Available online at www.sciencedirect.com
SCIENCE DIRECT®

J. Math. Anal. Appl. 303 (2005) 90–102

Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS
www.elsevier.com/locate/jmaa

Integral estimates for convergent positive series

Vlad Timofte

Département de Mathématiques, École Polytechnique Fédérale de Lausanne, 1015 Lausanne, Switzerland

Received 3 February 2004

Available online 24 November 2004

Submitted by H.R. Parks

Abstract

It is shown that for every $\alpha > 1$, we have

$$\sum_{k=n+1}^{\infty} \frac{1}{k^{\alpha}} = \frac{1}{(\alpha-1)(n+\theta_n)^{\alpha-1}}$$

for some strictly decreasing sequence $(\theta_n)_{n\geqslant 1}$ such that

$$\frac{1}{2} < \theta_n < \frac{1}{4}\left[1+\left(1+\frac{1}{2n+1}\right)^{\alpha}\right],$$

hence with $\lim_{n\to\infty}\theta_n = \frac{1}{2}$. This is only a particular case of more general new results on series defined by convex functions.

Keywords: Positive series; Convex function; Harmonic series; Partial sum

E-mail address: vlad.timofte@epfl.ch.

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2004.07.004

1. Introduction

Let $f:[1,\infty[\,\to\,]0,\infty[$ be a convex differentiable function, such that the series $\sum_{n\geqslant 1} f(n)$ converges. We will show that

$$\sum_{k=n+1}^{\infty} f(k) = \int_{n+\theta_n}^{\infty} f(t)\,dt \quad \text{for every } n \geqslant 1, \tag{1}$$

for some unique sequence $(\theta_n)_{n\geqslant 1} \subset\,]\frac{1}{2}, 1[$. Under reasonable assumptions the sequence is strictly decreasing to $\frac{1}{2}$. In this case, among all integral expressions $\int_{n+\alpha}^{\infty} f(t)\,dt$, the best asymptotic approximation for series' nth remainder is obtained for $\alpha = \frac{1}{2}$. As we shall see (Proposition 1 and Theorem 3), this "half integer" optimality is *strongly related to slow convergence* ($\lim_{n\to\infty} \frac{f(n+1)}{f(n)} = 1$) of the series. If the ratio test limit is less than 1, then $\frac{1}{2}$ is *no longer optimal*.

Let us recall that approximations for partial sums in terms of $n+\frac{1}{2}$ were used in [2] for the harmonic series (slowly divergent!), and in a hidden form in [3]. In the latter, for the alternating harmonic series (slowly convergent!), nth remainder's absolute value is expressed as

$$\left| \sum_{k=n+1}^{\infty} \frac{(-1)^{k-1}}{k} \right| = \frac{1}{2n+x_n}.$$

The main result from [3] states that the sequence $(x_n)_{n\geqslant 1}$ is strictly decreasing and provides good estimates for its convergence to 1. If we write this series as $\sum_{n\geqslant 1}(-1)^{n-1} g(n)$ for $g(x) = \frac{1}{x}$, then

$$\frac{1}{2n+x_n} = \frac{1}{2} g\left(n + \frac{x_n}{2}\right).$$

Thus the theorem from [3] actually has a half integer approximation nature. This was also pointed out in [4], where the results from [3] were generalized for Leibniz series defined by convex functions.

Our main results (Theorems 3, 6, and 9) are in the spirit of [3,4] and hold in particular for $f(x) = 1/x^{\alpha}$, with $\alpha > 1$, hence for all convergent generalized harmonic series. For instance, in the particular case $\alpha = 2$ we have

$$\frac{1}{n+\frac{1}{2}} > \frac{\pi^2}{6} - \sum_{k=1}^{n} \frac{1}{k^2} = \frac{1}{n+\theta_n}$$

for some strictly decreasing sequence $(\theta_n)_{n\geqslant 1}$, with

$$\theta_1 = \frac{6}{\pi^2-6} - 1 \approx 0.5505461$$

and

$$\frac{1}{2} < \theta_n < \frac{1}{2}\left[1 + \frac{1}{\sqrt{4(n+1)^2+1} + 2(n+1)}\right] < \frac{1}{2} + \frac{1}{8(n+1)}$$

(the first majorant of θ_n is given by (14)).

2. Existence and convergence of $(\theta_n)_{n\geqslant 1}$

Let us observe that (1) depends only on the restriction $f|_{[\frac{3}{2},\infty[}$. Therefore, we shall consider a continuous function $f : [1,\infty[\to]0,\infty[$, which is subject to the following conditions:

(i) the series $\sum_{n\geqslant 1} f(n)$ converges,
(ii) $f|_{[\frac{3}{2},\infty[}$ is convex.

Let us note that $f|_{[\frac{3}{2},\infty[}$ must be strictly decreasing. Set $S_n := \sum_{k=1}^{n} f(k)$ for every $n \in \mathbb{N}^* := \mathbb{N} \setminus \{0\}$ and $S := \lim_{n\to\infty} S_n$. Since $\int_1^\infty f(t)\,dt < \infty$ according to the integral test, we can define

$$F : [1,\infty[\to \mathbb{R}, \qquad F(x) = -\int_x^\infty f(t)\,dt.$$

Obviously, F is the unique primitive of f vanishing at infinity. Hence F is strictly increasing and $F|_{[\frac{3}{2},\infty[}$ is strictly concave.

