Probleme de optimizare de portofolii financiare in piete incomplete cu o filtratie discontinua

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Introducere

Probleme de optimizare a portofoliilor financiare masurata prin maximizarea utilitatii asteptate a valorii finale a portofoliilor financiare

$$V(x) = \sup_{\pi} E\left[U\left(X_T^{x,\pi}\right)\right]$$

au fost studiate in literatura de specialitate incepand cu Merton (1974) $\rightarrow$ intr-o piata completa. In acelasi context Karatzas, Shreve (1998)

In piete incomplete: Kramkov, Schachermayer (1999); Hu, Imkeller, Müller (2004); Morlais (2006); Lim, Quenez (2011); Jiao, Pham (2010)

Definirea problemei

Fie $(\Omega, \mathcal{F}, P)$ un camp de probabilitate
$\mathcal{F}_t := \mathcal{F}^B_{t+} \vee \mathcal{N}$, unde $\mathcal{F}^B_t = \sigma(B_s, 0 \leq s \leq t)$, cu B o miscare Browniana standard
$(\mathcal{F}_t) \to$ *filtratia de referinta* (*filtratia pietii*)

Fie τ o v.a. pozitiva, $H_t := 1_{(\tau \leq t)}$ si

$$\mathcal{G}_t := \mathcal{F}_t \vee \sigma(H_s, 0 \leq s \leq t) = \mathcal{F}_t \vee \sigma(\tau \wedge t)$$

Aceasta procedura s.n. *largirea progresiva a filtratiei*
$(\mathcal{G}_t) \to$ *filtratia largita* (*filtratia completa*)

Ne plasam intr-o piata financiara cu oportunitati de investitie in

- un activ fara risc (de ex. un depozit la termen) cu o *rata a dobanzii stohastice* $r(t)$ $\mathcal{G}$*-adaptata*, cu dinamica pretului

$$dS_t^0 = r_t S_t^0 dt \tag{1}$$

- un activ fara risc (de ex. un activ tranzactionat la bursa) cu dinamica pretului

$$dS_t = S_t(\nu_t dt + \sigma_t dB_t), \tag{2}$$

unde procesele (ν_t) si (σ_t) sunt $\mathcal{F}$-adaptate

Ipoteze

(H1) Toti coeficientii modelului sunt marginiti

(H2) Are loc asa numita proprietate de invarianta martingala (ipoteza **(H)**)

(H3) H_t admite un compensator absolut continuu, i.e. procesul compensat

$$M_t := H_t - \int_0^t \lambda_s^{\mathcal{G}} ds = H_t - \int_0^{t\wedge\tau} \lambda_s^{\mathcal{F}} ds$$

este un $\mathcal{G}$-martingal
$\lambda^{\mathcal{G}}$ s.n. *$\mathcal{G}$-intensitatea stohastica a lui τ*, iar $\lambda^{\mathcal{F}}$ s.n. *$\mathcal{F}$-intensitatea stohastica*
Presupunem ca $\lambda^{\mathcal{F}}$ este marginita

Formula (canonica) de descompunere a unui proces $\mathcal{G}$-adaptat ψ

$$\psi_t = \psi_t^0 1_{(0 \leq t < \tau)} + \psi_t^1(\tau) 1_{(t \leq \tau < T)},$$

unde ψ^0 este $\mathcal{F}$-adaptat si, pentru u fixat, procesul $(\psi_t^1(u))$ este $\mathcal{F}$-adaptat

Atunci

$$r_t = r_t^0 1_{(0 \leq t < \tau)} + r_t^1(\tau) 1_{(t \leq \tau < T)},$$

$$\lambda_t^{\mathcal{G}} = \lambda_t^0 1_{(0 \leq t < \tau)}$$

In mod evident $\lambda_t^0 = \lambda_t^{\mathcal{F}}$

Consideram un investitor, ce investeste capitalul $x > 0$ la momentul initial 0, pe un orizont de timp finit T
Valoarea portofoliului la orice moment de timp $t \in [0, T]$ este

$$X_t^{x,N} = N_t^0 S_t^0 + N_t S_t$$

Strategia este autofinantata, i.e.

