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J. Math. Anal. Appl. 284 (2003) 174–190

Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS

www.elsevier.com/locate/jmaa

On the positivity of symmetric polynomial functions. Part I: General results

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Received 24 July 2001

Submitted by R.M. Aron

Abstract

We prove that a real symmetric polynomial inequality of degree $d \geqslant 2$ holds on $\mathbf{R}_+^n$ if and only if it holds for elements with at most $\lfloor d/2 \rfloor$ distinct non-zero components, which may have multiplicities. We establish this result by solving a Cauchy problem for ordinary differential equations involving the symmetric power sums; this implies the existence of a special kind of paths in the minimizer of some restriction of the considered polynomial function. In the final section, extensions of our results to the whole space $\mathbf{R}^n$ are outlined. The main results are Theorems 5.1 and 5.2 with Corollaries 2.1 and 5.2, and the corresponding results for $\mathbf{R}^n$ from the last subsection. Part II will contain a discussion on the ordered vector space $\mathcal{H}_d^{[n]}$ in general, as well as on the particular cases of degrees $d = 4$ and $d = 5$ (finite test sets for positivity in the homogeneous case and other sufficient criteria).

Keywords: Minimizer; Symmetric function; Symmetric power sums

1. Introduction and notations

Excepting the results of Hilbert and Artin on Hilbert's 17th Problem and of Pólya on strictly positive forms, together with their further refinements, there are only particular results on positivity,[1] even for symmetric polynomials (small degree or number of variables or restricted form). Our paper deals with minimizers of symmetric polynomials or of some more general symmetric functions on "curved simplices." We thus prove that a real symmetric polynomial inequality of degree $d \geqslant 2$ holds on $\mathbf{R}_+^n := [0, \infty[^n$ if and only if

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[1] Which will mean $\geqslant 0$, according to the terminology of ordered vector spaces.

doi:10.1016/S0022-247X(03)00301-9

it holds for elements with at most $\lfloor d/2 \rfloor$ (the integer part) distinct non-zero components. A similar result holds for arbitrary even degree and $\mathbf{R}^n$. For the convenience of the reader, we included in Section 2 simplified versions of some of our results, in a purely polynomial setting.

In our entire discussion we require that $n \in \mathbf{N}$, $n \geqslant 2$. Since $\mathbf{R}$ is an infinite field we can and will always identify polynomials in $\mathbf{R}[X_1, \ldots, X_n]$ with real polynomial functions on $\mathbf{R}^n$. Let us denote[2] by $\Sigma^{[n]}$ the $\mathbf{R}$-algebra of all real symmetric polynomials on $\mathbf{R}^n$ (i.e., $\Sigma^{[n]} := \mathbf{R}[X_1, \ldots, X_n]^{S_n}$).

For each $d \in \mathbf{N}$, consider the following vector subspaces of $\Sigma^{[n]}$:

$$\Sigma_d^{[n]} := \left\{ f \in \Sigma^{[n]} \mid \deg(f) \leqslant d \right\},$$
$$\mathcal{H}_d^{[n]} := \{ f \in \Sigma^{[n]} \mid f \text{ is } d\text{-homogeneous} \} \subset \Sigma_d^{[n]}.$$

For all numbers $a, b \in \mathbf{R}$, set $a \wedge b := \min\{a, b\}$, $a \vee b := \max\{a, b\}$, and $\overline{a, b} := \mathbf{Z} \cap [a, b]$ (possibly empty!). Let $\lfloor a \rfloor$ denote the integer part of a.

For every $x = (x_1, \ldots, x_n) \in \mathbf{R}^n$, set

$$\operatorname{supp}(x) := \{ j \in \overline{1, n} \mid x_j \neq 0 \},$$
$$v(x) := \left| \{ x_j \mid j \in \overline{1, n} \} \right|, \qquad v^*(x) := \left| \left\{ x_j \mid j \in \operatorname{supp}(x) \right\} \right| = \left| \{ x_j \mid x_j \neq 0 \} \right|,$$

where $|A|$ stands for the number of elements (cardinal) of a finite set A. Both functions $v, v^* : \mathbf{R}^n \to \overline{0, n}$ are lower semi-continuous. It is worth pointing out that v^* is counting the non-zero distinct components of its argument, *without their multiplicities*.

For each $k \in \mathbf{N}^*$, the kth symmetric power sum is defined by

$$P_k : \mathbf{R}^n \to \mathbf{R}, \quad P_k(x) := \sum_{j=1}^{n} x_j^k.$$

Some of our functions are characterized by more parameters than we indicate; in an expression, say E, we may sometimes indicate a number $r \neq n$ of variables[3] in the form $E^{[r]}$.

For any $\sigma > 0$ and any continuous function $f : \mathbf{R}_+^n \setminus \{0_n\} \to \mathbf{R}$, define the following sets:

$$K_\sigma := P_1^{-1}\big(\{\sigma\}\big) \cap \mathbf{R}_+^n = \left\{ x \in \mathbf{R}_+^n \mid P_1(x) = \sigma \right\},$$
$$K_\sigma^s := \left\{ x \in K_\sigma \mid v^*(x) \leqslant s \right\} \quad (s \in \mathbf{N}^*),$$
$$M_\sigma(f) := \operatorname{minimizer}(f|_{K_\sigma}) = \left\{ \xi \in K_\sigma \mid f(\xi) = \min_{x \in K_\sigma} f(x) \right\}.$$

The simplex $K_\sigma \subset \mathbf{R}_+^n \setminus \{0_n\}$ is a compact set, $]0, \infty[\cdot K_\sigma = \mathbf{R}_+^n \setminus \{0_n\}$, and the restriction $f|_{K_\sigma}$ attains its minimum on $M_\sigma(f) \neq \emptyset$.

Definition 1.1. A path $\gamma : [a, b] \to \mathbf{R}^n$ is said to be an (s)-path ($s \in \mathbf{N}^*$) provided

$$P_i \circ \gamma \equiv (P_i \circ \gamma)(a), \quad \forall i \in \overline{1, s}, \qquad \operatorname{supp}\big(\gamma(t)\big) = \operatorname{supp}\big(\gamma(a)\big), \quad \forall t \in [a, b].$$

[2] We need our notations to be suggestive, explicit and condensed at the time.

[3] E.g., for the symmetric power sum $P_k : \mathbf{R}^r \to \mathbf{R}$, which acts on r variables.

For each $k \in \mathbf{N}^*$, the kth complete symmetric function h_k is the sum of all monomials of total degree k on $\mathbf{R}^n$. We take by convention $h_0 \equiv 1$. In [7], it is shown that h_{2k} is positive definite ($h_{2k} > 0$ on $\mathbf{R}^n \setminus \{0_n\}$) for every $k \in \mathbf{N}^*$.

For all $i, j \in \mathbf{N}^*$ with $i \leqslant j$, we may consider the vector-valued function

$$P_{(i,j)} : \mathbf{R}^n \to \mathbf{R}^{j-i+1}, \quad P_{(i,j)}(x) := \big(P_i(x), \dots, P_j(x)\big).$$

For all $J \subset \overline{1,n}$ and $x \in \mathbf{R}^n$, set $\check{J} := \overline{1,n} \setminus J$ and $x_J := (x_j)_{j \in J}$.

For every $k \in \overline{1,n}$, write $0_k := (0, \dots, 0) \in \mathbf{R}^k$ and $1_k := (1, \dots, 1) \in \mathbf{R}^k$. If $x \in \mathbf{R}^n$, it is convenient to write x_k for its kth component, as we already have done. Therefore, we avoid to denote vectors with symbols with lower indexes (however, upper indexes will be allowed) and $0_k, 1_k$ are *the only exceptions* to this rule.

2. Main results

The results of this paper are, to the best of my knowledge, all new (excepting those for which we explicitly mention the contrary). In this section, we will state only the most relevant of them.

Let $f \in \Sigma_d^{[n]}$ and $\xi \in M_\sigma(f)$ for some $\sigma > 0$. Since the case $d \in \{0, 1\}$ is trivial ($f = aP_1 + b$ for some $a, b \in \mathbf{R}$), we will assume here that $d \geqslant 2$.