Let us recall that any convex continuous $g : [a,b] \to \mathbb{R}$ satisfies the well-known Hadamard inequalities

$$g\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a}\int_a^b g(t)\,dt \leqslant \frac{g(a)+g(b)}{2}, \tag{2}$$

and both inequalities are strict if g is not an affine function.[1]

Proposition 1. *There exists a unique sequence $(\theta_n)_{n\geqslant 1} \subset [\frac{1}{2}, 1[$, such that*

$$S_n - S = F(n+\theta_n) \quad \text{for every } n \in \mathbb{N}^*. \tag{3}$$

This sequence depends only on the restriction $f|_{]\frac{3}{2},\infty[}$. We have the estimates

$$F\left(n+\frac{1}{2}\right) \leqslant S_n - S \leqslant F(n+1) - \frac{f(n+1)}{2}, \tag{4}$$

$$\frac{1}{2} \leqslant \theta_n < \frac{1}{4}\left[1 + \frac{f(n+\frac{1}{2})}{f(n+1)}\right], \tag{5}$$

for every $n \in \mathbb{N}^*$. In particular, if $\lim_{n\to\infty} \frac{f(n+1)}{f(n)} = 1$, then $\lim_{n\to\infty} \theta_n = \frac{1}{2}$.

Proof. Let us define the sequences $(X_n)_{n\geqslant 1}$ and $(Y_n)_{n\geqslant 1}$ by

$$X_n := S_n - S - F\left(n+\frac{1}{2}\right), \qquad Y_n := S_n - S - F(n+1) + \frac{f(n+1)}{2}.$$

[1] That is, $g(x) = \lambda x + \mu$ for some $\lambda, \mu \in \mathbb{R}$.

By (2) we deduce that $(X_n)_{n\geqslant 1}$ is decreasing and $(Y_n)_{n\geqslant 1}$ is increasing. As $\lim_{n\to\infty} X_n = \lim_{n\to\infty} Y_n = 0$, it follows that $Y_n \leqslant 0 \leqslant X_n$ for every $n \in \mathbb{N}^*$. We thus get (4), as well as the existence of a unique sequence $(\theta_n)_{n\geqslant 1} \subset [\frac{1}{2}, 1[$ satisfying (3), since F is continuous and strictly increasing.

It remains to prove (5). For every $n \in \mathbb{N}^*$, using (2) and (4) yields

$$\frac{f(n+\frac{1}{2}) - f(n+1)}{4} \geqslant \int_{n+\frac{1}{2}}^{n+1} f(t)\,dt - \frac{f(n+1)}{2} \geqslant S_n - S - F\left(n+\frac{1}{2}\right)$$
$$= F(n+\theta_n) - F\left(n+\frac{1}{2}\right)$$
$$\geqslant \left(\theta_n - \frac{1}{2}\right) f\left(n + \frac{2\theta_n + 1}{4}\right) \geqslant \left(\theta_n - \frac{1}{2}\right) f\left(n+\frac{3}{4}\right).$$

We thus get

$$\theta_n - \frac{1}{2} \leqslant \frac{f(n+\frac{1}{2}) - f(n+1)}{4f(n+\frac{3}{4})} < \frac{f(n+\frac{1}{2}) - f(n+1)}{4f(n+1)},$$

that is, (5). We also have

$$\theta_n < \frac{1}{4}\left[1 + \frac{f(n)}{f(n+1)}\right]$$

for every $n \geqslant 2$, which proves the last statement. □

Remark 2.

(a) If $f|_{]\frac{3}{2},\infty[}$ is differentiable or strictly convex, then (4) and (5) hold with strict inequalities.
(b) If $f|_{[\frac{3}{2},\infty[}$ is differentiable, then

$$0 < S_n - S - F\left(n+\frac{1}{2}\right) < -\frac{f'(n+\frac{1}{2})}{8} \quad \text{for every } n \in \mathbb{N}^*. \tag{6}$$

(a) follows from the strict inequalities $X_n > 0 > Y_n$. Suppose that $X_{n_0} = 0$ for some $n_0 \in \mathbb{N}^*$, that is, $X_{n+1} = X_n$ for $n \geqslant n_0$. It follows that $f|_{[n-\frac{1}{2},n+\frac{1}{2}]}$ is affine (equality in (2)) for every $n > n_0$. Thus, $f|_{]\frac{3}{2},\infty[}$ must be differentiable, since it is not strictly convex. We deduce that $f|_{[n_0+\frac{1}{2},\infty[}$ is affine, which is absurd, because $f > 0$ and $\lim_{n\to\infty} f(n) = 0$. Hence $X_n > 0$. The proof of the inequality $Y_n < 0$ is similar.