$$dX_t^{x,N} = N_t^0 dS_t^0 + N_t dS_t = \left(r_t X_t^{x,N} + N_t S_t (\nu_t - r_t) \right) dt + N_t S_t \sigma_t dB_t$$

Fie $\pi_t = \frac{N_t S_t}{X_t^{x,N}}$
Ecuatia de autofinantare se rescrie sub forma

$$dX_t^{x,\pi} = X_t^{x,\pi} \left[(r_t + \pi_t (\nu_t - r_t)) dt + \pi_t \sigma_t dB_t \right]$$

Definitia 1 *Multimea strategiilor admisibile $\mathcal{A}(x)$ consta in procesele $(\pi_t; 0 \le t \le T)$ $\mathcal{G}$-adaptate ce satisfac $E \int_0^T \pi_t^2 dt < \infty$.*

Formula explicita pentru procesul valorii portofoliului

$$X_t^\pi = x \exp\left(\int_0^t (r_s + \pi_s(\nu_s - r_s))ds\right) \mathcal{E}\left(\int_0^\cdot \pi_s \sigma_s dB_s\right)_t$$
$$= x \exp\left(\int_0^t (r_s + \pi_s(\nu_s - r_s)) - \frac{1}{2}\pi_s^2 \sigma_s^2)ds\right) \exp\left(\int_0^t \pi_s \sigma_s dB_s\right)$$

$\mathcal{E}(N) \rightarrow$ exponentiala stohastica a unui martingal (local) continuu de patrat integrabil N

$$\mathcal{E}(N)_t := \exp\left(N_t - \frac{1}{2}\langle N \rangle_t\right)$$

$\mathcal{E}(N)$ este solutia EDS

$$dY_t = Y_t dN_t$$

Definirea problemei

Consideram problema de optimizare

$$V(x) = \sup_{\pi \in \mathcal{A}(x)} E\left[U\left(X_T^{x,\pi}\right)\right] = \sup_{\pi \in \mathcal{A}(x)} J(\pi) \tag{3}$$

U este o functie de utilitate, i.e. este strict concava, strict crescatoare, diferentiabila si care satisface $\lim_{x \to 0} U'(x) = \infty$ si $\lim_{x \to \infty} U'(x) = 0$
Vom considera

- utilitate de tip logaritmic $\to$ $U(x) = \ln x$
- utilitate putere $\to$ $U(x) = \frac{x^p}{p}$, cu $0 < p < 1$

Utilitate logaritmica

$$U\left(X_T^{x,\pi}\right) = \ln x + \int_0^T \left(r_s + \pi_s(\nu_s - r_s) - \frac{1}{2}\pi_s^2\sigma_s^2\right) ds + \int_0^T \pi_s\sigma_s dB_s,$$

si

$$E\left[U\left(X_T^{x,\pi}\right)\right] = \ln x + E\int_0^T (r_s + \pi_s(\nu_s - r_s) - \frac{1}{2}\pi_s^2\sigma_s^2) ds$$

Problema de optimizare pe traiectorii

$$\pi_t^* := \frac{r_t - \nu_t}{\sigma_t^2} \tag{4}$$

Caracterizarea masurilor de probabilitate martingale echivalente

Fie $\mathcal{M}(P)$ multimea masurilor martingale echivalente (MME) cu P
Daca $Q \in \mathcal{M}(P) \rightarrow$ dens. R.N. $L_T^Q := \frac{dQ}{dP}|_{\mathcal{G}_T}$ si procesul dens. R.N. $L_t^Q = E(L_T^Q|\mathcal{G}_t)$
Conform teoremei de reprezentare predictibila a lui Kusuoka (ce se poate aplica unui $\mathcal{G}$- martingal de patrat integrabil), rezulta

$$dL_t^Q = \tilde{\theta}_t dB_t + \tilde{\gamma}_t dM_t = L_{t-}^Q(\theta_t dB_t + \gamma_t dM_t)$$

unde $\tilde{\theta}, \tilde{\gamma}$ sunt procese $\mathcal{G}$-predictibile

Propozitia 1

$$
\begin{aligned}
L_t^Q &= \mathcal{E}\left(\int_0^{\cdot} \theta_s dB_s\right)_t \mathcal{E}\left(\int_0^{\cdot} \gamma_s dM_s\right)_t \\
&= \exp\left(\int_0^t \theta_s dB_s - \frac{1}{2}\int_0^t \theta_s^2 ds\right) \\
&\quad \exp\left(\int_0^t \ln(1+\gamma_s) dH_s - \int_0^t \gamma_s \lambda_s^{\mathcal{G}} ds\right).
\end{aligned}
$$

