Uniqueness Theorem for a Thermomechanical Model of Shape Memory Alloys

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Abstract: In this paper, we give a uniqueness theorem concerning a thermomechanical model which describes the behavior of shape memory alloys and takes into account the nonisothermal character of the phase transformations, as well as the existence of the intrinsic dissipation.

Key Words: shape memory alloys, non-isothermal, intrinsic dissipation, abstract derivation structure

1. INTRODUCTION

The analyzed thermomechanical model of shape memory alloys (SMAs) is nonisothermal, considers a nonzero intrinsic dissipation, and is due to A. Chrysochoos [1]. This model was used by X. Balandraud, E. Ernst, and E. Soós to describe rheological properties of SMAs during direct and inverse phase transformations in [2–4]. Important qualitative properties of the model were established by us in [5].

For the convenience of the reader, we repeat the description of the model from [5], thus making our exposition self-contained. In Sections 4 and 5, we introduce the new notions of *increment point* and *decrement point* of a real continuous function and a new structure, that of *abstract derivation space*. We thus provide in Section 5 a more general setting for the initial differential system in the circular cylindrical case and a unitary study of it in Section 7. The SMA's operators defined in Section 6 will prove extremely useful for our theory. Our main results are Theorem 7.1 (the principle of optimum) with its corollary and Theorem 7.2 (the uniqueness of the solutions).

2. THE THERMOMECHANICAL MODEL OF SMA

The classical thermomechanical model we use in this paper to describe the behavior of the SMA contains three independent state variables. Along with the strain tensor ε and the

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absolute temperature T, we consider an internal variable β, which is an adimensional scalar field representing the volume fraction of the martensitic phase. Obviously, we must satisfy the restriction

$$0 \leq \beta \leq 1. \tag{2.1}$$

We assume that the system has a thermodynamic potential and that this potential is its specific free energy,

$$\psi = e - Ts. \tag{2.2}$$

Consequently, we may express the constitutive thermomechanical relations using the C^2-class function $\psi(\varepsilon, T, \beta)$:

$$\sigma = \rho \frac{\partial \psi}{\partial \varepsilon}, \tag{2.3}$$

$$s = -\frac{\partial \psi}{\partial T}, \tag{2.4}$$

$$B(\varepsilon, T, \beta) = -\frac{\partial \psi}{\partial \beta}. \tag{2.5}$$

Here, ρ is the mass density, e is the specific internal energy, σ represents the stress tensor, s the specific entropy, and B is the thermodynamical force associated with β.

The set of constitutive equations is completed by assuming for the heat flux vector q a classical Fourier law,

$$q = -K\nabla T, \tag{2.6}$$

where K is a positive material constant.

The evolution of the SMA is governed by the balance of momentum equation, considered here in its quasistatic form,

$$\operatorname{div} \sigma = 0, \tag{2.7}$$

and by the first law of thermodynamics,

$$\rho \dot{e} = \sigma \cdot \dot{\varepsilon} - \operatorname{div} q. \tag{2.8}$$

A superposed dot denotes the time derivative.

We neglect external body forces and external heat sources.

To describe the evolution of the SMA, the previous system must be completed with an equation for the internal variable β. This equation must be chosen such that any thermomechanical process satisfying the accordingly completed system identically satisfies the restriction (2.1) together with the second law of thermodynamics,

$$\rho\dot{s} \geq -\operatorname{div}\frac{q}{T}. \tag{2.9}$$

Using (2.2) and (2.8) in (2.9), we get

$$\sigma \cdot \dot{\varepsilon} - \rho s\dot{T} - \rho\dot{\psi} - \frac{1}{T}q \cdot \nabla T \geq 0. \tag{2.10}$$

Since

$$\delta_t := -\frac{1}{T}q \cdot \nabla T = \frac{K}{T}(\nabla T)^2 \geq 0\,, \tag{2.11}$$

inequality (2.10), and thus the second principle of thermodynamics, is satisfied every time when

$$\delta_i := \sigma \cdot \dot{\varepsilon} - \rho s\dot{T} - \rho\dot{\psi} = -\rho\frac{\partial\psi}{\partial\beta}\dot{\beta} = \rho B\dot{\beta} \geq 0. \tag{2.12}$$

Here, δ_i is the intrinsic dissipation.

We shall consider evolution equations leading to a nonzero internal dissipation. Let $\Phi : \mathbf{R} \to \mathbf{R}$ be a convex, continuous, nonnegative, and positively homogeneous of degree $r > 0$ function satisfying $\Phi(0) = 0$.

In this paper, we consider evolution equations for β based on the dissipation potential Φ:

$$r\rho B \in \partial\Phi(\dot{\beta}) + \partial I(\beta), \tag{2.13}$$

where I denotes the indicator function of the segment $[0, 1]$,

$$I(\beta) = \begin{cases} +\infty & \text{if } \beta \notin [0,1] \\ 0 & \text{if } \beta \in [0,1] \end{cases} \tag{2.14}$$

and ∂I its subdifferential,

$$\partial I(\beta) = \{x \in \mathbf{R} \,|\, I(\alpha) \geq I(\beta) + x(\alpha - \beta)\,, \forall\, \alpha \in \mathbf{R}\}. \tag{2.15}$$

The energy balance law (2.8) may be written as

$$\rho C\dot{T} + \operatorname{div} q = \delta_i + \rho T\frac{\partial^2\psi}{\partial\varepsilon\partial T} \cdot \dot{\varepsilon} + \rho T\frac{\partial^2\psi}{\partial\beta\partial T} \cdot \dot{\beta}, \tag{2.16}$$

where $C = -T\dfrac{\partial^2\psi}{\partial T^2}$ is the specific heat of the system.

Accordingly, any thermodynamic process which fulfills (2.13) and (2.8) will also satisfy

$$\rho C\dot{T} + \operatorname{div} q = \Phi(\dot{\beta}) + \rho T\frac{\partial^2\psi}{\partial\varepsilon\partial T} \cdot \dot{\varepsilon} + \rho T\frac{\partial^2\psi}{\partial\beta\partial T} \cdot \dot{\beta}, \tag{2.17}$$

and any such process which fulfills (2.13) and (2.17) will also satisfy (2.13) and (2.8). We may therefore replace (2.8) by (2.17) in the system of evolution equations.

Let T_0 be the absolute temperature (supposed constant) of the room in which the SMA is placed; the variation of the absolute temperature T of the sample with respect to T_0 will be denoted by θ, that is,

$$T = T_0 + \theta. \tag{2.18}$$

Our model is intended to describe the behavior of the SMA during quasistatic traction-compression tests. In this case, experimental data show that the variation of temperature θ is small compared to T_0 and that the norm of the strain is small with respect to 1. It has also been noticed that the classical thermoelastic coupling is characterized by a heat dilatation coefficient α of order $10^{-6}/^{o}\,C$ that is negligible with respect to the other terms from the heat propagation law.

Accordingly, we may attempt to describe the behavior of the SMA using a general quadratic specific free energy with zero thermal dilatation coefficient,

$$\begin{aligned} \rho\psi(\varepsilon,\theta,\beta) &= \frac{\lambda}{2}(\operatorname{tr}\varepsilon)^2 + \mu\,\varepsilon\cdot\varepsilon - \rho C\frac{\theta^2}{2T_0} - 2\mu\,\beta R\cdot\varepsilon \\ &+ \mu\beta^2 R\cdot R + \rho\frac{L}{T_0}(T_0+\theta-T_a)\beta + \rho\frac{L}{2T_0}\beta^2(T_a - T_m), \end{aligned} \tag{2.19}$$

where λ and μ are Lamé's coefficients, R is the structural tensor of the system, L represents the latent heat of the body, and T_a, $T_m < T_a$ are characteristic temperatures for the material. All the above scalars are positive, and the structural tensor is constant, symmetric, and traceless,

$$R^T = R \text{ and } \operatorname{tr} R = 0. \tag{2.20}$$

For the dissipation potential Φ, we let the simplest form ($r = 1$),

$$\Phi(\dot{\beta}) = K_p|\dot{\beta}|, \tag{2.21}$$

where K_p is a positive constant.