For (b) we combine (4), the second order Taylor expansion of $F(x)$ (at $n+1$, for $x = n+\frac{1}{2}$) with remainder in derivative form, and the monotony of f'.

Our next result provides a convergence test for the sequence $(\theta_n)_{n\geqslant 1}$, as well as the value of its limit. Let us define

$$L:[0,1]\to\mathbb{R},\qquad L(x)=\begin{cases}1, & \text{if } x=0,\\ \ln\bigl(\frac{x\ln x}{x-1}\bigr)/\ln x, & \text{if } x\in\,]0,1[,\\ \frac{1}{2}, & \text{if } x=1.\end{cases}$$

It is easy to check that L is continuous and $\frac{1}{2}\leqslant L\leqslant 1$. Hence $L([0,1])=[\frac{1}{2},1]$.

Theorem 3. *If* $\lim_{x\to\infty}\frac{f(x+t)}{f(x)}$ *exists*[2] *for every* $t\in[0,1]$, *then the sequence* $(\theta_n)_{n\geqslant 1}$ *converges. For* $a:=\lim_{n\to\infty}\frac{f(n+1)}{f(n)}\in[0,1]$, *we have*

$$\lim_{n\to\infty}\theta_n=L(a). \tag{7}$$

Proof. Let us first observe that $\omega(t):=\lim_{x\to\infty}\frac{f(x+t)}{f(x)}\in[0,1]$ exists for every $t\geqslant 0$ (we can obtain it as a finite product of limits as in our statement), and that $\omega:[0,\infty[\,\to[0,1]$ is decreasing, since so is $f|_{[\frac{3}{2},\infty[}$. It is easily seen that $\omega(t+s)=\omega(t)\omega(s)$ for all $t,s\geqslant 0$. It follows that $\omega(t)=a^t$ for every $t>0$, where $a=\omega(1)\in[0,1]$. To prove (7) we need to analyze three cases.

Case 1. If $a=1$, the conclusion follows by Proposition 1.

Case 2. If $a\in\,]0,1[$, then for every $n\in\mathbb{N}^*$ we have $z_n:=S_n-S-F(n+\theta)=F(n+\theta_n)-F(n+\theta)=(\theta_n-\theta)f(n+\lambda_n)$ for some $\lambda_n\in\,]\frac{1}{2},1[$, by the mean value theorem of Lagrange. We thus get

$$|\theta_n-\theta|=\frac{|z_n|}{f(n+\lambda_n)}\leqslant\frac{|z_n|}{f(n+1)}\quad\text{for every } n\in\mathbb{N}^*. \tag{8}$$

We next prove by applying Cesaro–Stolz theorem (0/0) that $\lim_{n\to\infty}\frac{z_n}{f(n+1)}=0$ for suitable θ. An easy computation leads for $n\geqslant 2$ to

$$\frac{z_n-z_{n-1}}{f(n+1)-f(n)}=\frac{1}{f(n+1)/f(n)-1}\times\left[1-\frac{f(n+\theta-1)}{f(n)}\int_0^1\frac{f(n+\theta-1+t)}{f(n+\theta-1)}\,dt\right].$$

As Lebesgue's theorem shows that

$$\lim_{n\to\infty}\int_0^1\frac{f(n+\theta-1+t)}{f(n+\theta-1)}\,dt=\int_0^1 a^t\,dt=\frac{a-1}{\ln a},$$

[2] For instance, if f is log-convex (that is, $\ln(f)$ is a convex function).

we have

$$\lim_{n\to\infty}\frac{z_n}{f(n+1)}=\lim_{n\to\infty}\frac{z_n-z_{n-1}}{f(n+1)-f(n)}=\frac{1}{a-1}\left(1-\frac{1}{a^{1-\theta}}\frac{a-1}{\ln a}\right).$$

Since this limit is 0 for $\theta=L(a)$, the conclusion follows by (8).

Case 3. If $a=0$, let $\varepsilon\in\,]0,1[$ and $\alpha:=1-\frac{\varepsilon}{2}\in\,]\frac{1}{2},1[$. As $\omega\equiv 0$, there exists $n_\varepsilon\in\mathbb{N}^*$, such that $\frac{f(x+\frac{\varepsilon}{2})}{f(x)}<\frac{\varepsilon}{3}$ for every $x\in[n_\varepsilon,\infty[$. Thus, $\frac{f(n+1)}{f(n+\alpha)}<\frac{\varepsilon}{3}$ for $n\geqslant n_\varepsilon$. If we prove that $\theta_n>1-\varepsilon$ for every $n\geqslant n_\varepsilon$, the assertion follows. On the contrary, suppose that $\theta_m\leqslant 1-\varepsilon<\alpha$ for some $m\geqslant n_\varepsilon$. By (3) and the concavity of $F<0$, it follows that

$$\begin{aligned}S-S_m&=-F(m+\theta_m)>F(m+\alpha)-F(m+\theta_m)\geqslant(\alpha-\theta_m)f(m+\alpha)\\&\geqslant\frac{\varepsilon}{2}f(m+\alpha)>\frac{3}{2}f(m+1).\end{aligned}$$