$\hat{B}_t := B_t - \int_0^t \theta_s ds$ *este o* $\mathcal{G}$*-miscare Browniana sub* Q, *iar*

$$
\hat{M}_t := M_t - \int_0^t \gamma_s \lambda_s^{\mathcal{G}} ds = H_t - \int_0^t (1+\gamma_s)\lambda_s^{\mathcal{G}} ds
$$

este un $\mathcal{G}$*-martingal discontinuu sub* Q, *ortogonal cu* $\hat{B}$

Fie $R_t := e^{-\int_0^t r_s ds}$, $\tilde{S}_t = R_t S_t$. Atunci

$$d\tilde{S}_t = \tilde{S}_t((\nu_t - r_t)dt + \sigma_t dB_t) = \sigma_t \tilde{S}_t(\mu_t dt + dB_t)$$

Q este o MME numai daca $\theta_t = -\mu_t = \frac{r_t - \mu_t}{\sigma_t}$

Propozitia 2 *Multimea $\mathcal{M}(P)$ este formata din toate masurile Q echivalente cu P, cu procesul dens. R.N.*

$$L_t^\gamma = \exp\left(-\int_0^t \mu_s dB_s - \frac{1}{2}\int_0^t \mu_s^2 ds\right) \exp\left(\int_0^t \ln(1+\gamma_s) dH_s - \int_0^t \gamma_s \lambda_s ds\right), \tag{5}$$

unde $(\gamma_t) \in \Gamma =$ *multimea proceselor $\mathcal{G}$-predictibile cu* $\gamma_t > -1$, $E(L_T^Q) = 1$

Cazul utilitatii putere. Abordarea prin problema duala

Pasii urmati

A. Definirea corespunzatoare a problemei duale

B. Rezolvarea problemei duale

C. Stabilirea legaturii dintre solutia problemei duale si problema primala (originala)

Fie U^* $\rightarrow$ conjugata convexa a fct. U (transformata Legendre-Fenchel)

$$U^*(y) := \sup_{x>0}(U(x) - xy), \; y > 0.$$

Supremumul este atins in punctul $I(y) := (U')^{-1}(y) = y^{\frac{1}{p-1}}$ si $U^*(y) = U(I(y)) = \frac{1-p}{p} y^{\frac{p}{p-1}}$

Formularea problemei duale

Fie $\tilde{X}_t = R_t X_t$ valoarea actualizata a valorii portofoliului

$$\tilde{X}_t^{x,\pi} = \tilde{X}_0^{x,\pi} + \int_0^t \pi_s \tilde{X}_s^{x,\pi}((\nu_s - r_s)ds + \sigma_s dB_s) = x + \int_0^t \pi_s \sigma_s \tilde{X}_s^{x,\pi} d\tilde{B}_s \tag{6}$$

$\tilde{X}_t^{x,\pi}$ este un martingal (local) marginit inferior $\Rightarrow$ $\tilde{X}_t^{x,\pi}$ este un supermartingal, deci

$$E^Q(\tilde{X}_T^{x,\pi}) = E^Q(R_T X_T^{x,\pi}) = E(R_T L_T^\gamma X_T^{x,\pi}) \leq x, \ \forall \ MME \ Q \tag{7}$$

Restrictia bugetara Problema (3) se rescrie

$$V(x) = \sup_{X_T \in \mathcal{X}} E\left[U\left(X_T\right)\right], \tag{8}$$

unde $\mathcal{X} = \{X_T \ v.a. \ \mathcal{G}_T - mas., pozitiva, satisf. \ (7)\}$