The constitutive equations are now

$$\sigma = \lambda\,\operatorname{tr}\varepsilon\mathbf{1} + 2\mu\,\varepsilon - 2\mu\,\beta R, \tag{2.22}$$

$$\rho s = \rho C\frac{\theta}{T_0} - \rho\frac{L}{T_0}\beta, \tag{2.23}$$

and

$$\rho B = 2\mu\, R\cdot\varepsilon - 2\mu\,\beta R\cdot R - \rho\frac{L}{T_0}(T_0+\theta-T_a+\beta\Delta T), \tag{2.24}$$

where

$$\Delta T = T_a - T_m > 0. \tag{2.25}$$

The energy balance law (2.17) becomes

$$\rho C\dot{T} + \operatorname{div} q = K_p|\dot{\beta}| + \rho T\frac{L}{T_0}\dot{\beta}. \tag{2.26}$$

To determine the form of the evolution equation for β, using (2.20) and (2.22), we express B as a function of σ, T and β as follows:

$$\rho B = \sigma \cdot R - \rho\frac{L}{T_0}(T_0 + \theta - T_a + \beta\Delta T). \tag{2.27}$$

Let us put

$$\begin{aligned} \sigma_R &:= \sigma \cdot R, \\ \sigma_R^{\pm} &:= \rho\frac{L}{T_0}(T_0 + \theta - T_a + \beta\Delta T) \pm K_p. \end{aligned}$$

Using (2.13), (2.21), and (2.27), we obtain the evolution system:

$$(\mathcal{E})^3 \quad - \quad \begin{cases} \text{If } \beta = 0, \text{ then } \sigma_R \le \sigma_R^+ \text{ and} \\ \qquad\qquad \dot{\beta} < 0 \Rightarrow \sigma_R \le \sigma_R^- \\ \text{If } 0 < \beta < 1, \text{ then } \sigma_R^- \le \sigma_R \le \sigma_R^+ \text{ and} \\ \qquad\qquad \begin{cases} \dot{\beta} < 0 \Rightarrow \sigma_R = \sigma_R^- \\ \dot{\beta} > 0 \Rightarrow \sigma_R = \sigma_R^+ \end{cases} \\ \text{If } \beta = 1, \text{ then } \sigma_R \ge \sigma_R^- \text{ and} \\ \qquad\qquad \dot{\beta} > 0 \Rightarrow \sigma_R \ge \sigma_R^+ \end{cases}$$

Concerning the above system, we make the following specifications:

Consider a nondegenerate interval $J \subset [0, \infty)$, with $\inf J = 0 \in J$, and the functions space:

$$\begin{aligned} \mathcal{D}_J \quad := \{ f : J \to \mathbf{R} \mid & \text{the set of all nonderivability points of } f \text{ is} \\ & \text{locally finite in } J, \text{ and } f \text{ has finite lateral derivatives at every point} \}. \end{aligned}$$

J is interpreted to be the time interval of the process's evolution. We assume that all functions which appear are in $\mathcal{D}_J$ and the equations involving their time derivatives are written in the sense of lateral derivatives.

3. THE CIRCULAR CYLINDRICAL CASE: ASSOCIATED ORDINARY DIFFERENTIAL SYSTEM

We shall analyze here the traction-compression problem for a cylindrical specimen. The authors of the model obtained in this case the next form of the heat propagation equation:

$$\dot{\theta} + \frac{1}{\tau}\theta = \Gamma|\dot{\beta}| + \frac{L}{C}\dot{\beta}, \tag{3.1}$$

where $\Gamma = K_p/\rho C$ and τ is a parameter which will be experimentally determined. Experimental data show that Γ is small, $\Gamma < L/C$.

Let us put

$$p = \frac{\rho L}{g T_0} \text{ and } q = \frac{K_p}{g}, \tag{3.2}$$

where g is the maximal axial strain due to the complete martensitic phase transformation, during a simple traction test.

Set

$$\sigma^{\pm} := p(T_0 - T_a + \theta + \beta \Delta T) \pm q. \tag{3.3}$$

From now on, σ and ε stand for axial components of the correspondent tensors, having for the traction-compression test in the circular cylindrical case the usual meaning.

The constitutive equation obtained in this case,

$$\sigma = E(\varepsilon - g\beta), \tag{3.4}$$

the heat propagation equation (3.1), the evolution system for β, together with (3.2), (3.3), and the initial conditions, lead us to the following system:

$$(\mathcal{T}) - \begin{cases} \dot{\theta} + \frac{1}{\tau}\theta = \Gamma|\dot{\beta}| + \frac{L}{C}\dot{\beta} \\ \sigma = E(\varepsilon - g\beta) \\ 0 \le \beta \le 1, \begin{cases} \text{If } \beta = 0, \text{ then } \sigma \le \sigma^+ \text{ and} \\ \qquad \dot{\beta} < 0 \Rightarrow \sigma \le \sigma^- \\ \text{If } 0 < \beta < 1, \text{ then } \sigma^- \le \sigma \le \sigma^+ \text{ and} \\ \qquad \begin{cases} \dot{\beta} < 0 \Rightarrow \sigma = \sigma^- \\ \dot{\beta} > 0 \Rightarrow \sigma = \sigma^+ \end{cases} \\ \text{If } \beta = 1, \text{ then } \sigma \ge \sigma^- \text{ and} \\ \qquad \dot{\beta} > 0 \Rightarrow \sigma \ge \sigma^+ \end{cases} \\ \beta(0) = 0,\ \theta(0) = 0,\ \varepsilon(0) = 0,\ \sigma(0) = 0 \end{cases}$$

The constants τ, $\Gamma, L, C, E, g, p, q, T_0, T_a, \Delta T$ are all positive, and $T_0 > T_a, \Gamma < L/C$.

4. THE GENERALIZATION OF THE EVOLUTION SYSTEM $(\mathcal{E})^3$

To remove the derivatives of β from $(\mathcal{E})^3$, we introduce and study a new notion:

Definition 4.1. Consider a nondegenerate interval $J_0 \subset \mathbf{R}$ and $u \in C(J_0)$. A point $t \in J_0$ is said to be an *increment point* (respectively, a *decrement point*) of u, iff for every neighborhood V of t, there exist $t_1, t_2 \in V \cap J_0$, such that $t_1 < t_2$ and $u(t_1) < u(t_2)$ (respectively, $u(t_1) > u(t_2)$). Let $M^+(u)$ (respectively, $M^-(u)$) denote the set of all these points.

Notations 4.1. Let J_0 and u be as above. Set:
$N_f(u) = \{t \in J_0 \mid u$ does not have a finite forward derivative at $t\}$,
$N_b(u) = \{t \in J_0 \mid u$ does not have a finite backward derivative at $t\}$,
and consider the function spaces:
$D_f(J_0) = \{u \in C(J_0) \mid N_f(u)$ is at most countable$\}$,
$D_b(J_0) = \{u \in C(J_0) \mid N_b(u)$ is at most countable$\}$,
$AC_{loc}(J_0) = \{u \in C(J_0) \mid u$ is locally absolutely continuous$\}$.

Proposition 4.1. For a nondegenerate interval $J_0 \subset \mathbf{R}$ and $u \in C(J_0)$, we have

1) $t \in J_0 \setminus M^+(u) \Leftrightarrow u$ is decreasing on a neighborhood of t.
   $t \in J_0 \setminus M^-(u) \Leftrightarrow u$ is increasing on a neighborhood of t.
2) If J_1 is a subinterval of J_0, then
   u is increasing on $J_1 \Leftrightarrow M^-(u) \cap \overset{o}{J_1} = \emptyset$.
   u is decreasing on $J_1 \Leftrightarrow M^+(u) \cap \overset{o}{J_1} = \emptyset$.
3) $M^+(u)$ and $M^-(u)$ are perfect sets in J_0, $M^-(u) = M^+(-u)$.
4) $M^+(u) \supset \overline{\{t \in J_0 \setminus N_f(u) \mid \dot{u}_f(t) > 0\}} \cap J_0$. For $u \in D_f(J_0) \cup AC_{loc}(J_0)$, these sets coincide. Analogous statements hold for the backward derivative and $u \in D_b(J_0)$, and also for $M^-(u)$.