Let us observe that

$$\frac{f(n+1)}{f(n)}<\frac{f(n+\frac{\varepsilon}{2})}{f(n)}<\frac{\varepsilon}{3}\quad\text{for } n>n_\varepsilon,$$

and hence

$$\frac{f(m+k)}{f(m+1)}\leqslant\left(\frac{\varepsilon}{3}\right)^{k-1}\quad\text{for every } k\in\mathbb{N}^*.$$

We thus get

$$S-S_m=\sum_{k=1}^{\infty}f(m+k)\leqslant\frac{f(m+1)}{1-\frac{\varepsilon}{3}}<\frac{3}{2}f(m+1),$$

a contradiction. We conclude that $\lim_{n\to\infty}\theta_n=1$. □

As Example 7 will show, all numbers from $[\frac{1}{2},1]$ are potential limits of the sequence $(\theta_n)_{n\geqslant 1}$.

3. Monotony of $(\theta_n)_{n\geqslant 1}$

The sequence $(\theta_n)_{n\geqslant 1}$ need not be monotone in general.

Example 4. Let us consider the function

$$f:[1,\infty[\,\to\mathbb{R},\qquad f(x)=\begin{cases}\frac{8x^2-25x+21}{4}, & x\in[1,\frac{3}{2}],\\ \frac{3-x}{4}, & x\in[\frac{3}{2},2],\\ \frac{1}{x^2}, & x\in[2,\infty[.\end{cases}$$

Then f is continuously differentiable and convex,

$$\sum_{n=1}^{\infty}f(n)=\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6},$$

$\lim_{n\to\infty}\theta_n=\frac{1}{2}$, but $\frac{1}{2}<\theta_1<\theta_2$. Therefore $(\theta_n)_{n\geqslant 1}$ is not monotone.

We have $f(n) = \frac{1}{n^2}$ for every $n \in \mathbb{N}^*$. Some easy computations show that $f \in C^1([1, \infty[)$ and

$$f'(x) = \begin{cases} \frac{16x-25}{4}, & x \in [1, \frac{3}{2}], \\ -\frac{1}{4}, & x \in [\frac{3}{2}, 2], \\ -\frac{2}{x^3}, & x \in [2, \infty[, \end{cases} \qquad F(x) = \begin{cases} -\frac{(x-3)^2+3}{8}, & x \in [\frac{3}{2}, 2], \\ -\frac{1}{x}, & x \in [2, \infty[. \end{cases}$$

We see that f' is increasing, that is, f is convex. That $\lim_{n\to\infty} \theta_n = \frac{1}{2}$ follows by Proposition 1. We have $S_1 - S = F(1 + \theta_1)$, $S_2 - S = F(2 + \theta_2)$, and so

$$\theta_1 = 2 - \sqrt{\frac{4\pi^2}{3} - 11} < 0.5305, \qquad \theta_2 = \frac{12}{2\pi^2 - 15} - 2 > 0.532.$$

Lemma 5. *Let $g : [a, b] \to \mathbb{R}$ be a continuous function and $c \in\,]a, b[$, such that*

$$\int_a^b g(t)\,dt = g(c)(b - a).$$

Assume $g|_{]a,b[}$ to be twice differentiable, with $g' \neq 0$ and $\frac{g''}{g'}$ monotone.[3] *If g and $\frac{g''}{g'}$ have opposite monotonies, then*

$$g(b) - g(a) \leqslant g'(c)(b - a). \tag{9}$$

If g and $\frac{g''}{g'}$ have the same monotony, then converse inequality holds in (9). *Strict inequality holds if $\frac{g''}{g'}$ is strictly monotone.*

Proof. We shall assume that $-g$ and $\frac{g''}{g'}$ are increasing on $]a, b[$, hence that $g' < 0$ (the proof is similar in all other cases). Fix a primitive $G : [a, b] \to \mathbb{R}$ of g, and define $u : [a, c] \times [c, b] \to \mathbb{R}$, $u(x, y) = G(y) - G(x) - g(c)(y - x)$.

Step 1. We first show that there is a unique function $\varphi : [a, c] \to [c, b]$ satisfying

$$u\bigl(x, \varphi(x)\bigr) = 0 \quad \text{for every } x \in [a, c]. \tag{10}$$

Let us observe that $\frac{\partial u}{\partial x}(x, y) = g(c) - g(x)$ and $\frac{\partial u}{\partial y}(x, y) = g(y) - g(c)$, and consequently the partial functions $u(x, \cdot) : [c, b] \to \mathbb{R}$ and $u(\cdot, y) : [a, c] \to \mathbb{R}$ are strictly decreasing for all fixed $x \in [a, c]$, $y \in [c, b]$. From this, it follows that $u(x, c) \geqslant u(c, c) = 0 = u(a, b) \geqslant u(x, b)$, with strict inequalities if $x \in\,]a, c[$. As u is continuous, there exists a unique solution $y =: \varphi(x) \in [c, b]$ of the equation $u(x, y) = 0$. We thus get the required implicit function $\varphi : [a, c] \to [c, b]$. Let us note that $\varphi(a) = b$, $\varphi(c) = c$, and $\varphi(]a, c[) \subset\,]c, b[$.

