from Algorithms 4.1–4.4 as functions of the number of levels J. We consider again the example in (202).

For $\sigma = 2$, $p = q = 2$ and $d = 1, 2, 3$, we get

$$\tilde{C}_1(J) = \begin{cases} CJS_{d,2}(J)^2 & \text{for Algorithms 4.1 and 4.2} \\ CJ^2 S_{d,2}(J)^2 & \text{for Algorithms 4.3 and 4.4} \end{cases} \tag{515}$$

and the error estimation is given in (506).

For $1 < q = \sigma < 2$, $p = 2$ and $d = 1, 2, 3$, we have

$$\tilde{C}_3(J) = \begin{cases} CJ^{\frac{(4-\sigma)(\sigma-1)}{\sigma}} S_{d,\sigma}(J)^2 & \text{for Algorithms 4.1 and 4.2} \\ CJ^{\frac{4(\sigma-1)}{\sigma}} S_{d,\sigma}(J)^2 & \text{for Algorithms 4.3 and 4.4} \end{cases} \tag{516}$$

and the error estimation is given in (509) with $\tilde{C}_2(J)$ in (510).

For $p = \sigma > 2$, $q = 2$ and $d = 1, 2, 3$, we get

$$\tilde{C}_3(J) = \begin{cases} CJ^{\frac{2\sigma-3}{\sigma-1}} S_{d,\sigma}(J)^{\frac{\sigma}{\sigma-1}} & \text{for Algorithms 4.1 and 4.2} \\ CJ^2 S_{d,\sigma}(J)^{\frac{\sigma}{\sigma-1}} & \text{for Algorithms 4.3 and 4.4} \end{cases} \tag{517}$$

and the error estimation is given in (513) with $\tilde{C}_2(J)$ in (514).

4.1.2 Multigrid methods for variational inequalities with contraction operators

Paper [29] is an attempt to introduce the multigrid method corresponding to Algorithm 3.15, in Section 3.1.9 (paper [28]), for the problems in Section 3.1.6 (paper [24]). This is an extension of the two-level method in [24] to more than two levels. The main difficulty is introduced by the convergence condition (80) in Theorem 3.6. Even if this condition seems to be a natural one, it being similar with the existence and uniqueness condition of the solution, (74) in Proposition 3.7, it will introduce an upper bound for the number of levels we can use in the multigrid method. Maybe another approach of the convergence proof or other conditions imposed to the operator T will solve this problem, but it remains an open problem so far.

We present in the following the results we have obtained in [29].

The framework is that one introduced in Section 3.1.9 (paper [28]), for $p = q = \sigma = 2$, and we consider problem (73),

$$u \in K : \langle F'(u), v - u \rangle - \langle T(u), v - u \rangle \geq 0, \text{ for any } v \in K.$$

and, for this problem, Algorithms 3.4–3.6 are generalized to algorithms similar with Algorithm 4.5. As in [28], the convex set of the problem is written as a sum of level convex sets, and we use Assumption 3.8 in the general convergence proof. As in [24], the introduced algorithms differ from one to another by the argument of the operator T. A direct application of the results in this paper is the convergence of the proposed algorithms for the quasi-linear inequalities.

To solve problem (73), we propose four algorithms. The first one can be written as,

Algorithm 4.5 *We start the algorithm with a $u^0 \in K$ and decompose it as in Assumption 3.8 with $w = u^0$, $u^0 = u_1^0 + \ldots + u_J^0$, $u_j^0 \in K_j$, $j = 1, \ldots, J$. At iteration $n+1$, $n \geq 0$, assuming that we have $u^n \in K$, we decompose it as in Assumption 3.8 with $w = u^n$, $u^n = u_1^n + \ldots + u_J^n$, $u_j^n \in K_j$, $j = 1, \ldots, J$. Then, for $j \in J, \ldots, 1$,*