Proof. 1) and 2) obviously hold.

3) We show that $(M^+(u))' \cap J_0 = M^+(u)$.

"$\subset$" Let $t \in (M^+(u))' \cap J_0$ and suppose $t \notin M^+(u)$. By (1) $\Rightarrow$ $(\exists)\, \delta > 0$, such that u is decreasing on $J_0 \cap (t-\delta, t+\delta)$. By (2) $\Rightarrow M^+(u) \cap (t-\delta, t+\delta) = \emptyset$ (contradiction with $t \in (M^+(u))'$).

"$\supset$" Let $t \in M^+(u)$ and suppose $t \notin (M^+(u))' \cap J_0 \Rightarrow t \notin (M^+(u))' \Rightarrow (\exists)\, \delta > 0$, such that $M^+(u) \cap (t-\delta, t+\delta) \setminus \{t\} = \emptyset \Rightarrow M^+(u) \cap (t-\delta, t) = M^+(u) \cap (t, t+\delta) = \emptyset$. By (2) $\Rightarrow u$ is decreasing on $J_0 \cap (t-\delta, t)$ and on $J_0 \cap (t, t+\delta)$, and therefore on $J_0 \cap (t-\delta, t+\delta)$, because u is continuous. By (1) $\Rightarrow t \notin M^+(u)$ (contradiction).

It follows that $(M^+(u))' \cap J_0 = M^+(u)$, hence $M^+(u)$ is a perfect set in J_0. It is clear that $M^-(u) = M^+(-u)$.

4) By (1) $\Rightarrow \{t \in J_0 \setminus N_f(u) \mid \dot{u}_f(t) > 0\} \subset M^+(u)$. By (3) $\Rightarrow M^+(u)$ is a closed set in $J_0 \Rightarrow \overline{\{t \in J_0 \setminus N_f(u) \mid \dot{u}_f(t) > 0\}} \cap J_0 \subset M^+(u)$.

For $u \in D_f(J_0) \cup AC_{loc}(J_0)$, let $t \in M^+(u)$, $\delta > 0$. Suppose $\dot{u}_f(s) \leq 0\ (\forall)\, s \in (J_0 \setminus N_f(u)) \cap (t-\delta, t+\delta) \Rightarrow u$ is decreasing on $J_0 \cap (t-\delta, t+\delta)$. By (1) $\Rightarrow t \notin M^+(u)$ (contradiction). $\square$

Comments. In Section 2, for $\beta, \theta \in \mathcal{D}_J$, we obtained the evolution system $(\mathcal{E})^3$. Proposition 4.1 (4) and $\mathcal{D}_J \subset D_f(J)$ make it obvious that this system becomes

$$(\mathcal{E})^3_{\mathcal{D}_J} \quad - \quad \begin{cases} \text{If } \beta(t) = 0, \text{ then } \sigma_R(t) \leq \sigma_R^+(t) \text{ and} \\ \qquad t \in M^-(\beta) \Rightarrow \sigma_R(t) \leq \sigma_R^-(t) \\ \text{If } 0 < \beta(t) < 1, \text{ then } \sigma_R^-(t) \leq \sigma_R(t) \leq \sigma_R^+(t) \text{ and} \\ \qquad \begin{cases} t \in M^-(\beta) \Rightarrow \sigma_R(t) = \sigma_R^-(t) \\ t \in M^+(\beta) \Rightarrow \sigma_R(t) = \sigma_R^+(t) \end{cases} \\ \text{If } \beta(t) = 1, \text{ then } \sigma_R(t) \geq \sigma_R^-(t) \text{ and} \\ \qquad t \in M^+(\beta) \Rightarrow \sigma_R(t) \geq \sigma_R^+(t) \end{cases}$$

Since the derivatives of β are no more present in $(\mathcal{E})^3_{\mathcal{D}_J}$, we can rephrase the system for functions β, θ in $C(J)$ or in an arbitrary vector subspace $X(J)$ of $C(J)$, obtaining a system which will be denoted by $(\mathcal{E})^3_{X(J)}$.

Theorem 4.1. *For $\beta, \theta \in C(J)$ satisfying $(\mathcal{E})^3_{C(J)}$ and $t \in J$, we have*

1) $t \in M^+(\beta) \Rightarrow \sigma_R(t) = \sigma_R^+(t)$.
2) $t \in M^-(\beta) \Rightarrow \sigma_R(t) = \sigma_R^-(t)$.

Moreover, in both of these cases, if β, θ, and σ are derivable at t, then $\dot{\sigma}_R(t) = \dot{\sigma}_R^+(t) = \dot{\sigma}_R^-(t)$.

Proof. 1) Let $t \in M^+(\beta)$ and suppose $\sigma_R(t) \neq \sigma_R^+(t)$. $\sigma_R, \sigma_R^+ \in C(J) \Rightarrow (\exists)\, \delta > 0$, such that $\sigma_R(s) \neq \sigma_R^+(s)\ (\forall)\, s \in J \cap (t-\delta, t+\delta)$. $t \in M^+(\beta) \Rightarrow (\exists)\, t_1, t_2 \in J \cap (t-\delta, t+\delta)$, such that $t_1 < t_2$, $\beta(t_1) < \beta(t_2)$. Set $s_1 = \sup\{s \in [t_1, t_2] \mid \beta(s) \leq \beta(t_1)\}$, $s_2 = \inf\{s \in [s_1, t_2] \mid \beta(s) \geq \beta(t_2)\} \Rightarrow t_1 \leq s_1 < s_2 \leq t_2$, $\beta(s_1) = \beta(t_1)$, $\beta(s_2) = \beta(t_2)$, $\beta(s_1) < \beta(s) < \beta(s_2) (\forall)\, s \in (s_1, s_2)$. By Proposition 4.1 (2), $s_1 < s_2$, $\beta(s_1) < \beta(s_2) \Rightarrow (\exists)\, s \in M^+(\beta) \cap (s_1, s_2)$. We have $s \in (s_1, s_2) \subsetneq (t_1, t_2) \subset J \cap (t-\delta, t+\delta) \Rightarrow \sigma_R(s) \neq \sigma_R^+(s)$, $\beta(s) \in (0, 1)$. From $(\mathcal{E})^3_{C(J)}$, $s \in M^+(\beta) \Rightarrow \sigma_R(s) = \sigma_R^+(s)$ (contradiction). We conclude that $\sigma_R(t) = \sigma_R^+(t)$.

2) The proof runs as in the case (1).

Assume now that β, θ, σ are derivable at $t \in M^+(\beta) \Rightarrow \sigma_R, \sigma_R^+, \sigma_R^-$ are derivable at t. By Proposition 4.1 (3) $\Rightarrow (\exists)\, (t_n)_{n \geq 0} \subset M^+(\beta) \setminus \{t\}$, such that $t_n \xrightarrow[n \to \infty]{} t$. We have $\sigma_R(t_n) = \sigma_R^+(t_n)\ (\forall)\, n \in \mathbf{N} \Rightarrow$

$$\dot{\sigma}_R(t) = \lim_{n \to \infty} \frac{\sigma_R(t_n) - \sigma_R(t)}{t_n - t} = \lim_{n \to \infty} \frac{\sigma_R^+(t_n) - \sigma_R^+(t)}{t_n - t} = \dot{\sigma}_R^+(t) = \dot{\sigma}_R^-(t). \square$$

Corollary 4.1. *For $X(J)$ a vector subspace of $C(J)$, the system $(\mathcal{E})^3_{X(J)}$ becomes*

$$(\mathcal{E})^3_{X(J)} \quad - \quad \begin{cases} \beta(t) > 0 \Rightarrow \sigma_R(t) \geq \sigma_R^-(t) \\ \beta(t) < 1 \Rightarrow \sigma_R(t) \leq \sigma_R^+(t) \\ t \in M^+(\beta) \Rightarrow \sigma_R(t) = \sigma_R^+(t) \\ t \in M^-(\beta) \Rightarrow \sigma_R(t) = \sigma_R^-(t) \end{cases}$$

Proof. Observe that if β and θ satisfy $(\mathcal{E})^3_{X(J)}$, they also satisfy $(\mathcal{E})^3_{C(J)}$. Theorem 4.1 now gives the conclusion. □

Corollary 4.2. (The phase transformation's inertia)
Let $\beta, \theta \in C(J)$ satisfy $(\mathcal{E})^3_{C(J)}$. Then, β is locally monotone, that is, for every $t \in J$, β is monotone on a neighborhood of t.