Avem o problema de extrem cu legaturi (restrictia bugetara). In conformitate cu teoria dualitatii convexe (cf. Luenberger (1969)), sau Rogers (2001), se defineste *functionala duala* (Lagrangeanul)

$$\begin{aligned} L(\gamma,\lambda) &:= \sup_{X_T\in\mathcal{X}} \left[E\big(U(X_T)\big) - \lambda\big(E(R_T L_T^{\gamma} X_T) - x\big)\right] \\ &= \lambda x + \sup_{X_T\in\mathcal{X}} E\big(U(X_T) - \lambda R_T L_T^{\gamma} X_T\big) \\ &= \lambda x + E\big(U^*(\lambda R_T L_T^{\gamma})\big) \\ &= \lambda x - \frac{1}{q}\lambda^q E\left((R_T L_T^{\gamma})^q\right), \end{aligned} \tag{9}$$

unde $q = \frac{p}{p-1}$. $\lambda \geq 0$
Observam ca supremumul mai sus se atinge in punctul $X_T = I(\lambda R_T L_T^{\gamma})$.

Problema duala

$$\inf_{\gamma \in \Gamma, \lambda} L(\gamma, \lambda) \tag{10}$$

Definim

$$\tilde{V}(\lambda) = \inf_{\gamma \in \Gamma} E\left((R_T L_T^{\gamma})^q\right) \tag{11}$$

Versiunea dinamica a fct. valoare $\tilde{V}(\lambda)$ este

$$\begin{aligned}
\tilde{V}(t, \lambda) &= \inf_{\gamma \in \Gamma} E\left[(R_T L_T^{\gamma})^q | \mathcal{G}_t\right] \\
&= \inf_{\gamma \in \Gamma_{0,t}} \left\{(R_t L_t^{\gamma})^q \inf_{\tilde{\gamma} \in \Gamma_{t,T}} E\left[\left(\frac{R_T L_T^{\tilde{\gamma}}}{R_t L_t^{\tilde{\gamma}}}\right)^q | \mathcal{G}_t\right]\right\} \\
&= \inf_{\gamma \in \Gamma_{0,t}} Y_t^{\gamma},
\end{aligned}$$

unde $Y_t^{\gamma} = (R_t L_t^{\gamma})^q \Phi_t$, cu procesul (Φ_t) un submartingal.

Dorim sa construim un proces (Φ_t), solutie a EDSR cu salturi

$$d\Phi_t = \varphi_t dt + \hat{\varphi}_t dB_t + \tilde{\varphi}_t dM_t, \; \Phi_T = 1$$

a.i. procesul Y_t^γ satisface

- Y_0^γ este constanta determinista independenta de γ
- Y^γ este un $\mathcal{G}$-submartingal, pentru orice $\gamma \in \Gamma$
- Exista $\gamma^* \in \Gamma$ a.i. Y^{γ^*} este un $\mathcal{G}$-martingal

Atunci

$$\begin{aligned} E\left[(R_T L_T^\gamma)^q\right] &= E\left[(R_T L_T^\gamma)^q \Phi_T\right] \geq E\left[(R_0 L_0^\gamma)^q \Phi_0\right] = \Phi_0 \\ &= E\left[(R_0 L_0^{\gamma^*})^q \Phi_0\right] = E\left[(R_T L_T^{\gamma^*})^q \Phi_T\right] = E\left[(R_T L_T^{\gamma^*})^q\right] \end{aligned}$$

ceea ce arata ca γ^* *este optimal*

Conform formulei lui Itô pentru procese cu salturi

$$
\begin{aligned}
dY_t^\gamma = (R_t L_t^\gamma)^q \Big[\Phi_t \Big(& - q r_t + \frac{1}{2} q(q-1)\mu_t^2 + \lambda_t((1+\gamma_t)^q - q\gamma_t - 1) \\
& + \varphi_t - q\mu_t \hat{\varphi}_t + \lambda_t \tilde{\varphi}_t \left((1+\gamma_t)^q - 1\right) \Big] dt \\
& - q Y_t^\gamma \mu_t dB_t \\
& + (R_t L_{t-}^\gamma)^q \left[(\Phi_{t-} + \tilde{\varphi}_t)(1+\gamma_t)^q - \Phi_{t-} \right] dM_t
\end{aligned}
$$