- we successively calculate, the corrections $w_j^{n+1} \in V_j$, $u_j^n + w_j^{n+1} \in K_j$, by the multiplicative algorithm: we first write $w_j^n = 0$, and for $i = 1, \ldots, I_j$, successively calculate $w_{ji}^{n+1} \in V_{ji}$, $u_j^n + w_j^{n+\frac{i-1}{I_j}} + w_{ji}^{n+1} \in K_j$, the solution of the inequality

$$\begin{aligned} &\langle F'(u^n + \sum_{k=j+1}^{J} w_k^{n+1} + w_j^{n+\frac{i-1}{I_j}} + w_{ji}^{n+1}), v_{ji} - w_{ji}^{n+1}\rangle \\ &+ \langle T(v_{ji}^{n+1}), v_{ji} - w_{ji}^{n+1}\rangle \geq 0 \end{aligned} \tag{518}$$

for any $v_{ji} \in V_{ji}$, $u_j^n + w_j^{n+\frac{i-1}{I_j}} + v_{ji} \in K_j$, and write $w_j^{n+\frac{i}{I_j}} = w_j^{n+\frac{i-1}{I_j}} + w_{ji}^{n+1}$. Above, the argument of T is

$$v_{ji}^{n+1} = u^n + \sum_{k=j+1}^{J} w_k^{n+1} + w_j^{n+\frac{i-1}{I_j}} + w_{ji}^{n+1}. \tag{519}$$

- then, we write, $u^{n+\frac{J-j+1}{J}} = u^{n+\frac{J-j}{J}} + w_j^{n+1}$.

The other three algorithms are variants of the above algorithm in which we change the argument of T, taking

$$v_{ji}^{n+1} = u^n + \sum_{k=j+1}^{J} w_k^{n+1} + w_j^{n+\frac{i-1}{I_j}} \text{ or } v_{ji}^{n+1} = u^n + \sum_{k=j+1}^{J} w_k^{n+1} \text{ or } v_{ji}^{n+1} = u^n. \tag{520}$$

Like inequality (73), inequality (518) is equivalent with a minimization problem. The global convergence of the above algorithms is proved by

Theorem 4.2 *We consider that V is a reflexive Banach, V_j, $j = 1, \ldots, J$, are closed subspaces of V, and V_{ji}, $i = 1, \ldots, I_j$, are some closed subspaces of V_j, $j = 1, \ldots, J$. Let K be a non empty closed convex subset of V decomposed* *as in (163) and which satisfies Assumption 3.8. Also, we assume that F is a Gâteaux differentiable functional on K and satisfies (38), the operator T satisfies (72), and F is coercive if K is not bounded. Also, we assume that for any $M > 0$*

$$\frac{\alpha_M}{2} - c_M C_1 (IJ)^{\frac{1}{2}} > 0 \tag{521}$$

and

$$\begin{aligned} \tilde{C}_M = {} & \frac{\alpha_M}{2} \\ & - \beta_M I(\max_{k=1,\cdots,J} \sum_{j=1}^{J} \beta_{kj}) \left(\frac{\tilde{C}_3}{\tilde{C}_2}\right)^{\frac{1}{2}} \left[(1 + C_1C_2 + C_3)\left(\frac{\tilde{C}_3}{\tilde{C}_2}\right)^{\frac{1}{2}} + 2C_2\right] \\ & - c_M (IJ)^{\frac{1}{2}} \left[C_1(1 + C_1C_2 + C_3)\frac{\tilde{C}_3}{\tilde{C}_2}\right. \\ & \left. + 2(1 + 2C_1C_2 + C_3)\left(\frac{\tilde{C}_3}{\tilde{C}_2}\right)^{\frac{1}{2}} + C_2\right] \geq 0 \end{aligned} \tag{522}$$

On these conditions, if u is the solution of problem (73), then there exists a constant $M > 0$ such that $||u||$, $||u^0||$ and $||u^n + \sum_{k=j+1}^{J} w_k^{n+1} + w_j^{n+\frac{i-1}{I_j}} + w_{ji}^{n+1}|| \leq M$, where $u^n + \sum_{k=j+1}^{J} w_k^{n+1} + w_j^{n+\frac{i-1}{I_j}} + w_{ji}^{n+1}$, $n \geq 0$, $j = 1, \ldots, J$, $i = 1, \ldots, I_j$, are the approximations of u obtained from Algorithm 4.5 or its variants, and we have the following error estimations