Proof. For every $t \in J$, we have $\sigma_R^+(t) \neq \sigma_R^-(t) \Rightarrow \sigma_R(t) \neq \sigma_R^+(t)$ or $\sigma_R(t) \neq \sigma_R^-(t) \Rightarrow t \notin M^+(\beta)$ or $t \notin M^-(\beta)$, which follows from Corollary 4.1. Proposition 4.1 (1) now completes the proof. □

Comments. Corollary 4.2 will prove extremely useful in the study of the heat propagation equation (3.1) (see also (2.26)). Corollaries 2.4 (the persistence of the phase) and 2.5 (the initial elasticity) from [5] remain valid, with similar proofs. Corollary 2.3 from [5] can be rephrased as follows:

Let $t, s \in J$ satisfy $t < s$, one of them being in $M^+(\beta)$ and the other in $M^-(\beta)$. Then, there exist $t', s' \in J$, such that $t \leq t' < s' \leq s$ and β is constant on $[t', s']$.

5. THE GENERALIZATION OF THE HEAT PROPAGATION EQUATION IN THE CIRCULAR CYLINDRICAL CASE

Consider $X(J)$ a vector subspace of $C(J)$. Following the notations of Sections 3 and 4, for the circular cylindrical case, $(\mathcal{E})^3_{X(J)}$ becomes

$$(\mathcal{E})_{X(J)} \quad - \quad \begin{cases} \beta(t) > 0 \Rightarrow \sigma(t) \geq \sigma^-(t) \\ \beta(t) < 1 \Rightarrow \sigma(t) \leq \sigma^+(t) \\ t \in M^+(\beta) \Rightarrow \sigma(t) = \sigma^+(t) \\ t \in M^-(\beta) \Rightarrow \sigma(t) = \sigma^-(t) \end{cases}$$

Note that Corollary 4.2 still holds for β and θ, satisfying $(\mathcal{E})_{X(J)}$. For $\beta, \theta \in \mathcal{D}_J$, under the assumptions of $(\mathcal{E})_{C(J)}$, an equivalent formulation of the heat propagation equation (3.1) is

$$(\mathcal{H})_{\mathcal{D}_J} - \left[\begin{array}{l} \text{For every subinterval } J_0 \text{ (of } J\text{), which is an interval of monotony} \\ \text{for } \beta, \text{ we have on } \overset{o}{J_0} \\ \qquad\qquad \dot{\theta} + \dfrac{1}{\tau}\theta = \left(\Gamma_0 + \dfrac{L}{C}\right)\dot{\beta}, \\ \text{where } \Gamma_0 = \begin{cases} \Gamma, & \text{if } \beta \text{ is increasing on } J_0 \\ -\Gamma, & \text{if } \beta \text{ is decreasing on } J_0 \end{cases} \\ \text{If } \beta \text{ is constant on } J_0, \text{ we make an arbitrary choice for } \Gamma_0 \in \{\Gamma, -\Gamma\}. \end{array}\right.$$

From now on, we will keep the above convention for Γ_0.

This condition can be rephrased by replacing $\mathcal{D}_J$ and the lateral derivatives by any of the spaces and derivatives from the examples below:

1) $C(J)$, with the derivation in the sense of distributions in $\mathcal{D}'(\overset{o}{J})$. Note here the natural inclusions $C(J) \subset C(\overset{o}{J}) \subset \mathcal{D}'(\overset{o}{J})$.
   Let us recall that a continuous function on an interval is an increasing function if and only if its derivative in the sense of distributions on the interior of that interval is a positive linear functional (that is, takes positive values for all positive test functions).
2) $AC_{loc}(J)$, with the derivation almost everywhere (a.e.) on $\overset{o}{J}$.
3) $D_f(J)$ (respectively, $D_b(J)$), with the forward (respectively, backward) derivation on $\overset{o}{J}$ where this one exists.
4) $D(J) := \{u \in C(J) \mid u$ has an at most countable set of nonderivability points$\}$, with the usual derivation on $\overset{o}{J}$ where this one exists.
5) a) $A_f(J) := \{u \in C(J) \mid u$ is a forward-analytic function$\}$, with the forward derivation on $\overset{o}{J}$.
   We call $u \in C(J)$ a forward-analytic function at $t \in J \setminus \{\sup J\}$, iff there exists $s \in J$, such that $s > t$ and u is an analytic function on $[t, s)$. We call u a forward-analytic function, iff u is a forward-analytic function at every $t \in J \setminus \{\sup J\}$.
   b) $A_b(J) := \{u \in C(J) \mid u$ is a backward-analytic function$\}$, with the backward derivation on $\overset{o}{J}$. The definitions are similar to those above.
   c) $A_l(J) := A_f(J) \cap A_b(J)$, with the forward and the backward derivations on $\overset{o}{J}$.

Each of the above spaces can be replaced by some vector subspaces of it, keeping the sense of the derivation.

To examine simultaneously all these cases, we now introduce the notion of *abstract derivation space*. Let $A(\mathbf{R})$ denote the space of all real analytic functions on $\mathbf{R}$.

Definition 5.1. Consider $X(J)$ an $A(\mathbf{R})$-submodule of $C(J)$ containing $A(\mathbf{R})$ and all the primitives of functions from $X(J)$, Y an $A(\mathbf{R})$-module, and $\zeta : X(J) \to Y$ an injective morphism of $A(\mathbf{R})$-modules. We call a Y-derivation on $X(J)$ any function $\partial : X(J) \to Y$, satisfying

i) ∂ is $\mathbf{R}$-linear and $\operatorname{Ker} \partial \subset \mathbf{R}$,

ii) $\partial(au) = \dot{a} \cdot \zeta\,(u) + a \cdot \partial u$ for all $a \in A(\mathbf{R})$ and $u \in X(J)$,
iii) $\partial u = \zeta\,(\dot{u})$ for every $u \in X(J) \cap C^1(J)$ with $\dot{u} \in X(J)$.

Under the above assumptions, the triplet $(X(J), \zeta\,, \partial)$ is said to be an abstract derivation space.

Proposition 5.1. The spaces from the examples (1) through (5) and $\mathcal{D}_J$ have natural structures of abstract derivation spaces.

Proof. We give only the main ideas of the proof. The details are left to the reader. We will denote by $F(T)$ the space of all real functions on the given set $T \neq \emptyset$. To simplify notation, we will use the same letter for a function on the interval J and for its restriction to $\overset{o}{J}$.

1) Consider $\zeta\,, \partial : C(J) \to \mathcal{D}'(\overset{o}{J})$, ζ the natural inclusion morphism, $\partial u = (\zeta\,(u))'$, where the derivative of $\zeta\,(u)$ is taken in the sense of distributions.
2) Define $Y := \{\hat{u} \,|\, u \in F(\overset{o}{J})\}$, $\hat{u}$ meaning the equivalence class of the function u, with respect to the equality a.e. Consider $\zeta\,, \partial : AC_{loc}\,(J) \to Y$, $\zeta\,(u) = \hat{u}$, $\partial u = \widehat{\dot{u}}$. Note that $\dot{u}$ exists a.e. for every $u \in AC_{loc}\,(J)$. Therefore, ∂ is well defined.
3) Define $Y := \{\hat{u} \,|\, u \in F(\overset{o}{J})\}$, $\hat{u}$ meaning the equivalence class of the function u, with respect to the relation on $F(\overset{o}{J})$ given by: $v \sim w \Leftrightarrow \{t \in \overset{o}{J} \,|\, v(t) \neq w(t)\}$ is at most countable. Consider $\zeta\,, \partial : D_f\,(J) \to Y$, $\zeta\,(u) = \hat{u}$, $\partial u = \widehat{\dot{u}_f}$. A similar construction works for $D_b(J)$.
4) The construction is similar to that used for the previous example.
5) a) Consider $\zeta\,, \partial : A_f\,(J) \to A_f\,(\overset{o}{J})$, $\zeta\,(u) = u$, $\partial u = \dot{u}_f$. (b) and (c) are similar to (a).