Theorem 2.1 (Of enlargement). *If $v^*(\xi) > \lfloor d/2 \rfloor$, then for each $\varepsilon > 0$, there exists an injective $(\lfloor d/2 \rfloor)$-path in $M_\sigma(f) \cap B(\xi, \varepsilon)$ which connects ξ to a point $\zeta \neq \xi$ satisfying*

$$v^*(\zeta) = \big|\operatorname{supp}(\zeta)\big|$$

(*i.e., all non-zero components of ζ are pairwise distinct*).

Theorem 2.2 (Of reduction). *There exists an $(\lfloor d/2 \rfloor)$-path in $M_\sigma(f)$ which connects ξ to a point ζ satisfying*

$$v^*(\zeta) \leqslant \lfloor d/2 \rfloor.$$

In particular, we have

$$\min_{x \in K_\sigma} f(x) = \min_{x \in K_\sigma^{\lfloor d/2 \rfloor}} f(x).$$

Note that the point $\zeta \in M_\sigma(f)$ from Theorems 2.1 and 2.2 also satisfies

$$\operatorname{supp}(\zeta) = \operatorname{supp}(\xi), \qquad P_i(\zeta) = P_i(\xi), \quad \forall i \in \overline{1, \lfloor d/2 \rfloor}.$$

The above results will be both stated and proved in a much more general setting in Section 5 (Theorems 5.1 and 5.2). For this, we will define there some new notions. The following corollary is a generalization of some known results on even symmetric forms of degree $2d$ from [3, Theorem 3.7, p. 567] (the case $2d = 6$) and [5, Theorem 2.3, p. 211] (the case $2d = 8$) and [5, Theorem 3.2, p. 215] (the case $2d = 10$, $n = 3$). Corollary 5.2 will be even more general.

Corollary 2.1.

(1) *A symmetric polynomial inequality of degree $d \in \mathbf{N}^*$ holds on $\mathbf{R}_+^n$ if and only if it holds on $\{x \in \mathbf{R}_+^n \mid v^*(x) \leqslant \lfloor d/2 \rfloor \vee 1\}$.*
(2) *A symmetric polynomial inequality of even degree $d \in \mathbf{N}^*$ holds on $\mathbf{R}^n$ if and only if it holds on $\{x \in \mathbf{R}^n \mid v(x) \leqslant (d/2) \vee 2\}$.*

Note that positivity for some $f \in \Sigma_d^{[n]}$ on $\mathbf{R}_+^n$ is equivalent to the positive (semi-)definiteness of $g \in \Sigma_{2d}^{[n]}$ defined by $g(x_1, \dots, x_n) := f(x_1^2, \dots, x_n^2)$.

3. Symmetric functions and curved simplices

It is well known (the fundamental theorem on symmetric functions) that every symmetric polynomial in n variables is uniquely expressible as a polynomial in the elementary symmetric functions $E_1, \dots, E_n$, and that the same is true for the complete symmetric functions $h_1, \dots, h_n$. In [10], it is shown that the algebra $\Sigma^{[n]}$ is also generated by every family of n monomial symmetric functions with degrees $1, 2, \dots, n$. The following result is well known, even if not exactly in this form (see [9, pp. 24–25]).

Theorem 3.1. *For every $f \in \Sigma_d^{[n]}$, there exists a unique polynomial $\tilde{f} : \mathbf{R}^{\bar{d}} \to \mathbf{R}$ ($\bar{d} := d \wedge n$), such that*

$$f = \tilde{f} \circ P_{(1,\bar{d})} = \tilde{f}(P_1, \dots, P_{\bar{d}}). \tag{1}$$

Moreover, f can be written in the form

$$f = g(P_1, \dots, P_{\lfloor d/2 \rfloor}) + \sum_{i=\lfloor d/2 \rfloor+1}^{d} g_i(P_1, \dots, P_{d-i}) \cdot P_i, \tag{2}$$

where g, g_i are polynomials and $g_i \equiv 0$ if $i > \bar{d}$. In particular, f depends only affinely[4] on each power sum P_i with $i > \lfloor d/2 \rfloor$.

Proof of Eq. (2). Let $f \in \Sigma_d^{[n]}$ be given. If f is expressed in the elementary symmetric functions $E_1, \dots, E_{\bar{d}}$, the most direct method to put it in the form (1) is to use the identities (see [9, p. 28])

$$k!E_k = \begin{vmatrix} P_1 & 1 & 0 & \dots & 0 \\ P_2 & P_1 & 2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ P_{k-1} & P_{k-2} & P_{k-3} & \dots & k-1 \\ P_k & P_{k-1} & P_{k-2} & \dots & P_1 \end{vmatrix}, \quad \forall k \in \overline{1, n}.$$

[4] By an affine function on a subset of a vector space, we mean a linear map plus a (constant) vector. Here, f is regarded (by a slight abuse) as an expression in $P_1, \dots, P_{\bar{d}}$.

In $\tilde{f}(P_1, \ldots, P_{\bar{d}})$, for reasons of degree, terms containing products as $P_i P_j$ with $i + j > d$ cannot appear. Therefore, power sums P_i with $i > \lfloor d/2 \rfloor$ can only occur in the first power. Now (2) follows easily. □

Note that in (2), we have $d - i \leqslant \lfloor d/2 \rfloor$ for every $i \geqslant \lfloor d/2 \rfloor + 1$, and so the polynomials g_i do not depend on P_k for $k \geqslant \lfloor d/2 \rfloor + 1$.

We can now summarize the idea of our construction. If $\gamma : [a, b] \to \mathbf{R}^n$ is an (s)-path for some fixed $s \geqslant \lfloor d/2 \rfloor$, and if $P_i \circ \gamma$ is an affine function for each $i \in \overline{s+1, d}$ (Lemma 4.1 will provide a method to find various such paths), then $f \circ \gamma$ is affine too, by (2). If a subset $K \subset \mathbf{R}^n$ can be characterized in terms of $P_1, \ldots, P_s$ (say by an equation of the form $\omega(P_1(x), \ldots, P_s(x)) = \sigma$) and if $\gamma(t_0) \in \text{minimizer}(f|_K)$ for some $t_0 \in]a, b[$, then $f \circ \gamma$ must be constant, and consequently, $\gamma([a, b]) \subset \text{minimizer}(f|_K)$. As we shall see, this pattern allows the construction of various kinds of (s)-paths and points in minimizer$(f|_K)$.

Our previous discussion motivates the introduction of the mathematical objects that we are studying: the classes of functions $\mathcal{F}_{d,s}^{[n]}$ and $\Omega_s^{[n]}$, the "ω-curved simplices" $K_\sigma(\omega) \subset \mathbf{R}_+^n$, and the minimizer $M_\sigma(f, \omega)$ of $f|_{K_\sigma(\omega)}$.

Definition 3.1. For all $d, s \in \mathbf{N}$, $d \geqslant s$, let us consider

(A) The space $\mathcal{G}_s^{[n]}$ of all functions $g : \mathbf{R}_+^n \setminus \{0_n\} \to \mathbf{R}$ of the form

$$g = \bar{g} \circ P_{(1,s)} \tag{3}$$

for some continuous map $\bar{g} :]0, \infty[^s \to \mathbf{R}$. We take by convention $\mathcal{G}_0^{[n]} := \mathbf{R}$.

(B) The space $\mathcal{F}_{d,s}^{[n]}$ of all functions $f : \mathbf{R}_+^n \setminus \{0_n\} \to \mathbf{R}$ of the form

$$f = g_s + \sum_{i=s+1}^{d} g_i P_i \tag{4}$$

for some $g_i \in \mathcal{G}_s^{[n]}$, $\forall i \in \overline{s, d}$.

(C) For each $s \in \mathbf{N}^*$, consider the subset $\Omega_s^{[n]} \subset \mathcal{G}_s^{[n]}$ of all functions $\omega : \mathbf{R}_+^n \setminus \{0_n\} \to]0, \infty[$ in $\mathcal{G}_s^{[n]}$, such that for every $x \in \mathbf{R}_+^n \setminus \{0_n\}$, the map

$$\omega_x :]0, \infty[\to]0, \infty[, \quad \omega_x(t) = \omega(tx),$$

is increasing and $\lim_{t \searrow 0} \omega_x(t) = 0$, $\lim_{t \to \infty} \omega_x(t) = \infty$.

Note that[5] $P_i \in \Omega_s^{[n]}$ for every $i \in \overline{1, s}$.

The following example shows that the functions from Definition 3.1 need not be polynomial.