Step 2. We next prove that φ is continuous, $\varphi|_{]a,c[}$ is differentiable, and

$$\varphi'(x) = \frac{g(x) - g(c)}{g(\varphi(x)) - g(c)} < 0 \quad \text{for every } x \in\,]a, c[. \tag{11}$$

[3] This is related to the convexity or concavity of $\ln(|g'|)$ on $]a, b[$.

The differentiability of $\varphi|_{]a,c[}$ and relation (11) follow by applying the implicit function theorem to u at every point $(x, \varphi(x)) \in\ ]a, c[\times]c, b[$. As by (11) $\varphi|_{]a,c[}$ is decreasing, both limits $\lambda_a := \lim_{x\searrow a} \varphi(x)$ and $\lambda_c := \lim_{x\nearrow c} \varphi(x)$ exist in $[c, b]$. Since passages to the limit in (10) lead to $u(a, \lambda_a) = 0 = u(a, \varphi(a))$ and $u(c, \lambda_c) = 0 = u(c, \varphi(c))$, by the uniqueness of φ we deduce that $\lambda_a = \varphi(a)$ and $\lambda_c = \varphi(c)$. We conclude that φ is continuous.

Step 3. We finally prove the required inequality from (9). The continuous function $h : [a, c] \to \mathbb{R},\ h(x) = g(\varphi(x)) - g(x) - g'(c)(\varphi(x) - x)$ is differentiable on $]a, c[$. For every $x \in\ ]a, c[$, using (11) leads to

$$\frac{h'(x)}{g(x)-g(c)} = \frac{g'(\varphi(x))-g'(c)}{g(x)-g(c)}\varphi'(x) - \frac{g'(x)-g'(c)}{g(x)-g(c)}$$
$$= \frac{g'(\varphi(x))-g'(c)}{g(\varphi(x))-g(c)} - \frac{g'(x)-g'(c)}{g(x)-g(c)} = \frac{g''(b_x)}{g'(b_x)} - \frac{g''(a_x)}{g'(a_x)} \geqslant 0$$

for some $x < a_x < c < b_x < \varphi(x)$, as follows by applying Cauchy's theorem for the differentiable functions g' and g. Hence h is increasing, and consequently $0 = h(c) \geqslant h(a) = g(b) - g(a) - g'(c)(b - a)$. □

Theorem 6. *Assume $f|_{]\frac{3}{2},\infty[}$ to be twice differentiable. If the function $\frac{f''}{f'}$ is monotone (strictly or not), then the sequence $(\theta_n)_{n\geqslant 1}$ has the opposite monotony. Furthermore, the limit $\lim_{n\to\infty} \frac{f(n+1)}{f(n)} =: a$ exists and (7) holds.*

Proof. We shall assume that $\frac{f''}{f'}$ is increasing on $]\frac{3}{2}, \infty[$. The proof is similar in the case of strict monotony. By (3), we have the recurrence relation

$$F(n + \theta_n) - F(n - 1 + \theta_{n-1}) = f(n) \quad \text{for every } n \geqslant 2. \tag{12}$$

Step 1. Let us show that there is a unique function $\Theta :]\frac{3}{2}, \infty[\ \to\ ]\frac{1}{2}, 1[$ satisfying

$$F\big(x + \Theta(x)\big) - F\big(x + \Theta(x) - 1\big) = f(x) \quad \text{for every } x \in \left]\frac{3}{2}, \infty\right[. \tag{13}$$

Define $v :]\frac{3}{2}, \infty[\times [\frac{1}{2}, 1] \to \mathbb{R}$, $v(x, y) = F(x + y) - F(x + y - 1) - f(x)$ and fix $x \in\ ]\frac{3}{2}, \infty[$. The partial function $v(x, \cdot)$ is strictly decreasing, since $\frac{\partial v}{\partial y}(x, y) = f(x + y) - f(x + y - 1) < 0$. As F is strictly concave, we have $v(x, 1) = F(x + 1) - F(x) - f(x) < 0$. By (2) we deduce that $v(x, \frac{1}{2}) \geqslant 0$. Assume that $v(x, \frac{1}{2}) = 0$, that is, $f|_{]x-\frac{1}{2},x+\frac{1}{2}[}$ is affine. Since $f''(x) = 0$ and $\frac{f''}{f'} \leqslant 0$ is increasing, it follows that $f''|_{[x,\infty[} \equiv 0$, hence that $f|_{[x,\infty[}$ is affine. This is absurd, because $f > 0$ and $\lim_{n\to\infty} f(n) = 0$. Thus, $v(x, \frac{1}{2}) > 0 > v(x, 1)$. As v is continuous, there exists a unique solution $y =: \Theta(x) \in\ ]\frac{1}{2}, 1[$ of the equation $v(x, y) = 0$. We thus get the required implicit function $\Theta :]\frac{3}{2}, \infty[\ \to\ ]\frac{1}{2}, 1[$.