For $\mathcal{D}_J$, define $\zeta\,, \partial : \mathcal{D}_J \to F(\overset{o}{J}) \times F(\overset{o}{J})$, $\zeta\,(u) = (u, u)$, $\partial u = (\dot{u}_f\,, \dot{u}_b)$.□

Comments. We can now consider the heat propagation equation in a more general setting. Let $X(J)$ be as in Definition 5.1. For every subinterval $J_0 \subset J$, set $X(J_0) := \{u|_{J_0} \,|\, u \in X(J)\}$. Assume that on each $X(J_0)$ (which also satisfies the requirements of Definition 5.1) is given an abstract derivation ∂_{J_0}, which always exists, at least in the sense of distributions (see Example 1). For simplicity, we will write ∂ and ζ instead of ∂_{J_0} and $\zeta_{\,J_0}$. From now on, $X(J)$ denotes such a structure (with a given abstract derivation on each $X(J_0)$), which will be called an *abstract derivation structure*. We now can replace $(\mathcal{H})_{\mathcal{D}_J}$ by

$$(\mathcal{H})_{X(J)} \; - \left[\begin{array}{l} \text{For every } J_0 \text{ as in } (\mathcal{H})_{\mathcal{D}_J}, \text{ we have in } (X(J_0), \zeta\,, \partial) \\ \partial\theta + \dfrac{1}{\tau}\zeta\,(\theta) = \left(\Gamma_0 + \dfrac{L}{C}\right)\partial\beta. \end{array}\right.$$

The generalized evolution system $(\mathcal{E})_{X(J)}$, the generalized heat propagation equation $(\mathcal{H})_{X(J)}$, the constitutive equation (3.4), the restriction (2.1), together with the initial conditions, lead us to a system which will be denoted by $(\mathcal{T})_{X(J)}$. Observe that $(\mathcal{T})_{X(J)}$ is a more general setting of $(\mathcal{T})$.

Our mathematical problem is the following: for a given $X(J)$ and a given σ in $C(J)$ (or in $X(J)$) with $\sigma(0) = 0$, we wish to investigate the system $(\mathcal{T})_{X(J)}$. It is required that

$\beta, \theta \in X(J), \varepsilon \in C(J)$. It is easily seen that we can ignore the constitutive equation (3.4) and the condition $\varepsilon(0) = 0$ from $(\mathcal{T})_{X(J)}$. Indeed, for β and θ satisfying all the other conditions, we get a solution of $(\mathcal{T})_{X(J)}$ with $\varepsilon = \dfrac{\sigma}{E} + g\beta \in C(J)$.

Theorem 5.1. *(The integral form of the heat propagation equation)*
For $\beta, \theta \in X(J)$, the following conditions are equivalent:

1) *β and θ satisfy $(\mathcal{H})_{X(J)}$.*
2) *For every J_0 as in $(\mathcal{H})_{\mathcal{D}_J}$ and for an arbitrary fixed $r \in J_0$, we have on J_0*

$$\theta(t) = \theta(r)e^{(r-t)/\tau} + \left(\Gamma_0 + \frac{L}{C}\right) \cdot \left[(\beta(t) - \beta(r)) - \frac{1}{\tau}e^{-t/\tau} \cdot \int_r^t e^{s/\tau}(\beta(s) - \beta(r))ds\right].$$

Proof. Let J_0, Γ_0 be as in $(\mathcal{H})_{\mathcal{D}_J}$ and $r \in J_0$. In $(X(J_0), \zeta, \partial)$, we have

$$\begin{aligned}
\partial\theta + \frac{1}{\tau}\zeta(\theta) &= \left(\Gamma_0 + \frac{L}{C}\right)\partial\beta \Leftrightarrow e^{t/\tau} \cdot \partial\theta + \left(\dot{\overline{e^{t/\tau}}}\right) \cdot \zeta(\theta) \\
&= \left(\Gamma_0 + \frac{L}{C}\right) e^{t/\tau} \cdot \partial\beta = \left(\Gamma_0 + \frac{L}{C}\right) e^{t/\tau} \cdot \partial(\beta - \beta(r)) \\
&\Leftrightarrow \partial(e^{t/\tau}\theta) = \left(\Gamma_0 + \frac{L}{C}\right) \\
&\cdot \left[\partial\left(e^{t/\tau}(\beta - \beta(r))\right) - \frac{1}{\tau}\zeta\left(e^{t/\tau}(\beta - \beta(r))\right)\right] \\
&= \left(\Gamma_0 + \frac{L}{C}\right) \cdot \left[\partial\left(e^{t/\tau}(\beta - \beta(r))\right) - \frac{1}{\tau}\partial\Big(\int_r^t e^{s/\tau}(\beta(s) - \beta(r))ds\Big)\right] \\
&= \left(\Gamma_0 + \frac{L}{C}\right) \cdot \partial\Big(e^{t/\tau}(\beta - \beta(r)) \\
&- \frac{1}{\tau}\int_r^t e^{s/\tau}(\beta(s) - \beta(r))ds\Big) \Leftrightarrow \text{The function}
\end{aligned}$$

$$J_0 \ni t \longrightarrow e^{t/\tau}\theta(t) - \left(\Gamma_0 + \frac{L}{C}\right)$$

$$\cdot\quad\left(e^{t/\tau}(\beta(t)-\beta(r))-\frac{1}{\tau}\int_r^t e^{s/\tau}(\beta(s)-\beta(r))ds\right)\in \mathbf{R}$$

is constant $\equiv e^{r/\tau}\theta(r)$, which establishes the formula. □

Notations 5.1. For a given $\Gamma_0 \in \{\Gamma,-\Gamma\}$, set $\alpha_0 = \dfrac{\Delta T}{\tau\,(\Gamma_0+L/C+\Delta T)}$, $\gamma_0 = \dfrac{1}{\tau}-\alpha_0$, $c_0=\dfrac{p\Delta T}{\tau\,\alpha_0}$. We have $\Gamma_0+\dfrac{L}{C}+\Delta T=\dfrac{\Delta T}{\tau\,\alpha_0}=\dfrac{c_0}{p}$, $\Gamma_0+\dfrac{L}{C}=\dfrac{\gamma_0\Delta T}{\alpha_0}$.
Since $\Gamma<L/C$, we have $\alpha_0,\gamma_0,c_0>0$.
For $r\in J$ and $\beta\in X(J)$, define $\chi_r^0=c_0(\beta-\beta(r))\in X(J)$.

Set $\eta_{\pm}=(T_0-T_a)\pm q/p$. The notation (3.3) now becomes $\sigma^{\pm}=p(\theta+\beta\Delta T+\eta_{\pm})$.
Let β and θ satisfy $(\mathcal{H})_{X(J)}$. For every $r\in J$, define $\sigma_r^{\pm}: J\to\mathbf{R}$, $\sigma_r^{\pm}(t)=p(\theta(r)e^{(r-t)/\tau}+\beta(r)\Delta T+\eta_{\pm})$.
Note that if β is constant on a subinterval J_0 of J, such that $r\in J_0$, then, by Theorem 5.1, we have on J_0: $\theta(t)=\theta(r)e^{(r-t)/\tau}$, $\sigma^{\pm}=\sigma_r^{\pm}$.