[5] In the new setting of Section 5.2, this holds only for even numbers $i \in \overline{2, s}$.

Example 3.1.

$$g = 2^{P_2} + P_1^2 \log\bigl(1 + P_1^4 + P_2\bigr) \in \mathcal{G}_2^{[n]} \setminus \Omega_2^{[n]},$$
$$f = \sqrt[3]{P_1 - P_2} \cdot P_4 + \log\bigl(P_1^2 + P_2\bigr) \cdot P_3 + P_1 \sin(P_1 P_2) \in \mathcal{F}_{4,2}^{[n]},$$
$$\omega = 2^{P_2} - 1 + P_1^2 \log\bigl(1 + P_1^4 + P_2\bigr) \in \Omega_2^{[n]} \subset \mathcal{G}_2^{[n]}.$$

It is convenient, by a minor abuse of notation, to write (3) and (4) as

$$g = \bar{g}(P_1, \ldots, P_s),$$
$$f = \bar{g}_s(P_1, \ldots, P_s) + \sum_{i=s+1}^{d} \bar{g}_i(P_1, \ldots, P_s) \cdot P_i. \tag{5}$$

We have the obvious relations

$$\mathcal{F}_{d,d}^{[n]} = \mathcal{G}_d^{[n]} \supset \mathcal{F}_{d,s}^{[n]}. \tag{6}$$

For $g \in \mathcal{G}_s^{[n]}$, $f \in \mathcal{F}_{d,s}^{[n]}$ we always assume the representations (3) and (4). Note that for each $f \in \mathcal{F}_{d,s}^{[n]}$ we also have $f = \bar{f} \circ P_{(1,d)}$, where

$$\bar{f} :]0, \infty[^s \times \mathbf{R}_+^{d-s} \to \mathbf{R}, \quad \bar{f}(y) = \bar{g}_s(y_1, \ldots, y_s) + \sum_{i=s+1}^{d} \bar{g}_i(y_1, \ldots, y_s) \cdot y_i. \tag{7}$$

Remark 3.1. (1) The family $(\mathcal{F}_{d,s}^{[n]})_{d,s}$ is increasing with respect to d and s. Every $\mathcal{G}_s^{[n]}$ is an $\mathbf{R}$-algebra and every $\mathcal{F}_{d,s}^{[n]}$ is a $\mathcal{G}_s^{[n]}$-module.

(2) We have the natural inclusion $\Sigma_d^{[n]} \subset \mathcal{F}_{d,\lfloor d/2\rfloor}^{[n]}$. If $d \leqslant 2s + 1$, then

$$\mathcal{H}_d^{[n]} \subset \Sigma_d^{[n]} \subset \mathcal{F}_{d,s}^{[n]}.$$

Proposition 3.1. *If $d \leqslant 2s + 1$, then $d \wedge n \leqslant 2(s \wedge n) + 1$ and*

$$\mathcal{F}_{d,s}^{[n]} = \mathcal{F}_{d\wedge n, s\wedge n}^{[n]}.$$

In particular, we have $\mathcal{G}_s^{[n]} = \mathcal{G}_{s\wedge n}^{[n]}$ and $\Omega_s^{[n]} = \Omega_{s\wedge n}^{[n]}$.

Proof. Set $\bar{d} := d \wedge n$, $\bar{s} := s \wedge n$. The inequality $\bar{d} \leqslant 2\bar{s} + 1$ is immediate and "$\supset$" clearly holds by Remark 3.1(1). To prove "$\subset$," we can assume that $n < d$ and hence $\bar{d} = n$, since otherwise $\bar{d} = d$ and $\bar{s} = s$. Fix $f \in \mathcal{F}_{d,s}^{[n]}$. By Theorem 3.1, we have for every $i \geqslant n$, $P_i = \tilde{P}_i \circ P_{(1,n)}$ for some unique polynomial $\tilde{P}_i : \mathbf{R}^n \to \mathbf{R}$, and so $P_i \in \mathcal{G}_n^{[n]}$. We need to consider two cases.

(1) If $s \geqslant n$, then replacing in (5) P_i by $\tilde{P}_i \circ P_{(1,n)}$ for each $i > n$ gives by (6) $f \in \mathcal{G}_n^{[n]} = \mathcal{F}_{n,n}^{[n]} = \mathcal{F}_{\bar{d},\bar{s}}^{[n]}$, since $\bar{s} = n$.

(2) If $s < n < d \leqslant 2s + 1$, let $i \in \overline{n+1, d}$ be fixed. By Theorem 3.1, the polynomial $\tilde{P}_i$ depends affinely on each P_j, $j \in \overline{s+1, n}$, since clearly $i \leqslant d \leqslant 2s + 1 < 2j$. Therefore,

$P_i \in \mathcal{F}_{n,s}^{[n]}$. As $P_i \in \mathcal{F}_{n,s}^{[n]}$, $\forall i \in \overline{n+1, d}$, and $\mathcal{F}_{n,s}^{[n]}$ is a $\mathcal{G}_s^{[n]}$-module, by (5) we deduce that $f \in \mathcal{F}_{n,s}^{[n]} = \mathcal{F}_{\bar{d},\bar{s}}^{[n]}$. $\square$

Definition 3.2. For all $\omega \in \Omega_s^{[n]}$, $\sigma > 0$, and for every continuous $f : \mathbf{R}_+^n \setminus \{0_n\} \to \mathbf{R}$, define the following sets:

$$K_\sigma(\omega) := \omega^{-1}\big(\{\sigma\}\big) = \big\{x \in \mathbf{R}_+^n \mid \omega(x) = \sigma\big\},$$
$$K_\sigma^s(\omega) := \big\{x \in K_\sigma(\omega) \mid v^*(x) \leqslant s\big\},$$
$$M_\sigma(f, \omega) := \text{minimizer}(f|_{K_\sigma(\omega)}) = \Big\{\xi \in K_\sigma(\omega) \mid f(\xi) = \min_{x \in K_\sigma(\omega)} f(x)\Big\}.$$

If $\omega = P_1$, we write these sets as K_σ, K_σ^s, $M_\sigma(f)$, that is without ω.

Lemma 3.1. *The set*[6] $K_\sigma(\omega)$ *is compact and* $]0,\infty[\, \cdot K_\sigma(\omega) = \mathbf{R}_+^n \setminus \{0_n\}$. *Therefore,* $M_\sigma(f, \omega) \neq \emptyset$.

Proof. The set $K := K_1(P_2) = \{x \in \mathbf{R}_+^n \mid P_2(x) = 1\}$ is clearly compact. Dini's theorem applied to the sequences of functions $(F_\nu)_{\nu \in \mathbf{N}^*}$, $(G_\nu)_{\nu \in \mathbf{N}^*}$,

$$F_\nu, G_\nu : K \to\,]0, \infty[\,, \quad F_\nu(x) = \omega(\nu^{-1}x), \quad G_\nu(x) = \frac{1}{\omega(\nu x)},$$

shows that $\exists \nu \in \mathbf{N}^*$, such that $F_\nu < \sigma$ and $G_\nu < \sigma^{-1}$ on K, that is

$$\omega(\nu^{-1}x) < \sigma < \omega(\nu x), \quad \forall x \in K. \tag{8}$$

We claim that $K_\sigma(\omega) \subset [\nu^{-1}, \nu] \cdot K =: H$. To see this, fix $z \in K_\sigma(\omega)$. For $\lambda := \sqrt{P_2(z)} > 0$ and $x := \lambda^{-1}z \in K$, we have $\omega(\lambda x) = \sigma$, which leads by (8) and the monotony of ω_x to $\lambda \in]\nu^{-1}, \nu[$. We thus get $z = \lambda x \in H$, which proves our claim. As $K_\sigma(\omega) \subset H \subset \mathbf{R}_+^n \setminus \{0_n\}$ and H is compact and $K_\sigma(\omega)$ is closed in $\mathbf{R}_+^n \setminus \{0_n\}$, we deduce that $K_\sigma(\omega)$ is compact. Now fix $x \in K$. Since ω is continuous, by (8) we must have $tx \in K_\sigma(\omega)$ for some $t \in [\nu^{-1}, \nu]$, and so $x \in]0, \infty[\, \cdot K_\sigma(\omega) =: T$. Hence, $K \subset T$ forces $\mathbf{R}_+^n \setminus \{0_n\} =]0, \infty[\, \cdot K \subset T$, which yields $T = \mathbf{R}_+^n \setminus \{0_n\}$. $\square$

It is to note that every point $x \in \mathbf{R}_+^n \setminus \{0_n\}$ is projectively represented in $K_\sigma(\omega)$. This is important if f is homogeneous.