$$\begin{aligned} & F(u^n) - F(u) + \langle T(u), u^n - u\rangle \\ & \leq \left(\frac{\tilde{C}_1}{\tilde{C}_1 + 1}\right)^n \left[F(u^0) - F(u) + \langle T(u), u^0 - u\rangle\right], \end{aligned} \tag{523}$$

$$||u^n - u||^2 \leq \frac{2}{\alpha_M}\left(\frac{\tilde{C}_1}{\tilde{C}_1 + 1}\right)^n \left[F(u^0) - F(u) + \langle T(u), u^0 - u\rangle\right], \tag{524}$$

where

$$\begin{aligned} \tilde{C}_1 = {} & \frac{\beta_M I(\max_{k=1,\cdots,J}\sum_{j=1}^{J}\beta_{kj})}{\tilde{C}_2^{\frac{1}{2}}}\left[\frac{1 + C_1C_2 + C_3}{\tilde{C}_2^{\frac{1}{2}}} + \frac{C_2}{\tilde{C}_3^{\frac{1}{2}}}\right] \\ & + \frac{c_M (IJ)^{\frac{1}{2}}}{\tilde{C}_2^{\frac{1}{2}}}\left[\frac{C_1(1 + C_1C_2 + C_3)}{\tilde{C}_2^{\frac{1}{2}}} + \frac{1 + 2C_1C_2 + C_3}{\tilde{C}_3^{\frac{1}{2}}}\right] \end{aligned} \tag{525}$$

with

$$\tilde{C}_2 = \frac{1}{2}[\frac{\alpha_M}{2} - c_M C_1 (IJ)^{\frac{1}{2}}] \text{ and } \tilde{C}_3 = \frac{2c_M^2 IJ}{\frac{\alpha_M}{2} - c_M C_1 (IJ)^{\frac{1}{2}}} \tag{526}$$

Remark 4.1 *If we write $x = \frac{c_M}{\frac{\alpha_M}{2}}$, then condition (521) can be written as*

$$1 - xC_1(IJ)^{\frac{1}{2}} > 0$$

and condition (522), as

$$1 - I\frac{\beta_M}{\alpha_M}(\max_{k=1,\cdots,J}\sum_{j=1}^{J}\beta_{kj})\frac{4x\,(IJ)^{\frac{1}{4}}}{1 - xC_1IJ}\left[(1 + C_1C_2 + C_3)\frac{2x\,(IJ)^{\frac{1}{4}}}{1 - xC_1IJ} + 2C_2\right]$$
$$- x(IJ)^{\frac{1}{2}}\left[C_1(1 + C_1C_2 + C_3)\frac{4x^2\,(IJ)^{\frac{1}{2}}}{(1 - xC_1IJ)^2}\right.$$
$$\left. + (1 + 2C_1C_2 + C_3)\frac{4x\,(IJ)^{\frac{1}{4}}}{1 - xC_1IJ} + C_2\right] \geq 0$$

These two conditions imply that there exists a $0 < \theta < 1$ such that (74) in Proposition 3.7 holds. However, as we already said, the two convergence conditions in the statement of Theorem 4.2 are stronger than the existence and uniqueness condition in that proposition, they asking that c_M to be small enough in comparison with α_M, and finally, they will impose, for a given problem, an upper bound for the number J of levels.

Now, we shall write the convergence rate of the multigrid Algorithm 3.15 and its variants in function of the number J of levels, and of the constants α_M, β_M and c_M. Constants C_1–C_3 have been found, in terms of J, in [28], for both multilevel and multigrid methods. From conditions (521) and (522), it follows that the constant $\tilde{C}_1$ can be taken of the form

$$\tilde{C}_1(J) = C\frac{\beta_M}{c_M}\frac{S_d(J)}{J^{\frac{1}{2}}} \tag{527}$$

where $S_d(J)$ is $S_{d,2}(J)$ in (197). Also, these two conditions are satisfied if

$$J^{\frac{1}{2}}S_d(J) \leq \frac{1}{C}\frac{\alpha_M^2}{c_M\beta_M} \tag{528}$$