Corollary 5.1. For $\beta,\theta\in X(J)$, the following conditions are equivalent:

1) β and θ satisfy $(\mathcal{H})_{X(J)}$.
2) For every $J_0\subset J$ and $r\in J_0$ as in the previous theorem, we have on J_0

$$(\sigma^+-\sigma_r^+)(t)=(\sigma^--\sigma_r^-)(t)=\chi_r^0(t)-\gamma_0e^{-t/\tau}\cdot\int_r^t e^{s/\tau}\chi_r^0(s)ds.$$

Proof. We have $\dfrac{(\sigma^{\pm}-\sigma_r^{\pm})(t)}{p}=\theta(t)-\theta(r)e^{(r-t)/\tau}+\Delta T(\beta(t)-\beta(r))\ \ (\forall)\,t\in J.$

Let $J_0\subset J$ and $r\in J_0$ as in (2). For every $t\in J_0$, the equality from Theorem 5.1 (2) is equivalent to $\dfrac{(\sigma^{\pm}-\sigma_r^{\pm})(t)}{p}=\left(\Gamma_0+\dfrac{L}{C}+\Delta T\right)(\beta(t)-\beta(r))$ $-\dfrac{1}{\tau}\left(\Gamma_0+\dfrac{L}{C}\right)e^{-t/\tau}\cdot\displaystyle\int_r^t e^{s/\tau}(\beta(s)-\beta(r))ds=\frac{1}{p}\left(\chi_r^0(t)-\gamma_0e^{-t/\tau}\cdot\int_r^t e^{s/\tau}\chi_r^0(s)ds\right).$
Theorem 5.1 now completes the proof. □

Observe that the conditions (2) from Theorem 5.1 and from its corollary are independent of the derivations considered on $X(J)$. Therefore, we can replace them by the derivations in the sense of distributions. It will cause no confusion if we use the same letter to designate a continuous function on an interval and the correspondent distribution. Thus, we have

Corollary 5.2. For $\beta,\theta\in X(J)$, the following conditions are equivalent:

1) β and θ satisfy $(\mathcal{H})_{X(J)}$.

2) For every J_0 as in $(\mathcal{H})_{\mathcal{D}_J}$, we have

$$\theta' + \frac{1}{\tau}\theta = \left(\Gamma_0 + \frac{L}{C}\right)\beta' \quad in\ \mathcal{D}'(\overset{o}{J_0}).$$

Corollary 5.3. Let σ, β, and θ satisfy $(\mathcal{T})_{X(J)}$. If one of the functions β, θ, and σ^+ allows a finite forward derivative at $t \in J$, then the same is true for the other two functions and we have

$$\dot{\theta}_f(t) + \frac{1}{\tau}\theta(t) = \Gamma|\dot{\beta}_f(t)| + \frac{L}{C}\dot{\beta}_f(t).$$

The corollary holds for the backward derivative too.

Proof. Let $t_0 \in J$. By Corollary 4.2, $\Rightarrow$ $(\exists)\,\delta > 0$, such that β is monotone on $J \cap (t_0 - \delta, t_0 + \delta)$. The equalities from Theorem 5.1 (2) and Corollary 5.1 (2) hold on $J_0 = J \cap (t_0 - \delta, t_0 + \delta)$. Therefore, β, θ, and σ^+ allow finite forward derivatives at t_0, iff one of them does so. Assume that β, θ, and σ^+ have this property. We multiply the formula from Theorem 5.1 (2) by $e^{t/\tau}$ on J_0, and we consider in the obtained equality the forward derivative at t_0, thus ending the proof. $\square$

Corollary 5.4. For $\beta, \theta \in C(J)$, the following conditions are equivalent:

1) (β, θ) is a solution of $(\mathcal{T})_{X(J)}$.
2) (β, θ) is a solution of $(\mathcal{T})_{C(J)}$ and $\beta, \theta \in X(J)$.

6. THE SMA's LINEAR OPERATORS

In this section, we introduce and study two linear operators which will play an important role in the study of our problem. Subsequently, J_0 will denote a nondegenerate interval and r a fixed element of it.

Definition 6.1. Define $U_r^0, V_r^0 : C(J_0) \to C(J_0)$,

$$U_r^0 v(t) = v(t) + \gamma_0 e^{-\alpha_0 t} \cdot \int_r^t e^{\alpha_0 s} v(s)ds, \quad V_r^0 u(t) = u(t) - \gamma_0 e^{-t/\tau} \cdot \int_r^t e^{s/\tau} u(s)ds$$

for all $u, v \in C(J_0)$ and $t \in J_0$.

For simplicity of notation, we will write $\alpha, \gamma, c, \chi_r, U_r, V_r$ instead of $\alpha_o, \gamma_0, c_0, \chi_r^0$, U_r^0, V_r^0 (all of them depending on $\Gamma_0 \in \{\Gamma, -\Gamma\}$) when no confusion can arise.

Proposition 6.1. U_r and V_r are invertible linear operators, and $(U_r)^{-1} = V_r$. Moreover, U_r and V_r are continuous with respect to the topology of $C(J_0)$ given by the uniform convergence on compact subsets of J_0.

Proof. It is easily seen that U_r and V_r are linear and continuous. For $v \in C(J_0)$ and $t \in J_0$, we have

$$
\begin{aligned}
V_r U_r v(t) &= U_r v(t) - \gamma e^{-t/\tau} \cdot \int_r^t e^{s/\tau} U_r v(s) ds \\
&= U_r v(t) - \gamma e^{-t/\tau} \cdot \Big[\int_r^t e^{s/\tau} v(s) ds + \gamma \int_r^t e^{s/\tau} \cdot e^{-\alpha s} \Big(\int_r^s e^{\alpha \xi} v(\xi) d\xi \Big) ds \Big] \\
&= U_r v(t) - \gamma e^{-t/\tau} \cdot \int_r^t e^{s/\tau} v(s) ds - \gamma e^{-t/\tau} \int_r^t \dot{(e^{\gamma s})} \cdot \Big(\int_r^s e^{\alpha \xi} v(\xi) d\xi \Big) ds \\
&= U_r v(t) - \gamma e^{-t/\tau} \cdot \int_r^t e^{s/\tau} v(s) ds - \gamma e^{-t/\tau} \cdot \Big[\Big(e^{\gamma s} \cdot \int_r^s e^{\alpha \xi} v(\xi) d\xi \Big) \Big|_r^t \\
&- \int_r^t e^{\gamma s} \cdot e^{\alpha s} v(s) ds \Big] = U_r v(t) - \gamma e^{-\alpha t} \cdot \int_r^t e^{\alpha s} v(s) ds = v(t) .
\end{aligned}
$$

Similarly, we get $U_r V_r u(t) = u(t)$ $(\forall) u \in C(J_0)$, $t \in J_0$. Therefore, U_r and V_r are invertible, $(U_r)^{-1} = V_r$. □

Remark 6.1. Every abstract derivation space of functions on J_0 is an invariant vector subspace of $C(J_0)$ for the operators U_r and V_r.

The proof is straightforward.

Remark 6.2. For $u, v \in C(J_0)$, the following conditions are equivalent:

1) $u = U_r v$.
2) $v = V_r u$.
3) $u(r) = v(r)$ and $u' + \alpha u = v' + \dfrac{1}{\tau} v$ in $\mathcal{D}'(\overset{o}{J_0})$.

Proof. Proposition 6.1 gives (1) $\Leftrightarrow$ (2). To prove (1) $\Leftrightarrow$ (3), define $u_1, v_1 \in C(J_0)$, $u_1(t) = e^{\alpha t} u(t)$, $v_1(t) = e^{\alpha t} U_r v(t)$. We have the following equivalent conditions: $u = U_r v \Leftrightarrow u_1 = v_1 \Leftrightarrow u_1(r) = v_1(r)$ and $u_1' = v_1'$ in $\mathcal{D}'(\overset{o}{J_0})$. An easy computation shows that in $\mathcal{D}'(\overset{o}{J_0})$, we have the equivalence: $u_1' = v_1' \Leftrightarrow u' + \alpha u = v' + \dfrac{1}{\tau} v$. □

In the remainder of this section, we assume that $r = \inf J_0$.