Step 2. We next prove that Θ is decreasing. Applying the implicit function theorem to v at every point $(x, \Theta(x)) \in]\frac{3}{2}, \infty[\times]\frac{1}{2}, 1[$ shows that Θ is differentiable. We also have

$$\Theta'(x) = -\frac{\frac{\partial v}{\partial x}(x, \Theta(x))}{\frac{\partial v}{\partial y}(x, \Theta(x))} = \frac{f(x + \Theta(x)) - f(x + \Theta(x) - 1) - f'(x)}{f(x + \Theta(x) - 1) - f(x + \Theta(x))} \leqslant 0,$$

the last inequality being a consequence of Lemma 5 (applied for $g = f$ and $a = x + \Theta(x) - 1,\ b = x + \Theta(x),\ c = x$). Hence Θ is decreasing.

Step 3. We continue by showing that $(\theta_n)_{n\geqslant 1}$ is decreasing and satisfies[4]

$$\theta_n \leqslant \Theta(n+1) \quad \text{for every } n \in \mathbb{N}^*. \tag{14}$$

Set $T_n := S_n - F(n + \Theta(n))$ for every $n \in \mathbb{N}^*$. The following equivalent statements hold, since so does the last one:

$$\begin{aligned} T_{n+1} \geqslant T_n \quad &\overset{(13)}{\Longleftrightarrow} \quad F\big(n + \Theta(n)\big) \geqslant F\big(n + 1 + \Theta(n+1)\big) - f(n+1) \\ &\qquad\qquad = F\big(n + \Theta(n+1)\big) \\ &\overset{F\uparrow}{\Longleftrightarrow} \quad \Theta(n) \geqslant \Theta(n+1). \end{aligned}$$

Hence the sequence $(T_n)_{n\geqslant 1}$ is increasing. As $\lim_{n\to\infty} T_n = S$, we have $T_n \leqslant S$ for every $n \in \mathbb{N}^*$. The following equivalent statements hold for every $n \geqslant 2$, since so does the last one:

$$\begin{aligned} \theta_{n-1} \geqslant \theta_n \quad &\overset{F\uparrow,(12)}{\Longleftrightarrow} \quad F(n + \theta_{n-1}) \geqslant F(n + \theta_n) = F(n - 1 + \theta_{n-1}) + f(n) \\ &\Longleftrightarrow \quad v(n, \theta_{n-1}) \geqslant 0 = v\big(n, \Theta(n)\big) \\ &\overset{v(n,\cdot)\downarrow}{\Longleftrightarrow} \quad \theta_{n-1} \leqslant \Theta(n) \\ &\overset{(3),F\uparrow,(13)}{\Longleftrightarrow} \quad S_{n-1} - S = F(n - 1 + \theta_{n-1}) \leqslant F\big(n - 1 + \Theta(n)\big) \\ &\qquad\qquad = F\big(n + \Theta(n)\big) - f(n) \\ &\Longleftrightarrow \quad T_n \leqslant S. \end{aligned}$$

We conclude that $(\theta_n)_{n\geqslant 1}$ is decreasing, and that (14) holds.

The last part of our statement follows by Theorem 3 if we prove that the limit $\lim_{x\to\infty} \frac{f(x+t)}{f(x)}$ exists for each $t \in]0, 1]$. The function $\rho_t :]\frac{3}{2}, \infty[\to]0, 1]$, $\rho_t(x) = \frac{f'(x+t)}{f'(x)}$ is increasing, since

$$\rho_t'(x) = \rho_t(x)\left(\frac{f''(x+t)}{f'(x+t)} - \frac{f''(x)}{f'(x)}\right) \geqslant 0.$$

Hence $\lim_{x\to\infty} \rho_t(x)$ exists, and so $\lim_{x\to\infty} \frac{f(x+t)}{f(x)}$ exist too, by l'Hôpital's rule. □

[4] With strict inequality, if $\frac{f''}{f'}$ is strictly increasing.

Example 7.

(a) For $f(x) = \frac{1}{x^\alpha}$ $(\alpha > 1)$, we have $\lim_{n\to\infty} \theta_n = \frac{1}{2} = L(1)$, and $(\theta_n)_{n\geqslant 1}$ is strictly decreasing.
(b) For $f(x) = a^x$ $(a \in]0,1[)$, we have $\lim_{n\to\infty} \theta_n = L(a)$, and $(\theta_n)_{n\geqslant 1}$ is constant.
(c) For $f(x) = \mathrm{e}^{-x^2}$, we have $\lim_{n\to\infty} \theta_n = 1 = L(0)$, and $(\theta_n)_{n\geqslant 1}$ is strictly decreasing.

That the above statements hold is clear by Theorem 6.

4. An iterative method

Let us observe that for every $n \in \mathbb{N}^*$, the expression

$$S_n - S - F\left(n + \frac{1}{2}\right) = \sum_{k=n+1}^{\infty} \left[\int_{k-\frac{1}{2}}^{k+\frac{1}{2}} f(t)\,dt - f(k) \right]$$

is the nth remainder of a convergent series associated to a function $g : [\frac{3}{2}, \infty[\to [0, \infty[$. If g is convex, then inequalities (4) may be applied to this new series. Furthermore, under suitable assumptions we may repeat this argument again. This reasoning justifies our following construction.