We can conclude,

Corollary 4.1 *We assume that F is a Gâteaux differentiable functional which is coercive and satisfies (154), the operator T satisfies (72). Also, we assume that for any $M > 0$ condition (528) holds.*

On these conditions, if u is the solution of problem (73), then there exists a constant $M > 0$ such that $||u||$, $||u^0||$ and $||u^n + \sum_{k=j+1}^{J} w_k^{n+1} + w_j^{n+\frac{i-1}{I_j}} + w_{ji}^{n+1}|| \leq M$, where $u^n + \sum_{k=j+1}^{J} w_k^{n+1} + w_j^{n+\frac{i-1}{I_j}} + w_{ji}^{n+1}$, $n \geq 0$, $j = 1, \dots, J$, $i = 1, \dots, I_j$, are the approximations of u obtained from multigrid Algorithm 3.15 or its variants, and we have the following error estimation

$$||u^n - u||_1^2 \leq \tilde{C}_0 \left(1 - \frac{1}{1 + \tilde{C}_1(J)}\right)^n \tag{529}$$

where $\tilde{C}_1(J)$ is given in (527) and $\tilde{C}_0$ is a constant independent of J.

4.2 Domain decomposition methods for Navier-Stokes equation and for saddle point problems

The convex set of the problems we introduced so far are of the one- and two-obstacle type, or a little more general, they have Property 3.1 in Section 3.1.3 or Property 3.2 in Section 3.1.4, in the case of the finite elements. In this section we succintly discuss the application of the Schwarz methods to problems whose convex set is not of these types, like Navier-Stokes problem or saddle point problems. It is evident that the verification of the assumptions made in the general convergence theory, Assumption 3.7, for instance, can not be made by using unity partitions associated to the decomposition of the domain, as in the previous sections. First, we recall and make some remarks on the Schwarz method for the Navier-Stokes equation whose convergence has been proved in Section 3.1.6. At the end, we introduce a saddle point formulation of the plasticity problem with hardening. In [2], the iterative Uzawa's method (which decouples the stresses and the hardening parameter from the displacements) associated with the Schwarz method have been used to solve this problem. We hope to provide more direct domain decomposition methods.

4.2.1 Navier-Stokes problem

In [24] (section 3.1.6), we have proved, as a consequence of the general convergence result for inequalities with contraction operators, that Schwarz method converges for Navier-Stokes problem. In fact, we have introduced in [24] five Schwarz methods for the Navier-Stokes problem. More precisely, we have written the problem in a weak form as: find $\mathbf{u} \in V$ such that

$$a(\mathbf{u}; \mathbf{u}, \mathbf{v}) = \langle \mathbf{f}, \mathbf{v} \rangle \text{ for any } \mathbf{v} \in V,$$

where

$$V = \{\mathbf{v} \in H_0^1(\Omega)^d : \mathrm{div}\mathbf{v} = 0\}$$

For a decomposition of the domain $\Omega = \cup_{i=1}^m \Omega_i$, we associate with the subdomain Ω_i the subspace of V,

$$V_i = \{\mathbf{v}_i \in H_0^1(\Omega_i)^d : \mathrm{div}\mathbf{v}_i = 0\}.$$

We proved that the following three multiplicative algorithms are geometrically convergent: *we start the algorithms with an arbitrary initial guess* $\mathbf{u}^0 \in V$*, and, at each iteration* $n \geq 1$ *and on each subdomain* $i = 1, \cdots, m$*, we solve*

$$\mathbf{w}_i^{n+1} \in V_i : a(\mathbf{u}^{n+\frac{i-1}{m}} + \mathbf{w}_i^{n+1}; \mathbf{u}^{n+\frac{i-1}{m}} + \mathbf{w}_i^{n+1}, \mathbf{v}_i) = \langle \mathbf{f}, \mathbf{v}_i \rangle \textit{ for any } \mathbf{v}_i \in V_i,$$

$$\mathbf{w}_i^{n+1} \in V_i : a(\mathbf{u}^{n+\frac{i-1}{m}}; \mathbf{u}^{n+\frac{i-1}{m}} + \mathbf{w}_i^{n+1}, \mathbf{v}_i) = \langle \mathbf{f}, \mathbf{v}_i \rangle \textit{ for any } \mathbf{v}_i \in V_i,$$