Lemma 6.1. Consider $v_1, v_2 \in \mathcal{D}_{J_0}$ and set $v_{\min} = v_1 \wedge v_2$, $v_{\max} = v_1 \vee v_2$, $u_j = U_r v_j$ for $j \in \{1,2\}$, $u_{\min} = U_r v_{\min}$, $u_{\max} = U_r v_{\max}$. Then, for every $t \in J_0 \setminus \{\sup J_0\}$, we have $(\dot{\overline{u_{\min}}})_f(t) \geq (\dot{u}_1)_f(t) \wedge (\dot{u}_2)_f(t)$, $(\dot{\overline{u_{\max}}})_f(t) \leq (\dot{u}_1)_f(t) \vee (\dot{u}_2)_f(t)$.

Proof. We first observe that $v_{\min} \in D_f(J_0)$. By Proposition 5.1 and Remark 6.1, $\Rightarrow u_{\min} \in D_f(J_0)$. For every $j \in \{1,2\}$, we have $u_j = U_r v_j$, and therefore

$$(\dot{u}_j)_f(t) = (\dot{v}_j)_f(t) + \gamma v_j(t) - \alpha\gamma e^{-\alpha t} \int_r^t e^{\alpha s} v_j(s)ds \text{ on } J_0 \setminus \{\sup J_0\}.$$

A similar equality holds for $u_{\min}$ and $v_{\min}$ instead of u_j and v_j.

Fix $t \in J_0 \setminus \{\sup J_0\}$. $v_{\min} = v_1 \wedge v_2 \Rightarrow (\exists) j \in \{1,2\}$, such that $v_{\min}(t) = v_j(t)$,

$$\begin{aligned} (\dot{\overline{v_{\min}}})_f(t) &= (\dot{v}_j)_f(t) \Rightarrow (\dot{\overline{u_{\min}}})_f(t) \\ &= (\dot{u}_j)_f(t) + \alpha\gamma e^{-\alpha t} \int_r^t e^{\alpha s}(v_j(s) \\ &- v_{\min}(s))ds \geq (\dot{u}_j)_f(t) \geq (\dot{u}_1)_f(t) \wedge (\dot{u}_2)_f(t). \end{aligned}$$

To prove the second inequality of the lemma, we can apply the first one to the functions $-v_1$ and $-v_2$ instead of v_1 and v_2. □

Proposition 6.2. Consider $v_1, v_2 \in C(J_0)$. If $U_r v_1$ and $U_r v_2$ are increasing (respectively, decreasing) functions, then so is $U_r(v_1 \wedge v_2)$ (respectively, $U_r(v_1 \vee v_2)$).

Proof. Assume that $u_j := U_r v_j$ is increasing $(\forall) j \in \{1,2\}$. For each $j \in \{1,2\}$, approaching the plot of the function u_j by open polygons gives the sequence $(u_j^n)_{n\in\mathbf{N}}$, such that u_j^n is increasing $(\forall) n \in \mathbf{N}$ and $u_j^n \xrightarrow[n\to\infty]{\text{u.c.}} u_j$, where "u.c." means the uniform convergence on compact subsets of J_0. By Proposition 6.1, $\Rightarrow v_j^n := V_r u_j^n \xrightarrow[n\to\infty]{\text{u.c.}} V_r u_j = v_j$ $(\forall) j \in \{1,2\} \Rightarrow v_1^n \wedge v_2^n \xrightarrow[n\to\infty]{\text{u.c.}} v_1 \wedge v_2$. Proposition 6.1 now gives $U_r(v_1^n \wedge v_2^n) \xrightarrow[n\to\infty]{\text{u.c.}} U_r(v_1 \wedge v_2)$. By Remark 6.1 and Lemma 6.1, $\Rightarrow (\dot{\overline{U_r(v_1^n \wedge v_2^n)}})_f(t) \geq (\dot{u}_1^n)_f(t) \wedge (\dot{u}_2^n)_f(t) \geq 0$ $(\forall) t \in J_0 \setminus \{\sup J_0\} \Rightarrow U_r(v_1^n \wedge v_2^n)$ is increasing $(\forall) n \in \mathbf{N}$. It follows that $U_r(v_1 \wedge v_2)$ is increasing. □

7. THE PRINCIPLE OF OPTIMUM AND THE UNIQUENESS THEOREM

In this section, we consider a given solution (β, θ) of $(\mathcal{T})_{X(J)}$ and a nondegenerate subinterval J_0 of J. Unless otherwise stated, we assume that β is monotone on J_0 and $r := \inf J_0 \in J_0$.

Recall that the operators U_r and V_r defined on $C(J_0)$ depend on the value of $\Gamma_0 \in \{\Gamma, -\Gamma\}$. We require Γ_0 to be as in $(\mathcal{H})_{\mathcal{D}_J}$.

Notations 7.1. For a given function $\omega \in C(J_0)$, set

$$A_r^+(\omega) \quad = \quad \{v \in C(J_0) \,|\, v(r) \geq 0, \ v \geq \omega, \ U_r v \text{ is increasing}\}, \quad \text{where } \Gamma_0 = \Gamma,$$

$$A_r^-(\omega) \quad = \quad \{v \in C(J_0) \,|\, v(r) \leq 0, \ v \leq \omega, \ U_r v \text{ is decreasing}\}, \quad \text{where } \Gamma_0 = -\Gamma.$$

It will cause no confusion if we use the same letter to designate a member of $C(J)$ and its restriction to J_0.

Theorem 7.1. *(The principle of optimum)*

1) *If on J_0 we have $\sigma \leq \sigma^+$ and β is increasing, then $\sigma^+ - \sigma_r^+ = \min A_r^+(\sigma - \sigma_r^+)$.*
2) *If on J_0 we have $\sigma \geq \sigma^-$ and β is decreasing, then $\sigma^- - \sigma_r^- = \max A_r^-(\sigma - \sigma_r^-)$.*

Proof. By Corollary 5.4, $\Rightarrow (\beta, \theta)$ is a solution of $(\mathcal{T})_{C(J)}$.

1) To shorten notation, set $w_r = \sigma^+ - \sigma_r^+ \in C(J_0)$. Corollary 5.1 gives $w_r = V_r \chi_r$. By Proposition 6.1, $\Rightarrow U_r w_r = \chi_r$ is increasing on J_0. It follows that $w_r \in A_r^+(\sigma - \sigma_r^+)$. It remains to prove that w_r is a minorant of $A_r^+(\sigma - \sigma_r^+)$. Fix $v \in A_r^+(\sigma - \sigma_r^+)$. Set $w = w_r \wedge v \in C(J_0) \Rightarrow w(r) = 0$. By Proposition 6.2, $\Rightarrow u := U_r w$ is increasing on J_0. Observe that U_r is a positive operator. We have $\sigma - \sigma_r^+ \leq w \leq w_r \Rightarrow u = U_r w \leq U_r w_r = \chi_r$. It follows that $u(r) = 0$, $(u - \chi_r)(r) = 0$, u is increasing.

We next show that $u - \chi_r$ is increasing. It suffices to prove that $(u - \chi_r)' \geq 0$ in $\mathcal{D}'(\overset{o}{J_0})$. We have $u - \chi_r = U_r(w - w_r)$. By Remark 6.2, $\Rightarrow$
$(u - \chi_r)' + \alpha(u - \chi_r) = (w - w_r)' + \dfrac{1}{\tau}(w - w_r)$ in $\mathcal{D}'(\overset{o}{J_0})$. Fix $\varphi \in \mathcal{D}'(\overset{o}{J_0})$, $\varphi \geq 0 \Rightarrow$

$$< (u - \chi_r)', \varphi >= \alpha \int_{J_0} (\chi_r - u)\varphi - \int_{J_0} (w - w_r)\left(\dot{\varphi} - \frac{1}{\tau}\varphi\right)$$

$$= \alpha \int_{J_0} (\chi_r - u)\varphi - \int_{J_0 \setminus M^+(\beta)} (w - w_r)\left(\dot{\varphi} - \frac{1}{\tau}\varphi\right),$$