4. Construction of (s)-paths

The following lemma is the last important ingredient of our construction. Its interest is that it allows one to vary continuously the symmetric power sums, keeping some of them constant. *All properties* (i)–(vii) of the function given by this lemma are essential and will be used in Section 5.

[6] For Lemma $\widetilde{3.1}$ from Section 5.2, the sets $K_\sigma(\omega)$, $M_\sigma(f, \omega)$, $\mathbf{R}_+^n$ will be replaced by $\tilde{K}_\sigma(\omega)$, $\tilde{M}_\sigma(f, \omega)$, $\mathbf{R}^n$.

Lemma 4.1. *For every $\xi \in]0,\infty[^n$ with $r := v^*(\xi) \geqslant 2$, there exists a bounded interval $I =]\alpha,\beta[$ and a function $\varphi : \bar{I} \to \mathbf{R}^n_+$, satisfying*

(i) $t_0 := P_r(\xi) \in I$ *and* $\varphi(t_0) = \xi$;
(ii) φ *is continuous on* $\bar{I}$ *and* $\varphi(I) \subset]0,\infty[^n$;
(iii) $P_i \circ \varphi \equiv P_i(\xi)$ *for each* $i < r$, *but* $P_r(\varphi(t)) = t$ *for every* $t \in \bar{I}$;
(iv) $v^*(\varphi(t)) = r$ *for every* $t \in I$, *but* $v^*(\varphi(\alpha)) < r$ *and* $v^*(\varphi(\beta)) < r$;
(v) $\varphi(\alpha) \in]0,\infty[^n$ *or* $\varphi(\beta) \in]0,\infty[^n$;
(vi) $\varphi \in \mathcal{C}^\infty(I,\mathbf{R}^n)$, $\varphi'_j \neq 0$ *on* I, $\forall j \in \overline{1,n}$, *and* $(P_k \circ \varphi)' > 0$, $\forall k \geqslant r$;
(vii) *If* $r = n$, *then* $P_k \circ \varphi$ *is an affine function for each* $k < 2n$ *and*

$$(P_k \circ \varphi)' \equiv \frac{k}{n} h_{k-n}(\xi), \quad \forall k \in \overline{n, 2n-1}.$$

Proof. We can assume, by permuting coordinates in $\mathbf{R}^n$ if necessary, that $\xi = (\zeta_1 1_{n_1}, \ldots, \zeta_r 1_{n_r}) =: (\zeta_j 1_{n_j})_{j\in\overline{1,r}}$ for some $\zeta = (\zeta_j)_{j\in\overline{1,r}} \in]0,\infty[^r$ with $\zeta_1 < \zeta_2 < \cdots < \zeta_r$ and some $n_j \in \mathbf{N}^*$ $(j \in \overline{1,r})$, with $\sum_{j=1}^r n_j = n$. Set $t_0 := P_r(\xi)$, $D := \{z \in]0,\infty[^r \mid z_1 < z_2 < \cdots < z_r\} \subset \mathbf{R}^r$ and define the function $f = (f_1,\ldots,f_r) : \mathbf{R} \times D \to \mathbf{R}^r$ by

$$f_j(t,z) = \left(n_j r \prod_{k\neq j}(z_j - z_k)\right)^{-1}, \quad \forall j \in \overline{1,r}.$$

According to the Cauchy–Lipschitz theorem, there exists a unique maximal solution $\psi : I \to D$ (I an open interval, $t_0 \in I$) of the Cauchy problem

$$\psi' = f(t,\psi), \quad \psi(t_0) = \zeta. \tag{9}$$

Hence, $\psi \in \mathcal{C}^\infty(I,D)$, $\psi_1 < \psi_2 < \cdots < \psi_r$, and

$$\psi'_j = \left(n_j r \prod_{k\neq j}(\psi_j - \psi_k)\right)^{-1}, \quad \forall j \in \overline{1,r}. \tag{10}$$

Consequently, all ψ_j are strictly monotone. Now define

$$F : \mathbf{R}^r \to \mathbf{R}^n, \quad F(z) := (z_j 1_{n_j})_{j\in\overline{1,r}}, \qquad \varphi = F \circ \psi : I \to]0,\infty[^n.$$

Hence, $\varphi \in \mathcal{C}^\infty(I,\mathbf{R}^n)$, $\varphi(t_0) = F(\zeta) = \xi$, $\psi_1 = \varphi_1 \leqslant \varphi_2 \leqslant \cdots \leqslant \varphi_n$, and $v^*(\varphi(t)) = v^*(\psi(t)) = r$, $\forall t \in I$. The components of φ are the same as those of ψ, with multiplicities. We claim that (10) is equivalent to

$$\begin{cases} \sum_{j=1}^r n_j \psi_j^{i-1} \psi'_j \equiv 0, & \forall i \in \overline{1,r-1}, \\ \sum_{j=1}^r n_j \psi_j^{r-1} \psi'_j \equiv r^{-1}. \end{cases} \tag{11}$$

Indeed, solving (11) via Cramers rule and Vandermonde determinants leads to (10). By the definitions of F and φ, it follows that (11) is equivalent to

$$\begin{cases} (P_i \circ \varphi)' \equiv 0, & \forall i \in \overline{1,r-1}, \\ (P_r \circ \varphi)' \equiv 1, \end{cases}$$

which leads to $P_i \circ \varphi \equiv P_i(\xi)$, $\forall i \in \overline{1, r-1}$ and $(P_r \circ \varphi)(t) = t$, $\forall t \in I$, since $\varphi(t_0) = \xi$ and $P_r(\xi) = t_0$. We next show that I is a bounded interval and that φ can be continuously extended to $\bar{I}$. Indeed, I is bounded since

$$0 < t = P_r\bigl(\varphi(t)\bigr) < P_1\bigl(\varphi(t)\bigr)^r = P_1(\xi)^r, \quad \forall t \in I. \tag{12}$$

Therefore, $I =]\alpha, \beta[$ for some $\alpha, \beta \in \mathbf{R}$. Since by (10) and

$$0 < \psi_j < \sum_{i=1}^{r} n_i \psi_i = P_1 \circ \varphi \equiv P_1(\xi), \quad \forall j \in \overline{1, r}, \tag{13}$$

the components of ψ are strictly monotone and bounded, there exist some continuous extensions $\bar{\psi} : \bar{I} \to \bar{D} \subset \mathbf{R}_+^r$, $\bar{\varphi} : \bar{I} \to \mathbf{R}_+^n$ of ψ and φ. We have

$$\bar{\varphi}(t) = \bigl(\bar{\psi}_j(t) 1_{n_j}\bigr)_{j \in \overline{1,r}}, \quad v^*\bigl(\bar{\varphi}(t)\bigr) = v^*\bigl(\bar{\psi}(t)\bigr) \leqslant r, \quad \forall t \in \bar{I}.$$

Clearly, $\bar{\varphi}$ satisfies (i)–(iii) and the first parts of (iv) and (vi).

(iv) If $v^*(\bar{\psi}(\alpha)) = r$, then $\bar{\psi}(\alpha) \in \bar{D} \setminus \partial D = D$, and so ψ cannot be a maximal solution of (9). Therefore, $v^*(\bar{\varphi}(\alpha)) = v^*(\bar{\psi}(\alpha)) < r$ and similarly $v^*(\bar{\varphi}(\beta)) < r$. Consequently, $\bar{\varphi}$ satisfies (iv).

(v) If $\{\bar{\varphi}(\alpha), \bar{\varphi}(\beta)\} \cap]0, \infty[^n = \emptyset$, then $0 \leqslant \bar{\varphi}_1 \leqslant \cdots \leqslant \bar{\varphi}_n$ clearly forces $\bar{\varphi}_1(\alpha) = \bar{\varphi}_1(\beta) = 0$, which contradicts that $\bar{\varphi}_1 = \bar{\psi}_1$ is strictly monotone. Hence, $\bar{\varphi}$ satisfies (v).