$$\mathbf{w}_i^{n+1} \in V_i : a(\mathbf{u}^n; \mathbf{u}^{n+\frac{i-1}{m}} + \mathbf{w}_i^{n+1}, \mathbf{v}_i) = \langle \mathbf{f}, \mathbf{v}_i \rangle \textit{ for any } \mathbf{v}_i \in V_i,$$

respectively, and then we update

$$\mathbf{u}^{n+\frac{i}{m}} = \mathbf{u}^{n+\frac{i-1}{m}} + \mathbf{w}_i^{n+1}.$$

Also, the following two additive algorithms are convergent: *we start the algorithms with an arbitrary initial guess* $\mathbf{u}^0 \in V$*, and, at each iteration* $n \geq 1$ *and on each subdomain* $i = 1, \cdots, m$*, we solve*

$$\mathbf{w}_i^{n+1} \in V_i : a(\mathbf{u}^n + \mathbf{w}_i^{n+1}; \mathbf{u}^n + \mathbf{w}_i^{n+1}, \mathbf{v}_i) = \langle \mathbf{f}, \mathbf{v}_i \rangle \textit{ for any } \mathbf{v}_i \in V_i,$$

$$\mathbf{w}_i^{n+1} \in V_i : a(\mathbf{u}^n; \mathbf{u}^n + \mathbf{w}_i^{n+1}, \mathbf{v}_i) = \langle \mathbf{f}, \mathbf{v}_i \rangle \textit{ for any } \mathbf{v}_i \in V_i,$$

respectively, and then we update

$$\mathbf{u}^{n+1} = \mathbf{u}^n + \varrho \sum_{i=1}^{m} \mathbf{w}_i^{n+1},$$

with $0 < \varrho \leq 1/m$.

We can prove that Assumption 3.7 holds for the above defined spaces V and V_i, $i = 1, \dots, m$. The checking of this assumption in finite element spaces is more complicated, it depends on the type of the finite elements. Moreover, if we use two- or multilevel methods we have to carefully chose the numerical definition of the divergence. Otherwise, by summing corrections on different levels, we finally get that the condition of zero divergence holds only on the coarsest space. This study is not made so far. To my knowledge, except the result in [24], there exists in the literature only one paper dealing with the application of the Schwarz method to this problem, [59]. In this paper, the proofs are given only for the Schwarz method with two subdomains and does not analyze the method for the discretized problem.

4.2.2 Saddle point problems in elasto-plasticity

We give in the following the weak formulation of the elasto-plastic problem with hardening introduced in [49]–[51]. As we already said, we intend to propose a Schwarz method to directly solve this problem, without using the Uzawa's method, as in the case of the contact problems with friction, in Section 3.1.7, where the fixed-point iteration has been avoided.

We consider a body occupying the open set $\Omega \subset \mathbf{R}^d$, $(d = 2, 3)$, with sufficiently smooth boundary. The boundary is divided in two open parts Γ_D and Γ_F such that $\partial\Omega = \bar{\Gamma}_D \cup \bar{\Gamma}_F$ and $\text{meas}(\Gamma_D) > 0$. The body is submitted to a body force density $\mathbf{f} = (f_i)$ in Ω and to a surface force density $\mathbf{g} = (g_i)$ on Γ_F. The velocity is given, on $\Gamma_D \times (0, T)$, $\mathbf{v} = \mathbf{v}_0$.

Let us first introduce some spaces.