Definim

$$
\begin{aligned}
A_t^\gamma := (R_t L_t^\gamma)^q \Big[& \varphi_t - q r_t \Phi_t + \frac{1}{2} q(q-1)\mu_t^2 \Phi_t - \lambda_t \Phi_t \\
& - q\mu_t \hat{\varphi}_t - \lambda_t \tilde{\varphi}_t + \lambda_t \Big((\Phi_t + \tilde{\varphi}_t)(1+\gamma_t)^q - q\Phi_t \gamma_t \Big) \Big]
\end{aligned}
$$

A_t^γ isi atinge valoarea minima in punctul

$$\gamma_t^* := \left(\frac{\Phi_t}{\Phi_t + \tilde{\varphi}_t}\right)^{\frac{1}{q-1}} - 1 > -1$$

Punem conditia ca $A_t^{\gamma^*} = 0$. Rezulta

$$\varphi_t = qr_t\Phi_t - \frac{1}{2}q(q-1)\mu_t^2\Phi_t + \lambda_t\Phi_t + q\mu_t\hat{\varphi}_t + \lambda_t\tilde{\varphi}_t - \lambda_t\big((\Phi_t + \tilde{\varphi}_t)(1+\gamma_t^*)^q - q\Phi_t\gamma_t^*\big)$$

Teorema 2

Valoarea finala optimala este data de

$$X_T^* = I\left(\lambda^* R_T L_T^{\gamma^*}\right) = \left(\lambda^* R_T L_T^{\gamma^*}\right)^{\frac{1}{p-1}}, \tag{12}$$

unde multiplicatorul lui Lagrange

$$\lambda^* = \left(\frac{x}{E[(R_T L_T^{\gamma^*})^q]}\right)^{\frac{1}{q-1}}$$

Strategia optimala este data de

$$\pi_t^* = \frac{\frac{\hat{\varphi}_t}{\Phi_t} - (q-1)\mu_t}{\sigma_t} \tag{13}$$

Demonstratia teoremei se bazeaza pe urmatorii pasi

- X_T^* este admisibil pentru problema primala (i.e. satisface restrictia bugetara)
- $$\sup_{X_T} E(U(X_T)) = \inf_{\gamma \in \Gamma, \lambda} L(\gamma, \lambda)$$

A doua afirmatie va rezulta din egalitatea

$$E(U(X_T^*)) = L(\gamma^*, \lambda^*)$$

Stim ca valoarea actualizata a portofoliului $\tilde{X}_t^* = R_t X_t^*$ este un $\mathcal{G}$-martingal sub orice MME Q. Rezulta

$$\begin{aligned} R_t X_t^* &= E^{Q^*}(R_T X_T^* | \mathcal{G}_t) = (\lambda^*)^{\frac{1}{p-1}} E^{Q^*}\left(R_T (R_T L_T^{\gamma^*})^{\frac{1}{p-1}} | \mathcal{G}_t\right) \\ &= (\lambda^*)^{\frac{1}{p-1}} \frac{1}{L_t^{\gamma^*}} E(R_T L_T^{\gamma^*} (R_T L_T^{\gamma^*})^{\frac{1}{p-1}} | \mathcal{G}_t) \\ &= (\lambda^*)^{\frac{1}{p-1}} \frac{1}{L_t^{\gamma^*}} E((R_T L_T^{\gamma^*})^q | \mathcal{G}_t) \\ &= (\lambda^*)^{\frac{1}{p-1}} \frac{1}{L_t^{\gamma^*}} E((R_T L_T^{\gamma^*})^q \Phi_T | \mathcal{G}_t) \\ &= (\lambda^*)^{\frac{1}{p-1}} \frac{1}{L_t^{\gamma^*}} (R_t L_t^{\gamma^*})^q \Phi_t \end{aligned} \tag{14}$$

Deducem

$$X_t^* = (\lambda^*)^{q-1} \left(R_t L_t^{\gamma^*}\right)^{q-1} \Phi_t \tag{15}$$

O problema de optimizare cu un activ supus riscului de credit

Trei tipuri de active financiare

- un activ fara risc avand dinamica pretului

$$dR(t) = R(t)r(t)dt$$

- un activ supus riscului de piata avand dinamica pretului

$$dS(t) = S(t)(\mu(t)dt + \sigma(t)dW(t))$$

- o obligatiune emisa de o societate privata, ce poate da default la un moment aleator de timp τ