(vi) Fix $k \geqslant r$. Since $P_k \circ \varphi = \sum_{j=1}^{r} n_j \psi_j^k$, it follows that the components of ψ' satisfy, on I, (11) and the additional equation

$$\sum_{j=1}^{r} n_j \psi_j^{k-1} \psi_j' = k^{-1} (P_k \circ \varphi)'.$$

Since the algebraic system of $r+1$ equations in the r components of ψ' is consistent, Rouché's theorem yields

$$\begin{vmatrix} 1 & 1 & \ldots & 1 & 0 \\ \psi_1 & \psi_2 & \ldots & \psi_r & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \psi_1^{r-1} & \psi_2^{r-1} & \ldots & \psi_r^{r-1} & r^{-1} \\ \psi_1^{k-1} & \psi_2^{k-1} & \ldots & \psi_r^{k-1} & k^{-1}(P_k \circ \varphi)' \end{vmatrix} = 0.$$

Developing this determinant with respect to the last column, it follows that

$$k^{-1} (P_k \circ \varphi)' \prod_{1 \leqslant i < j \leqslant r} (\psi_j - \psi_i) = r^{-1} \bigl(h_{k-r}^{[r]} \circ \psi\bigr) \prod_{1 \leqslant i < j \leqslant r} (\psi_j - \psi_i)$$

(for the cofactor of r^{-1}, see [9, pp. 40–41, relations (3.1) and (3.4)]). Since $\psi_1 < \cdots < \psi_r$, we can cancel and obtain

$$(P_k \circ \varphi)' = \frac{k}{r} \bigl(h_{k-r}^{[r]} \circ \psi\bigr) > 0. \tag{14}$$

(vii) Assume that $r = n$ and fix $k < 2n$. Since for $k \leqslant n$ the conclusion follows from (iii), assume that $k > n$. As $r = n$ forces $n_j = 1$ for all j and hence $\psi = \varphi$, (14) becomes $(P_k \circ \varphi)' = kn^{-1}(h_{k-n} \circ \varphi)$. By Theorem 3.1, we have $h_{k-n} = Q \circ P_{(1,k-n)}$ for some polynomial $Q:\mathbf{R}^{k-n}\to\mathbf{R}$. Now using (iii) and $k-n<n=r$ leads to $h_{k-n}\circ\varphi\equiv Q(P_{(1,k-n)}(\xi))=h_{k-n}(\xi)$, which proves (vii). □

For some other aspects concerning the variation of $P_{(1,n)}$ on $\mathbf{R}_+^n$, see [15].

5. Minimizers on ω-curved simplices

5.1. Symmetric inequalities on $\mathbf{R}_+^n$

Context. Unless otherwise stated, we will consider in this subsection the following setting:

$$f\in\mathcal{F}_{d,s}^{[n]},\quad \omega\in\Omega_s^{[n]},\quad \sigma>0,\quad \xi\in M_\sigma(f,\omega).$$

Furthermore, excepting Lemma 5.1, we will require that $d\leqslant 2s+1$.

Note that we must have $s\geqslant 1$ (see the definition of $\Omega_s^{[n]}$).

Our intention is to investigate the set $M_\sigma(f,\omega)$. An *important particular case* is $f\in\Sigma_d^{[n]}$, $\omega=P_1$. If f and ω are both homogeneous, the value of σ is irrelevant. Thus, our results also provide information on the behavior of polynomials $f\in\mathcal{H}_d^{[n]}$ on $\mathbf{R}_+^n$.

The following remark is important.

Remark 5.1. If $\gamma:[a,b]\to\mathbf{R}_+^n\setminus\{0_n\}$ is an (s)-path, then every function from $\mathcal{G}_s^{[n]}\supset\Omega_s^{[n]}$ is constant on $\gamma([a,b])$. If $f\in\mathcal{F}_{d,s}^{[n]}$, then

$$f\circ\gamma=g_s(\zeta)+\sum_{i=s+1}^{d}g_i(\zeta)(P_i\circ\gamma),\quad \forall\zeta\in\gamma\big([a,b]\big).$$

Theorem 5.1 (Of enlargement). *If $v^*(\xi)>s$, then for every $\varepsilon>0$, the point ξ is connected by an injective (s)-path in $M_\sigma(f,\omega)\cap B(\xi,\varepsilon)$ to a point $\zeta\neq\xi$ satisfying*[7]

$$v^*(\zeta)=\big|\operatorname{supp}(\zeta)\big|. \tag{15}$$

Proof. Fix $\varepsilon>0$ and set $M:=M_\sigma(f,\omega)$, $r:=v^*(\xi)>s$, $J:=\overline{1,r}$. As f and ω are symmetric, we can assume that $v^*(\xi_J)=r$. For $\xi_J\in]0,\infty[^r$, let $t_0:=P_r^{[r]}(\xi_J)$, $I:=]\alpha,\beta[$, $\varphi:\bar I\to\mathbf{R}_+^r$ be as in Lemma 4.1 (with n replaced by r) and define $\psi:\bar I\to\mathbf{R}_+^n$, $\psi(t):=(\varphi(t),\xi_{\check J})$. From the properties of φ, we see that ψ is continuous and injective, $\psi(t_0)=\xi$, and

$$P_{(1,r-1)}\circ\psi\equiv P_{(1,r-1)}(\xi),\quad (P_r\circ\psi)(t)=t+P_r^{[n-r]}(\xi_{\check J}),\quad \forall t\in\bar I, \tag{16}$$

$$\operatorname{supp}\big(\psi(t)\big)=\operatorname{supp}(\xi),\quad \forall t\in I. \tag{17}$$

Moreover, $P_i\circ\psi=P_i^{[r]}\circ\varphi+P_i^{[n-r]}(\xi_{\check J})$ is an affine function for each $i<2r$, since $P_i^{[r]}\circ\varphi$ is so, according to Lemma 4.1(vii). By (16) and Remark 5.1, we also deduce

[7] For Theorem $\widetilde{5.1}$ from Section 5.2, we will replace v^* by v and (15) by $v(\zeta)=n$.

that every function from $\mathcal{G}_s^{[n]} \supset \Omega_s^{[n]}$ is constant on $\psi(\bar{I}) \ni \xi$, since $s < r$. Therefore, $\omega \circ \psi \equiv \omega(\xi) = \sigma$ and

$$\psi(\bar{I}) \subset K_\sigma(\omega), \qquad f \circ \psi = g_s(\xi) + \sum_{i=s+1}^{d} g_i(\xi)(P_i \circ \psi). \tag{18}$$

Here $i \leqslant d < 2s + 2 \leqslant 2r$ shows that $f \circ \psi$ is an affine function (since all $P_i \circ \psi$ involved in the sum are so). But $t_0 \in I$ is a minimum point for $f \circ \psi$, and so $(f \circ \psi)'(t_0) = 0$ forces $f \circ \psi \equiv f(\xi)$, which leads to $\psi(\bar{I}) \subset M$. By Lemma 4.1, all r components of φ are continuous and injective functions, and consequently there exists $t_1 \in I$, $t_1 > t_0$, such that

$$v^*\big(\psi(t)\big) = r + v^*(\xi_{\check{J}}), \qquad \big\|\psi(t) - \xi\big\| < \frac{\varepsilon}{n}, \qquad \forall t \in]t_0, t_1]. \tag{19}$$

(1) If $v^*(\xi) = |\operatorname{supp}(\xi)|$, then $\xi_{\check{J}} = 0_{n-r}$, and so $\zeta := \psi(t_1) \in M$ and the (s)-path $\psi|_{[t_0,t_1]}$ have the requested properties, by (16), (17), and (19).