$V = \{\mathbf{v} \in (H^1(\Omega))^3 : \mathbf{v} = 0 \text{ on } \Gamma_D\}$,
$L = (L^2(\Omega))^3,\ H = \{\boldsymbol{\tau} \in (L^2(\Omega))^9 : \tau_{ij} = \tau_{ji},\ 1 \leq i, j \leq 3\},\ \Lambda = (L^2(\Omega))^m$,
$H_\Gamma = (H^{\frac{1}{2}}(\Gamma_F))^3$ and H'_Γ its dual

where $m \geq 1$ is the dimension of the hardening parameter. Also, we define the convex set

$$P = \{(\boldsymbol{\tau}, \boldsymbol{\eta}) \in H \times \Lambda : \mathcal{F}(\boldsymbol{\tau}, \boldsymbol{\eta}) \leq 0 \text{ a.e. in } \Omega\}$$

where $\mathcal{F} : \mathbf{R}^9 \times \mathbf{R}^m \to \mathbf{R}$ is a continuous and convex function which describe the hardening surface, $\mathcal{F}(\boldsymbol{\tau}(x), \boldsymbol{\eta}(x)) = 0$, and is supposed to be differentiable on this surface. The weak form of the elasto-plastic problem with hardening can be written as: *find the velocity* $\mathbf{v}$*, the stress* $\boldsymbol{\sigma}$*, and the hardening parameter* $\boldsymbol{\zeta}$*,* $((\boldsymbol{\sigma}, \boldsymbol{\zeta}), \mathbf{v}) : (0, T) \to (H \times \Lambda) \times V$*, such that a.e. in* $(0, T)$ *we have* $(\boldsymbol{\sigma}, \boldsymbol{\zeta}) \in P$*,*

$$a(\dot{\boldsymbol{\sigma}}, \boldsymbol{\tau}-\boldsymbol{\sigma}) - (\boldsymbol{\varepsilon}(\mathbf{v}), \boldsymbol{\tau}-\boldsymbol{\sigma})_H + \alpha(\dot{\boldsymbol{\zeta}}, \boldsymbol{\eta}-\boldsymbol{\zeta})_L - (\mathbf{l}, \boldsymbol{\tau}-\boldsymbol{\sigma})_H \geq 0 \textit{ for any } (\boldsymbol{\tau}, \boldsymbol{\eta}) \in P,$$

$$(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\mathbf{w}))_H = (\mathbf{g}, \mathbf{w})_{H'_\Gamma, H_\Gamma} + (\mathbf{f}, \mathbf{w})_L \textit{ for any } \mathbf{w} \in V,$$

and

$$\boldsymbol{\sigma}(0) = 0, \ \boldsymbol{\zeta}(0) = 0 \ \textit{a.e. in } \Omega.$$

Above, a is the bilinear form generated by the tensor of the elastic constants A, $\alpha > 0$ is a constant which depends on the type of hardening, and $\mathbf{l} = \boldsymbol{\varepsilon}(\mathbf{v}_0)$.

By introducing the convex set

$$K(t) = \{(\boldsymbol{\tau}, \boldsymbol{\eta}) \in P : (\boldsymbol{\tau}, \boldsymbol{\varepsilon}(\mathbf{w}))_H = (\mathbf{g}, \mathbf{w})_{H'_{\Gamma, H_\Gamma}} + (\mathbf{f}, \mathbf{w})_L \text{ for any } \mathbf{w} \in V\}$$

the above problem reduces to a problem formulated only in stresses and the hardening parameter: *find the stress* $\boldsymbol{\sigma}$ *and the hardening parameter* $\boldsymbol{\zeta}$*,* $(\boldsymbol{\sigma}, \boldsymbol{\zeta}) : (0, T) \to H \times \Lambda$*, such that a.e. in* $(0, T)$ *we have* $(\boldsymbol{\sigma}(t), \boldsymbol{\zeta}(t)) \in K(t)$*,*

$$a(\dot{\boldsymbol{\sigma}}, \boldsymbol{\tau} - \boldsymbol{\sigma}) + \alpha(\dot{\boldsymbol{\zeta}}, \boldsymbol{\eta} - \boldsymbol{\zeta})_L - (\mathbf{l}, \boldsymbol{\tau} - \boldsymbol{\sigma})_H \geq 0 \textit{ for any } (\boldsymbol{\tau}(t), \boldsymbol{\eta}(t)) \in K(t),$$

and

$$\boldsymbol{\sigma}(0) = 0, \ \boldsymbol{\zeta}(0) = 0 \ \textit{a.e. in } \Omega.$$

The above problems have a unique solution.

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