because $(\forall)\, t \in J_0 \cap M^+(\beta)$, $\sigma(t) = \sigma^+(t)$ and consequently $w(t) = w_r(t)$. By Proposition 4.1 (3), $\Rightarrow \overset{o}{J_0} \setminus M^+(\beta)$ is an open subset of $\mathbf{R} \Rightarrow (\exists)\, ((a_i, b_i))_{i \in I}$, an at most countable family of mutually disjoint open intervals with limiting points in $(J_0 \cap M^+(\beta)) \cup (\bar{J}_0 \setminus \overset{o}{J_0})$, such that $\overset{o}{J_0} \setminus M^+(\beta) = \bigcup\limits_{i \in I} (a_i, b_i)$. Let (a, b) denote one of these intervals $\Rightarrow$

$$\begin{aligned}
\int_a^b (w - w_r)\left(\dot{\varphi} - \frac{1}{\tau}\varphi\right) &= \int_a^b (w - w_r) d\varphi - \frac{1}{\tau}\int_a^b (w - w_r)\varphi \\
&= (w - w_r)\varphi \Big|_a^b - \int_a^b \varphi d(w - w_r) - \frac{1}{\tau}\int_a^b (w - w_r)\varphi \\
&= -\int_a^b \varphi dw - \frac{1}{\tau}\int_a^b w\varphi + \int_a^b (\dot{w}_r + \frac{1}{\tau} w_r)\varphi \\
&= -\int_a^b e^{-t/\tau}\varphi(t) d\left(e^{t/\tau} w(t)\right) + \alpha \int_a^b \chi_r \varphi,
\end{aligned}$$

because on (a, b), β is constant, which gives $\sigma^+ = \sigma_a^+ \in C^1((a, b))$ and, by Remark 6.2, $\dot{w}_r + \dfrac{1}{\tau} w_r = \dot{\chi}_r + \alpha\chi_r = \alpha\chi_r$. Note that $e^{t/\tau} w(t) = e^{t/\tau} u(t) - \gamma \cdot \int_r^t e^{s/\tau} u(s) ds \ (\forall) t \in J_0 \Rightarrow \int_a^b e^{-t/\tau}\varphi(t) d\left(e^{t/\tau} w(t)\right) = \int_a^b \varphi du + \alpha \int_a^b u\varphi \Rightarrow$

$$\int_a^b (w - w_r)\left(\dot{\varphi} - \frac{1}{\tau}\varphi\right) = -\int_a^b \varphi du + \alpha \int_a^b (\chi_r - u)\varphi.$$

It follows that

$$\int\limits_{J_0 \setminus M^+(\beta)} (w - w_r)\left(\dot{\varphi} - \frac{1}{\tau}\varphi\right) = -\int\limits_{J_0 \setminus M^+(\beta)} \varphi du + \alpha \int\limits_{J_0 \setminus M^+(\beta)} (\chi_r - u)\varphi.$$

We conclude that

$$< (u - \chi_r)', \varphi >= \int\limits_{J_0 \setminus M^+(\beta)} \varphi du + \alpha \int\limits_{J_0 \cap M^+(\beta)} (\chi_r - u)\varphi \geq 0.$$

We have thus proved that $(u-\chi_r)' \geq 0$ in $\mathcal{D}'(\overset{o}{J_0}) \Rightarrow u-\chi_r$ is increasing. $(u-\chi_r)(r) = 0 \Rightarrow u-\chi_r \geq 0$. Hence, $u-\chi_r \equiv 0 \Rightarrow U_r w_r = \chi_r = u = U_r w$. By Proposition 6.1, $\Rightarrow w_r = w = w_r \wedge v \Rightarrow v \geq w_r$. Therefore, $w_r = \min A_r^+(\sigma - \sigma_r^+)$.

2) Taking $w_r = \sigma^- - \sigma_r^- \in C(J_0)$ and $w = w_r \vee v$ for $v \in A_r^-(\sigma - \sigma_r^-)$, we see at once that the proof runs as above. □

Corollary 7.1. 1) Under the hypotheses of Theorem 7.1 (1), we have

$$\chi_r = \min\{u \in C(J_0) \,|\, u(r) \geq 0,\ V_r u \geq \sigma - \sigma_r^+,\ u \text{ is increasing}\}.$$

2) Under the hypotheses of Theorem 7.1 (2), we have

$$\chi_r = \max\{u \in C(J_0) \,|\, u(r) \leq 0,\ V_r u \leq \sigma - \sigma_r^-,\ u \text{ is decreasing}\}.$$

Proof. 1) Set $B = \{u \in C(J_0) \,|\, u(r) \geq 0,\ V_r u \geq \sigma - \sigma_r^+,\ u \text{ is increasing}\}$, $A = A_r^+(\sigma - \sigma_r^+)$. Proposition 6.1 gives $B = U_r(A)$. By Theorem 7.1 (1), $\Rightarrow \sigma^+ - \sigma_r^+ = \min A$. U_r is a positive linear operator $\Rightarrow U_r(\sigma^+ - \sigma_r^+) = \min B$. By Corollary 5.1 and Proposition 6.1, we also have $\chi_r = U_r(\sigma^+ - \sigma_r^+)$, which completes the proof.

2) The proof runs as in the case (1). □

Theorem 7.2. *(The uniqueness theorem)*
For every abstract derivation structure $X(J)$, the system $(\mathcal{T})_{X(J)}$ has at most one solution with $\beta, \theta \in X(J)$ and $\varepsilon \in C(J)$.

Proof. Let (β_1, θ_1) and (β_2, θ_2) be solutions of $(\mathcal{T})_{X(J)}$ and set $r := \sup\{s \in J \,|\, \beta_1 = \beta_2 \text{ on } [0,s]\}$. Suppose $r < \sup J \Rightarrow r \in J, \beta_1 = \beta_2$ on $[0,r]$. By Corollary 4.2 and Theorem 5.1, $\Rightarrow \theta_1 = \theta_2$ on the compact interval $[0,r]$. Therefore, $(\sigma^+)_1 = (\sigma^+)_2$ on $[0,r]$ and $(\sigma_r^+)_1 = (\sigma_r^+)_2$ on J. We need to consider the following three cases:

1) $(\sigma^-)_1(r) \neq \sigma(r) \neq (\sigma^+)_1(r) \Rightarrow (\exists)\, \delta > 0$, such that on $J \cap (r-\delta, r+\delta)$ we have $(\sigma^-)_j \neq \sigma \neq (\sigma^+)_j$, and consequently β_j is constant $(\forall) j \in \{1,2\}$. $\beta_1(r) = \beta_2(r) \Rightarrow \beta_1 = \beta_2$ on $J \cap [0, r+\delta)$ (contradiction).
2) $\sigma(r) = (\sigma^+)_1(r) = (\sigma^+)_2(r) \Rightarrow (\exists)\, \delta > 0$, such that on $J \cap (r-\delta, r+\delta)$ we have $\sigma > (\sigma^-)_j$, and consequently β_j is increasing $(\forall) j \in \{1,2\}$. We will now consider two subcases.
   i) $\beta_1(r) = \beta_2(r) = 1 \Rightarrow \beta_1 = \beta_2 \equiv 1$ on $J \cap [r, r+\delta) \Rightarrow \beta_1 = \beta_2$ on $J \cap [0, r+\delta)$ (contradiction).
   ii) $\beta_1(r) = \beta_2(r) < 1 \Rightarrow (\exists)\, \delta_0 \in (0,\delta]$, such that on $J_0 := J \cap [r, r+\delta_0)$ we have $\beta_j < 1$, and consequently $\sigma \leq (\sigma^+)_j$ $(\forall) j \in \{1,2\}$. By Corollary 7.1 (1), $\Rightarrow (\chi_r)_1 = (\chi_r)_2$ on J_0, which gives $\beta_1 = \beta_2$ on $J_0 \Rightarrow \beta_1 = \beta_2$ on $J \cap [0, r+\delta_0)$ (contradiction).
3) $\sigma(r) = (\sigma^-)_1(r) = (\sigma^-)_2(r)$. As in the previous case, we obtain a contradiction.

By (1), (2), (3), $\Rightarrow r = \sup J \Rightarrow \beta_1 = \beta_2$. Theorem 5.1 now gives $\theta_1 = \theta_2$, which completes the proof. □

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