(2) If $v^*(\xi) < |\operatorname{supp}(\xi)|$, then $\xi_{\check{J}} \neq 0_{n-r}$. Consider $\xi^1 := \psi(t_1) \in M$ and $\psi^1 := \psi|_{[t_0,t_1]}$. By (16), (17), and (19), we get

$$v^*(\xi^1) > v^*(\xi), \qquad v^*\big(\psi^1(t)\big) = \begin{cases} v^*(\xi), & t = t_0, \\ v^*(\xi^1), & t \in]t_0, t_1], \end{cases} \tag{20}$$

$$\operatorname{supp}(\xi^1) = \operatorname{supp}(\xi), \qquad P_{(1,s)}(\xi^1) = P_{(1,s)}(\xi), \qquad \|\xi^1 - \xi\| < \frac{\varepsilon}{n}.$$

If $v^*(\xi^1) < |\operatorname{supp}(\xi^1)|$, we can apply the above arguments again, with ξ replaced by ξ^1, to obtain an injective (s)-path $\psi^2 : [\theta_0, \theta_1] \to M$ with $\psi^2(\theta_0) = \xi^1$, $\psi^2(\theta_1) =: \xi^2$, and properties of ψ^2, ξ^1, ξ^2 that are similar to those of ψ^1, ξ, ξ^1. By (20), we have

$$v^*\big(\psi^1(t)\big) = v^*(\xi^1) < v^*(\xi^2) = v^*\big(\psi^2(\theta)\big), \quad \forall t \in]t_0, t_1], \ \forall \theta \in]\theta_0, \theta_1],$$

and so the (s)-path given by the union of ψ^1 and ψ^2 (defined on the interval $[t_0, t_1 + \theta_1 - \theta_0]$) is injective. Thus, an induction argument finally proves, in at most $|\operatorname{supp}(\xi)| - r$ steps, the existence of $\zeta \in M$ which satisfies (15). The union of (s)-paths obtained at each step of the induction is injective. □

Lemma 5.1. *If $v^*(\xi) \geqslant d > s$, then $P_{(s+1,d)}(\xi)$ is a local minimum point for the partial function*[8]

$$\bar{f}\big(P_{(1,s)}(\xi), \cdot\big) : \mathbf{R}_+^{d-s} \to \mathbf{R}$$

(*see* (7)). *In particular, we have $g_i(\xi) = 0$ for every $i \in \overline{s+1, d}$.*

Proof. As f and ω are both symmetric and $v^*(\xi) \geqslant d$, we can certainly assume that $\xi_1 > \cdots > \xi_d$. For $\alpha := P_{(1,s)}(\xi) \neq 0_s$, $\beta := P_{(s+1,d)}(\xi)$, we shall prove that β is a local minimum point for $\bar{f}(\alpha, \cdot) : \mathbf{R}_+^{d-s} \to \mathbf{R}$. By hypothesis, we have $f(x) \geqslant f(\xi)$, $\forall x \in U_\xi \cap K_\sigma(\omega)$ for some neighborhood (nbd) U_ξ of ξ. Set $\zeta := (\xi_j)_{j \in \overline{1,d}} \in]0, \infty[^d$, $\eta := (\xi_j)_{j \in \overline{d+1,n}} \in \mathbf{R}_+^{n-d}$ and choose nbd V_ζ of ζ with $V_\zeta \subset]0, \infty[^d$ and $V_\zeta \times \{\eta\} \subset U_\xi$.

[8] For Lemma $\widetilde{5.1}$ from Section 5.2, we replace $v^*(\xi)$ by $v(\xi)$ and $\mathbf{R}_+^{d-s}$ by $\mathbf{R}^{d-s}$.

That $P_{(1,d)}(V_\zeta \times \{\eta\})$ is nbd of (α, β) follows by an application of the inverse mapping theorem for the function $P_{(1,d)}(\cdot, \eta) : V_\zeta \to]0, \infty[^d$ at ζ. Now choose nbd W_β of β with $\{\alpha\} \times W_\beta \subset P_{(1,d)}(V_\zeta \times \{\eta\})$. We claim that $\bar{f}(\alpha, z) \geqslant \bar{f}(\alpha, \beta)$, $\forall z \in W_\beta$. To prove this, fix $z \in W_\beta$. By the choice of W_β, we have $(\alpha, z) = P_{(1,d)}(x)$ for some $x \in V_\zeta \times \{\eta\} \subset U_\xi$. Hence, $P_{(1,s)}(x) = \alpha = P_{(1,s)}(\xi)$ and $\omega = \bar{\omega} \circ P_{(1,s)}$ lead to $\omega(x) = \bar{\omega}(\alpha) = \omega(\xi) = \sigma$, and so $x \in U_\xi \cap K_\sigma(\omega)$. We thus get $\bar{f}(\alpha, z) = f(x) \geqslant f(\xi) = \bar{f}(\alpha, \beta)$. To prove the last part, observe that

$$\bar{f}(\alpha, z) = g_s(\xi) + \sum_{i=s+1}^{d} g_i(\xi) z_i, \quad \forall z = (z_i)_{i \in \overline{s+1,d}} \in W_\beta,$$

depends affinely on z. As β is a minimum for $\bar{f}(\alpha, \cdot)$, we must have $g_i(\xi) = 0$, $\forall i \in \overline{s+1, d}$. $\square$

The following corollary provides in the case $v^*(\xi) > s$, $d > s$ some equations involving $\xi \in M_\sigma(f, \omega)$ and the “coefficients” $g_i \in \mathcal{G}_s^{[n]}$ of f. If $f \in \Sigma_d^{[n]}$, then all g_i are symmetric polynomials: g_d is constant, $g_{d-1} = \alpha P_1 + \beta$ for some $\alpha, \beta \in \mathbf{R}$, etc. Only Corollary 5.1(1), which is a refined version of Lemma 5.1, will be needed in the proof of Theorem 5.2.

Corollary 5.1. *If $v^*(\xi) > s$, then*[9]

(1) $|\operatorname{supp}(\xi)| \geqslant d > s \Rightarrow g_i(\xi) = 0$, $\forall i \in \overline{s+1, d}$;
(2) $|\operatorname{supp}(\xi)| = d - 1 > s \Rightarrow d g_d(\xi) P_1(\xi) + (d-1) g_{d-1}(\xi) = 0$.

Proof. Applying Theorem 5.1 for every $\varepsilon > 0$ shows the existence of a sequence $(\zeta^k)_{k \geqslant 1} \subset M_\sigma(f, \omega)$ satisfying

$$\lim_{k \to \infty} \zeta^k = \xi, \quad \operatorname{supp}(\zeta^k) = \operatorname{supp}(\xi), \quad v^*(\zeta^k) = \left|\operatorname{supp}(\xi)\right|, \quad \forall k \in \mathbf{N}^*. \tag{21}$$

(1) For every $k \in \mathbf{N}^*$, we have $g_i(\zeta^k) = 0$, $\forall i \in \overline{s+1, d}$, by Lemma 5.1. A passage to the limit ($k \to \infty$) now establishes the claimed equalities.

(2) As (21) shows that $v^*(\zeta^k) = |\operatorname{supp}(\zeta^k)| = |\operatorname{supp}(\xi)| = d - 1$, $\forall k \in \mathbf{N}^*$, it suffices to prove the required equality for each member of the sequence $(\zeta^k)_{k \geqslant 1}$; then the assertion follows by a passage to the limit. Therefore, we can assume that $v^*(\xi) = |\operatorname{supp}(\xi)|$ and that $\operatorname{supp}(\xi) = \overline{1, d-1}$ by the symmetry of f and ω. Define $\varphi : \bar{I} \to \mathbf{R}_+^r$, $\psi : \bar{I} \to \mathbf{R}_+^n$ exactly as in the proof of Theorem 5.1, with $r = d - 1$ and $J = \operatorname{supp}(\xi)$. As there, we obtain (18) and $f \circ \psi \equiv f(\xi)$. As $P_i^{[d-1]} \circ \varphi$ is constant for every $i < d - 1$, we get

$$0 \equiv (f \circ \psi)' = \sum_{i=s+1}^{d} g_i(\xi) \big(P_i^{[d-1]} \circ \varphi\big)' \equiv (d-1)^{-1} \sum_{i=d-1}^{d} i g_i(\xi) h_{i-d+1}^{[d-1]}(\xi_J).$$

Indeed, for $i \in \{d-1, d\}$ we can apply Lemma 4.1(vii), since $v^*(\xi) = d - 1$ and $d - 1 \leqslant i \leqslant 2(d-1) - 1$ (we have $d \geqslant 3$, because $d - 1 > s \geqslant 1$). $\square$

[9] For Corollary $\widetilde{5.1}$ from Section 5.2, we replace $v^*(\xi)$ by $v(\xi)$ and $|\operatorname{supp}(\xi)|$ by n.