For every $a \in \mathbb{R}$, let $\mathcal{F}_a$ denote the real vector space consisting of all continuous functions $h : [a, \infty[\to \mathbb{R}$. Let us consider the linear operator

$$J_a : \mathcal{F}_a \to \mathcal{F}_{a+\frac{1}{2}}, \qquad J_a h(x) = \int_{x-\frac{1}{2}}^{x+\frac{1}{2}} h(t)\,dt - h(x).$$

Set $\mathcal{F} := \bigcup_{a\in\mathbb{R}} \mathcal{F}_a$ and define $J : \mathcal{F} \to \mathcal{F}$, such that $J|_{\mathcal{F}_a} = J_a$ for every $a \in \mathbb{R}$. The result of $J(Jh)$ will be written as J^2h, and so on. The needed properties of J are collected in the following lemma.

Lemma 8. *Let $h \in \mathcal{F}_a$.*

(a) *For all $m, n \in \mathbb{N}$ with $m > n \geqslant a - \frac{1}{2}$, we have*

$$-\sum_{r=n+1}^{m} h(r) + \int_{n+\frac{1}{2}}^{m+\frac{1}{2}} h(t)\,dt = \sum_{r=n+1}^{m} Jh(r).$$

(b) *If h vanishes at infinity, then so does Jh.*
(c) *If h is continuously differentiable, then so is Jh and $(Jh)' = J(h')$.*
(d) *If h is strictly convex, then $Jh > 0$.*

(e) *If h is twice differentiable, then for every $x \geqslant a + \frac{1}{2}$, there exists $\xi \in]-\frac{1}{2}, \frac{1}{2}[$, such that*

$$Jh(x) = \frac{h''(x+\xi)}{24}.$$

Proof. Properties (b)–(d) are obvious, and (a) follows by a trivial computation. To prove (e), let us observe that for every $x \geqslant a + \frac{1}{2}$, a third order Taylor expansion of $[0, \frac{1}{2}] \ni u \mapsto \int_{x-u}^{x+u} h(t)\,dt \in \mathbb{R}$ at 0 shows that

$$Jh(x) = \int_{x-\frac{1}{2}}^{x+\frac{1}{2}} h(t)\,dt - h(x) = \frac{h''(x+\eta) + h''(x-\eta)}{48}$$

for some $\eta \in]0, \frac{1}{2}[$. As h'' has the intermediate value property, we must have $\frac{h''(x+\eta)+h''(x-\eta)}{2} = h''(x+\xi)$ for some $\xi \in [-\eta, \eta] \subset]-\frac{1}{2}, \frac{1}{2}[$. □

Theorem 9. *Assume f to be $2p+2$ times continuously differentiable ($p \in \mathbb{N}$), with $f^{(2p+2)} > 0$. Set*

$$\sigma_p := \sum_{k=0}^{p} (-1)^k J^k F, \qquad \varepsilon_p(n) := J^p F(n+1) - J^p F\left(n + \frac{1}{2}\right) - \frac{J^p f(n+1)}{2}.$$

Then for every $n \geqslant \frac{p+1}{2}$ we have

$$0 < (-1)^{p+1}\left[S - S_n + \sigma_p\left(n + \frac{1}{2}\right)\right] < \varepsilon_p(n) < -\frac{f^{(2p+1)}(n - \frac{p-1}{2})}{8 \cdot 24^p}. \tag{15}$$

Note that $\sigma_0 = F,\ \sigma_1 = F - JF,\ \sigma_2 = F - JF + J^2F$, and so on.

Proof. We can assume that $p \in \mathbb{N}^*$, since otherwise the conclusion follows by Remark 2(b). Fix $n \in \mathbb{N}^*,\ n \geqslant \frac{p+1}{2}$.

Step 1. We first prove the equality

$$(-1)^{p+1}\left[S - S_n + \sigma_p\left(n + \frac{1}{2}\right)\right] + J^p F\left(n + \frac{1}{2}\right) = -\sum_{r=n+1}^{\infty} J^p f(r). \tag{16}$$

Fix $m > n$. Repeated application of Lemma 8(b, c) yields $\lim_{x\to\infty} J^k F(x) = 0$ and $(J^k F)' = J^k f \in \mathcal{F}_{1+\frac{k}{2}}$. By Lemma 8(a) we deduce that

$$-\sum_{r=n+1}^{m} J^k f(r) + J^k F\left(m + \frac{1}{2}\right) - J^k F\left(n + \frac{1}{2}\right) = \sum_{r=n+1}^{m} J^{k+1} f(r),$$

hence that the series $\sum_{r\geqslant n+1} J^k f(r)$ converges for every $k\in\{0,1,\ldots,p\}$, since it does so for $k=0$ ($J^0 f = f$). We thus get

$$-\sum_{r=n+1}^{\infty} J^k f(r) - J^k F\left(n+\frac{1}{2}\right) = \sum_{r=n+1}^{\infty} J^{k+1} f(r) \quad \text{for } k\in\{0,1,\ldots,p-1\}.$$

Summation of the above equalities multiplied by $(-1)^{k+1}$ leads to (16).