Se presupune ca la momentul aparitiei defaultului (daca acesta se produce) cumparatorul obligatiunii primeste o compensatie data de un procent $(1 - L(t))$ din valoarea obligatiunii chiar inainte de producerea defaultului
Pretul activului la momentul t este

$$D(t, T) = 1_{(\tau > t)} E^Q \left(e^{-\int_t^T (r(s) + \tilde{\lambda}(s) L(s)) ds} X | \mathcal{F}_t \right) := 1_{(\tau > t)} B(t, T), \tag{16}$$

unde $\hat{r}(t) := r(t) + \tilde{\lambda}(t) L(t)$

$$z(\tau) = (1 - L(\tau)) D(\tau-, T) = (1 - L(\tau)) B(\tau, T)$$

Dinamica procesului $D(t, T)$ este data de

$$dD(t, T) = D(t-, T) \left[(\hat{r}(t) + \theta(t)\beta(t) - \lambda(t)) dt + \beta(t) dW(t) - dM(t \right. \tag{17}$$

unde $\theta(t) = \frac{\mu(t) - r(t)}{\sigma(t)}$ si $\beta(t) := \theta(t) + q(t)$

Dinamica portofoliului

$$\begin{aligned} dX_t^\pi = X_{t-}^\pi \Big[&\big(r(t) + \pi_S(t)(\mu(t) - r(t)) \\ &+ \pi_D(t)\big(\lambda(t)(\gamma(t)L(t) + L(t) - 1) + \theta(t)\beta(t)\big)\big) dt \\ &+ \big(\pi_S(t)\sigma(t) + \pi_D(t)\beta(t)\big) dW(t) - \pi_D(t) dM(t) \Big] \end{aligned} \tag{18}$$

Dinamica portofoliului pre-default

$$\begin{aligned} dX_t^{\underline{\pi}} = X_t^{\underline{\pi}} \Big[&\big(r(t) + \underline{\pi}_S(t)\sigma(t)\theta(t) \\ &+ \pi_D(t)(\lambda(t)(\gamma(t)L(t) + L(t) - 1) \\ &+ \theta(t)\beta(t)))dt + (\underline{\pi}_S(t)\sigma(t) + \pi_D(t)\beta(t)\big)\, dW(t) \Big], \end{aligned} \tag{19}$$

pentru $t < \tau \wedge T$

Dinamica portofoliului post-default

$$dX_t^{\overline{\pi}} = X_t^{\overline{\pi}} \left[r(t) + \overline{\pi}_S(t)\sigma(t)\theta(t) \right] dt + \overline{\pi}_S(t)\sigma(t)dW(t), \quad (20)$$

pentru $t \geq \tau \wedge T$. Conditia initiala este

$$X^{\overline{\pi}}(\tau) = X^{\underline{\pi}}(\tau)(1 - \underline{\pi}_D(\tau)).$$

Ipoteza principala: τ admite o densitate conditionala, i.e. presupunem ca pentru fiecare $t \in [0, T]$ si $s \geq 0$, exista o familie de variabile aleatoare $(\alpha_t(s))$, cu $\alpha_t(s)$ $\mathcal{F}_t$-adaptata, a.i.

$$P(\tau \leq s | \mathcal{F}_t) = \int_0^s \alpha_t(u) du$$

Functia valoare a problemei de optimizare *pre-default*

$$\underline{V}(x) = \sup_{\underline{\pi}} \overline{J}(\tau, X_\tau^{\underline{\pi}}(1 - \underline{\pi}_D(\tau)), \overline{\pi}^*), \tag{21}$$

unde

$$\overline{J}(\tau, \eta, \overline{\pi}^*) = E\left[\eta^p \exp\left(p(G_T^* - G_\tau^*)\right)\right],$$

si $G_t^* := \int_0^t h_s(\overline{\pi}_s^*) ds$

Versiunea dinamica a functiei valoare $\underline{V}(x)$

$$\underline{V}(t, x) = \sup_{\underline{\pi} \in \underline{\mathcal{A}}(t,x)} E \int_t^T (X_s^{\underline{\pi}})^p \, (1 - \underline{\pi}_D(s))^p \exp\left(p(G_T^* - G_s^*)\right) \alpha_s ds$$

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