Theorem 5.2 (Of reduction). *The point ξ is connected by an (s)-path in $M_\sigma(f,\omega)$ to a point ζ satisfying*[10]

$$v^*(\zeta) \leqslant s. \tag{22}$$

In particular, we have

$$\min_{x\in K_\sigma(\omega)} f(x) = \min_{x\in K_\sigma^s(\omega)} f(x).$$

Proof. Assume that $v^*(\xi) > s$ and hence $n > s$, since otherwise the conclusion is trivial. There is no loss of generality in assuming that $d > s$. Indeed, if $d = s$, then $f \in \mathcal{F}_{s,s}^{[n]} \subset \mathcal{F}_{s+1,s}^{[n]}$ and we replace d by $s+1 > d$. It suffices to prove that ξ is connected by an (s)-path in $M := M_\sigma(f,\omega)$ to a point ξ^1 satisfying

$$v^*(\xi^1) < v^*(\xi). \tag{23}$$

Set $r := v^*(\xi) > s$, $J := \operatorname{supp}(\xi)$, $m := |J| \geqslant r$. As f and ω are symmetric, we can assume that $J = \overline{1,m}$, and so $\xi = (\xi_J, 0_{n-m})$. For $\xi_J \in]0,\infty[^m$, let $t_0 := P_r^{[r]}(\xi_J) = P_r(\xi)$, $I :=]\alpha,\beta[$, $\varphi : \bar{I} \to \mathbf{R}_+^m$ be as in Lemma 4.1 (with n replaced by m) and define $\psi : \bar{I} \to \mathbf{R}_+^n$, $\psi(t) := (\varphi(t), 0_{n-m})$. We claim that $f \circ \psi$ is a constant function. The properties of φ yield

$$\psi(t_0) = \xi, \quad P_{(1,r-1)} \circ \psi \equiv P_{(1,r-1)}(\xi), \quad (P_r \circ \psi)(t) = t, \quad \forall t \in \bar{I}. \tag{24}$$

By (24) and Remark 5.1, every function from $\mathcal{G}_s^{[m]} \supset \Omega_s^{[m]}$ is constant on $\varphi(\bar{I})$, since $r > s$. Define $f^{[m]} \in \mathcal{F}_{d,s}^{[m]}$ and $\omega^{[m]} \in \Omega_s^{[m]}$ by

$$f^{[m]}(y) = f(y, 0_{n-m}), \qquad \omega^{[m]}(y) = \omega(y, 0_{n-m}).$$

Hence, $f \circ \psi = f^{[m]} \circ \varphi$ and $\xi_J \in M_\sigma(f^{[m]}, \omega^{[m]})$. By Proposition 3.1, we have $f^{[m]} \in \mathcal{F}_{\bar{d},s}^{[m]}$ for $\bar{d} := d \wedge m > s$, and consequently

$$f^{[m]} = q_s + \sum_{i=s+1}^{\bar{d}} q_i P_i^{[m]}$$

for some $q_i \in \mathcal{G}_s^{[m]}$, $\forall i \in \overline{s,\bar{d}}$. As $|\operatorname{supp}(\xi_J)| = m \geqslant \bar{d} > s$ and $v^*(\xi_J) > s$, Corollary 5.1(1) shows that $q_i(\xi_J) = 0$, $\forall i \in \overline{s+1,\bar{d}}$. Since all q_i $(i \in \overline{s,\bar{d}})$ are constant on $\varphi(\bar{I}) \ni \xi_J$, we have $f^{[m]} \circ \varphi \equiv q_s(\xi_J)$, which proves our claim. Therefore, $f \circ \psi \equiv (f \circ \psi)(t_0) = f(\xi)$. That $\psi(\bar{I}) \subset K_\sigma(\omega)$ follows from $\omega \circ \psi = \omega^{[m]} \circ \varphi \equiv (\omega^{[m]} \circ \varphi)(t_0) = \omega(\xi) = \sigma$. We thus conclude that $\psi(\bar{I}) \subset M$. But φ satisfies (iv) and (v) from Lemma 4.1, and so there exists $t_1 \in \{\alpha,\beta\}$, with $v^*(\psi(t_1)) < r = v^*(\xi)$ and $\operatorname{supp}(\psi(t_1)) = \operatorname{supp}(\xi)$. Hence, $\xi^1 := \psi(t_1) \in M$ satisfies (23) and it is connected to ξ by the (s)-path $\psi|_{[t_0\wedge t_1, t_0\vee t_1]}$ in M. If $v^*(\xi^1) > s$, we can apply all above arguments again, with ξ replaced by ξ^1. Thus, an obvious induction completes the proof. □

[10] For Theorem $\widetilde{5.2}$ from Section 5.2, we will replace (22) by $v(\zeta) \leqslant s$.

Corollary 5.2. *Consider* $f, g \in \mathcal{F}_{d,s}^{[n]}$, $\omega \in \Omega_s^{[n]}$, $\sigma > 0$, *with* $g > 0$ *on* $K_\sigma(\omega)$. *If* $d \leqslant 2s+1$, *then*

$$\min_{x \in K_\sigma(\omega)} \frac{f(x)}{g(x)} = \min_{x \in K_\sigma^s(\omega)} \frac{f(x)}{g(x)}.$$

Proof. For $\alpha := \min_{x \in K_\sigma(\omega)}(f(x)/g(x))$ and $h := f - \alpha g \in \mathcal{F}_{d,s}^{[n]}$, we clearly have $M_\sigma(h, \omega) = \text{minimizer}((f/g)|_{K_\sigma(\omega)})$, and the conclusion follows by applying Theorem 5.2 for h, ω and σ. □

Proof of Corollary 2.1(1). Fix $d \in \mathbf{N}^*$, $f \in \Sigma_d^{[n]}$, set $s := \lfloor d/2 \rfloor \vee 1$, and assume that $f > 0$ on $A := \{x \in \mathbf{R}_+^n \mid v^*(x) \leqslant s\}$. We shall prove that $f > 0$ on $\mathbf{R}_+^n$. To show this, fix $x \in \mathbf{R}_+^n \setminus A$ and set $\sigma := P_1(x)$. Since $0_n \in A$, we have $x \neq 0_n$, and so $\sigma > 0$. As $f \in \mathcal{F}_{d,s}^{[n]}$ by Remark 3.1(2) and $P_1 \in \Omega_s^{[n]}$, Theorem 5.2 shows that $\exists \zeta \in M_\sigma(f) \cap A \neq \emptyset$. But $x \in K_\sigma$ now leads to $f(x) \geqslant f(\zeta) > 0$. We conclude that $f > 0$ on $\mathbf{R}_+^n$. The reasoning for a non-strict inequality (i.e., $f \geqslant 0$ on A) is similar.

Corollary 2.1(2) will be proved in Section 5.2. □

Corollary 5.3. *If* $f \in \Sigma_3^{[n]}$, *then for every* $\sigma > 0$ *we have*

$$\min_{x \in K_\sigma} f(x) = \min_{1 \leqslant k \leqslant n} f(\sigma k^{-1} 1_k, 0_{n-k}).$$

Proof. Since $f \in \mathcal{F}_{3,1}^{[n]}$ and $P_1 \in \Omega_1^{[n]}$, the conclusion follows immediately by Theorem 5.2. □

In the homogeneous case we obtain the following corollary, which is known in the context of even symmetric sextics (see [3, Theorem 3.7]).

Corollary 5.4. *If* $f \in \mathcal{H}_d^{[n]}$ *and* $d \leqslant 3$, *then*

$$f \geqslant 0 \text{ on } \mathbf{R}_+^n \quad \Leftrightarrow \quad f(1_k, 0_{n-k}) \geqslant 0, \quad \forall k \in \overline{1, n}.$$

Remark 5.2. For $d \geqslant 4$ and $n \geqslant 2$, the previous equivalence is no longer true. Indeed, $f := P_2 P_{d-2} - P_1 P_{d-1} \in \mathcal{H}_d^{[n]}$ satisfies $f(1_k, 0_{n-k}) = 0$ for every $k \in \overline{1, n}$, but $f(x) = -\sum_{1 \leqslant i < j \leqslant n} x_i x_j (x_i - x_j)(x_i^{d-3} - x_j^{d-3}) \leqslant 0$ on $\mathbf{R}_+^n$ and $f(2, 1, 0_{n-2}) < 0$.