Step 2. We next show the inequalities

$$J^p F\left(n+\frac{1}{2}\right) < -\sum_{r=n+1}^{\infty} J^p f(r) < J^p F\left(n+\frac{1}{2}\right) + \varepsilon_p(n). \tag{17}$$

Let us observe that $J^p f \in \mathcal{F}_{1+\frac{p}{2}}$ is convex, since according to Lemma 8(c, e), for every $x \geqslant 1+\frac{p}{2}$ we have

$$\left(J^p f\right)''(x) = J^p(f'')(x) = \frac{f^{(2p+2)}(x+\xi)}{24^p} > 0$$

for some $\xi \in \left]-\frac{p}{2}, \frac{p}{2}\right[$. Therefore $J^p f$ is strictly convex and differentiable, and consequently it can be extended to a function $g : [1,\infty[\to]0,\infty[$ keeping these properties. For the convergent series $\sum_{s\geqslant 1} g(s+n-1)$, applying (4) for $s=1$ now gives

$$\int_{\frac{3}{2}}^{\infty} g(t+n-1)\,dt > \sum_{s=2}^{\infty} g(s+n-1) > \int_{2}^{\infty} g(t+n-1)\,dt + \frac{g(2)}{2},$$

which yields (17), since $g|_{[1+\frac{p}{2},\infty[} = J^p f$ and $n+\frac{1}{2} \geqslant 1+\frac{p}{2}$.

Step 3. We finally prove (15). The first two estimates are just a combination of (16) and (17). Thus it remains to show the last inequality. As for Remark 2(b) we deduce that

$$\varepsilon_p(n) < -\frac{(J^p f)'(n+\frac{1}{2})}{8}.$$

As $f^{(2p+1)}$ is strictly increasing and a repeated application of Lemma 8(c, e) yields

$$\left(J^p f\right)'\left(n+\frac{1}{2}\right) = J^p(f')\left(n+\frac{1}{2}\right) = \cdots = \frac{f^{(2p+1)}(n+\frac{1}{2}+\xi)}{24^p}$$

for some $\xi \in \left]-\frac{p}{2}, \frac{p}{2}\right[$, the inequality follows. □

Let us note that the last expression of (15) provides an a priori error estimate; for fixed $\varepsilon > 0$, it can be used to find suitable p, n. The following example shows that the error made by using $S_n - \sigma_p(n+\frac{1}{2})$ as an approximation for S may be surprisingly small even for small values of n and p.

Example 10. We shall apply Theorem 9 for $f : [1, \infty[\to]0, \infty[$, $f(x) = \frac{1}{x^3}$ and $p = 1$. Some easy computations show that

$$f'''(n) = -\frac{60}{n^6}, \qquad F(x) = -\frac{1}{2x^2}, \qquad Jf(x) = \frac{8x^2 - 1}{x^3(2x+1)^2(2x-1)^2},$$
$$JF(x) = -\frac{1}{2x^2(2x-1)(2x+1)}, \qquad \sigma_1\left(n + \frac{1}{2}\right) = -\frac{(2n+1)^2 - 2}{2n(n+1)(2n+1)^2},$$
$$\varepsilon_1(n) = \frac{10(n+1)^2 - 1}{2n(n+1)^3(2n+1)^2(2n+3)^2}.$$

By Theorem 9, we have

$$0 < S - S_n + \sigma_1\left(n + \frac{1}{2}\right) < \varepsilon_1(n) < \frac{5}{16n^6} \quad \text{for every } n \in \mathbb{N}^*.$$

Let us note that $\varepsilon_1(2) = \frac{89}{132300} < 0.7 \cdot 10^{-3}$, $\varepsilon_1(4) = \frac{83}{3267000} < 0.3 \cdot 10^{-4}$, and $\varepsilon_1(12) < 0.8 \cdot 10^{-7}$.

For high precision approximations (80 correct digits) for sums of generalized harmonic series with exponent $\alpha \in \{2, 3, \ldots, 251\}$ we refer the reader to [1].

References

[1] M. Carmignani, G. Puleo, A. Tortorici Macaluso, Calculating to high precision the Euler–Mascheroni constant and generalized harmonic series. First applications to the calculation of the function $\Gamma(x)$, Atti Accad. Sci. Lett. Arti Palermo Parte I (4) 40 (1980/81) 211–223 (in Italian).

[2] D.W. DeTemple, S.H. Wang, Half integer approximations for the partial sums of the harmonic series, J. Math. Anal. Appl. 160 (1991) 149–156.

[3] L. Tóth, J. Bukor, On the alternating series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$, J. Math. Anal. Appl. 282 (2003) 21–25.

[4] V. Timofte, On Leibniz series defined by convex functions, J. Math. Anal. Appl. 300 (2004) 160–171.