5.2. *Symmetric inequalities on* $\mathbf{R}^n$

Our theory (results and proofs) for $\mathbf{R}_+^n$ can be adapted to the study of symmetric polynomials or more general symmetric functions on $\mathbf{R}^n$. This can be done according to the following translation table (TT):

Symbol	To be substituted by	For every
$\mathbf{R}_+^k$	$\mathbf{R}^k$	$k \in \mathbf{N}^*$
$]0,\infty[^k$	$\mathbf{R}^k \setminus \{0_k\}$	$k \in \mathbf{N}^*$
$v^*(x)$	$v(x)$	$x \in \mathbf{R}^k$, $k \in \mathbf{N}^*$
$\operatorname{supp}(x)$	$\overline{1,k}$	$x \in \mathbf{R}^k$, $k \in \mathbf{N}^*$

Substitutions for $]0,\infty[^k$ by $\mathbf{R}^k \setminus \{0_k\}$ will be made in our previous statements and proofs only if the exponent is present (even if $k=1$), but $]0,\infty[$ will remain unchanged (e.g., in Definition 3.1(C), Lemma 3.1, etc.).

We first need the following correspondent of Lemma 4.1.

Lemma 5.2. *For every $\xi \in \mathbf{R}^n$ with $r := v(\xi) \geqslant 3$, there exists a bounded interval $I =]\alpha,\beta[$ and a function $\varphi : \bar{I} \to \mathbf{R}^n$, satisfying*

- (i) *$t_0 := P_r(\xi) \in I$ and $\varphi(t_0) = \xi$;*
- (ii) *φ is continuous on $\bar{I}$;*
- (iii) *$P_i \circ \varphi \equiv P_i(\xi)$ for each $i < r$, but $P_r(\varphi(t)) = t$ for every $t \in \bar{I}$;*
- (iv) *$v(\varphi(t)) = r$ for every $t \in I$, but $v(\varphi(\alpha)) < r$ and $v(\varphi(\beta)) < r$;*
- (vi) *$\varphi \in \mathcal{C}^\infty(I,\mathbf{R}^n)$ and $\varphi_j' \neq 0$ on I for every $j \in \overline{1,n}$;*
- (vii) *If $r = n$, then $P_k \circ \varphi$ is an affine function for each $k < 2n$ and*

$$(P_k \circ \varphi)' \equiv \frac{k}{n} h_{k-n}(\xi), \quad \forall k \in \overline{n, 2n-1}.$$

Proof. Replacing in the proof of Lemma 4.1 Eqs. (12) and (13) by

$$|t| = \bigl|P_r\bigl(\varphi(t)\bigr)\bigr| < P_2\bigl(\varphi(t)\bigr)^{r/2} = P_2(\xi)^{r/2}, \quad \forall t \in I, \tag{25}$$

$$\psi_j^2 < \sum_{i=1}^r n_i \psi_i^2 = P_2 \circ \varphi \equiv P_2(\xi), \quad \forall j \in \overline{1,r}, \tag{26}$$

then removing item (v) and the inequality from (14), and finally reading all with (TT) gives a valid proof. The hypothesis $r \geqslant 3$ is needed in (25). $\square$

Translating Definitions 1.1, 3.1, and 3.2 by (TT) leads to a *new notion* of (s)-path, *new classes of functions* $\tilde{\mathcal{G}}_s^{[n]}, \tilde{\mathcal{F}}_{d,s}^{[n]}, \tilde{\Omega}_s^{[n]}$ (we require that $d \geqslant s \geqslant 2$ for this new definition) consisting of functions defined on $\mathbf{R}^n \setminus \{0_n\}$ and *new sets* $\tilde{K}_\sigma(\omega), \tilde{K}_\sigma^s(\omega), \tilde{M}_\sigma(f,\omega)$. Let us consider the following setting:

$$f \in \tilde{\mathcal{F}}_{d,s}^{[n]}, \quad \omega \in \tilde{\Omega}_s^{[n]}, \quad \sigma > 0, \quad \xi \in \tilde{M}_\sigma(f,\omega), \quad d \leqslant 2s+1.$$

An *important particular case* is $f \in \Sigma_d^{[n]}$, $\omega = P_2$.

Claim 5.1. Translating by (TT) from Sections 3 and 5.1 *all results and proofs* which were given before the proof of Corollary 2.1(1) leads to *new results with valid proofs*.[11]

[11] We verified this by making all changes for the concerned results into the “tex.” file of this article. The result after typesetting makes sense and all works.

We will now refer to a translated result by adding "$\tilde{}$" over its number. Rewriting with (TT) and verifying those proofs that do not use Lemma 4.1 poses no problem. All other proofs were based on the initial assumption $v^*(\xi) > s$, which, upon translation by (TT), turns into $v(\xi) > s$. As $s \geqslant 2$, we have $v(\xi) \geqslant 3$, and therefore in the new setting we can use Lemma 5.2 instead of Lemma 4.1. The proof of Theorem $\widetilde{5.2}$ simplifies (we will have $J = \overline{1,n}$, $m := |J| = n \geqslant r := v(\xi) > s$, hence $f^{[m]} = f$, $\omega^{[m]} = \omega$, etc.). The detailed verification of the rewritten proofs is left to the reader. We thus get the correspondents of Theorems 5.1 and 5.2 and of Corollary 5.2.

Theorem 5.3 (Of enlargement). *If $v(\xi) > s$, then for every $\varepsilon > 0$, the point ξ is connected by an injective (s)-path in $\tilde{M}_\sigma(f,\omega) \cap B(\xi,\varepsilon)$ to a point $\zeta \neq \xi$ with pairwise distinct components.*

Theorem 5.4 (Of reduction). *The point ξ is connected by an (s)-path in $\tilde{M}_\sigma(f,\omega)$ to a point ζ satisfying $v(\zeta) \leqslant s$. In particular, we have*

$$\min_{x\in\tilde{K}_\sigma(\omega)} f(x) = \min_{x\in\tilde{K}_\sigma^s(\omega)} f(x).$$

Corollary 5.5. *Consider $f, g \in \tilde{\mathcal{F}}_{d,s}^{[n]}$, $\omega \in \tilde{\Omega}_s^{[n]}$, $\sigma > 0$, with $g > 0$ on $\tilde{K}_\sigma(\omega)$. If $d \leqslant 2s+1$, then*

$$\min_{x\in\tilde{K}_\sigma(\omega)} \frac{f(x)}{g(x)} = \min_{x\in\tilde{K}_\sigma^s(\omega)} \frac{f(x)}{g(x)}.$$

Proof of Corollary 2.1(2). Fix $d \in \mathbf{N}^*$, $f \in \Sigma_d^{[n]}$, set $s := (d/2) \vee 2$, and assume that $f > 0$ on $A := \{x \in \mathbf{R}^n \mid v(x) \leqslant s\}$. We shall prove that $f > 0$ on $\mathbf{R}^n$. To show this, fix $x \in \mathbf{R}^n \setminus A$ and set $\sigma := P_2(x)$. Since $0_n \in A$, we have $x \neq 0_n$, and so $\sigma > 0$. As $f \in \tilde{\mathcal{F}}_{d\vee 2,s}^{[n]}$ by Remark $\widetilde{3.1}$ and $P_2 \in \tilde{\Omega}_s^{[n]}$, Theorem 5.4 shows that $\exists\zeta \in M_\sigma(f) \cap A \neq \emptyset$. But $x \in K_\sigma$ now leads to $f(x) \geqslant f(\zeta) > 0$. We conclude that $f > 0$ on $\mathbf{R}^n$. The proof for a non-strict inequality (i.e., $f \geqslant 0$ on A) is similar. □

The last part of the following corollary is an equivalent form of Theorem 2.3 in [5].

Corollary 5.6. *If $f \in \mathcal{H}_4^{[n]}$, then*

$$f \geqslant 0 \text{ on } \mathbf{R}^n \quad\Leftrightarrow\quad f(t\cdot 1_k, 1_{n-k}) \geqslant 0, \quad \forall t \in [-1,1],\ \forall k \in \overline{1,n-1},$$
$$f \geqslant 0 \text{ on } \mathbf{R}_+^n \quad\Leftrightarrow\quad f(t\cdot 1_k, 1_l, 0_{n-k-l}) \geqslant 0, \quad \forall t \in [0,1],\ \forall k,l \in \mathbf{N},\ k+l \leqslant n.$$

For the results of Hilbert and Artin on Hilbert's 17th Problem and of Pólya on strictly positive forms, we refer the reader to [1,6] and [4]. Bounds for the exponent from Pólya's theorem are given in [8,13]. Various symmetric inequalities can be found especially in [4, 11], but also in [2,7,12,14].

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