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Representation of a class of locally convex (M)-lattices

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ABSTRACT

We prove a representation theorem for Hausdorff locally convex (M)-lattices which are Dedekind σ-complete, and whose topologies are order σ-continuous and monotonically complete. These turn out to be the weighted spaces $c_0(T,\mathcal{H})$, defined in the paper for $T \neq \emptyset$ and $\mathcal{H} \subset \mathbb{R}_+^T$. We also characterize the dual of $c_0(T,\mathcal{H})$, as the space $l^1(T,\mathcal{H})$ defined in the last section. The known representation (on $c_0(T)$) of Banach (M)-lattices with order continuous norm follows as a particular case. We obtain these results by first proving a new general isomorphism theorem, which seems to be of independent interest. Our notion of "monotonic topological completeness" is weaker than the usual completeness and seems to be very convenient in the framework of topological ordered vector spaces.

1. INTRODUCTION AND NOTATIONS

As for prerequisites, the reader is expected to be familiar with notions, basic properties, and some results on ordered vector spaces from [2–5,7].

By *ordered topological vector space* (OTVS*) we mean any partially ordered topological vector space. *No connection between ordering and topology* is assumed.[3] On the other hand, by *topological ordered vector space* (TOVS) we mean, as usual, any OTVS* with locally full topology. Following [2], we will use the

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[3] The star from our notation OTVS* is intended to recall this convention.

notation X_τ for an OTVS* X with topology τ. If two such spaces X_τ and Y_η are isomorphic as OTVS* (algebraically, topologically, and as ordered sets), we will write this as $X_\tau \simeq Y_\eta$. The aim of this paper is to describe a class of locally convex (M)-lattices; these are TOVS and lattices, with the topology defined by a family of solid (M)-seminorms. Recall that a seminorm p defined on a vector lattice X is called an (M)-*seminorm*, if and only if

$$p(x \vee y) = p(x) \vee p(y) \quad \text{for all } x, y \in X_+.$$

The seminorm p is said to be *solid*, if and only if

$$|x| \leqslant |y| \Longrightarrow p(x) \leqslant p(y) \quad (x, y \in X).$$

For arbitrary set $T \neq \emptyset$, the function spaces

$$\begin{aligned}
\mathbb{R}^T &= \{f \mid f : T \to \mathbb{R} \text{ is a function}\},\\
c_0(T) &= \bigl\{f \in \mathbb{R}^T \mid f^{-1}(\mathbb{R} \setminus [-\varepsilon, \varepsilon]) \text{ is finite for every } \varepsilon > 0\bigr\},\\
c_{00}(T) &= \bigl\{f \in \mathbb{R}^T \mid f^{-1}(\mathbb{R} \setminus \{0\}) \text{ is finite}\bigr\},\\
l^\infty(T) &= \bigl\{f \in \mathbb{R}^T \mid f \text{ is bounded}\bigr\},\\
l^p(T) &= \left\{f \in \mathbb{R}^T \;\middle|\; \sum_{t \in T} |f(t)|^p < \infty\right\} \quad (p > 0),
\end{aligned}$$

are vector lattices (Riesz spaces in [4,7]) with respect to the pointwise ordering. We have the obvious inclusions

$$c_{00}(T) \subset l^p(T) \subset c_0(T) \subset l^\infty(T) \subset \mathbb{R}^T.$$

Definition 1. For any subset $\mathcal{H} \subset \mathbb{R}^T_+$ satisfying the restriction

$$\bigcap_{h \in \mathcal{H}} h^{-1}\bigl(\{0\}\bigr) = \emptyset, \tag{1}$$

let us consider the vector space

$$c_0(T, \mathcal{H}) := \bigl\{\psi \in \mathbb{R}^T \mid \psi \cdot \mathcal{H} \subset c_0(T)\bigr\}.$$

On $c_0(T, \mathcal{H})$ we consider the pointwise ordering, as well as the topology θ defined by the family of solid (M)-seminorms $(\| \|_h)_{h \in \mathcal{H}}$, where

$$\| \|_h : c_0(T, \mathcal{H}) \to \mathbb{R}_+, \quad \|\psi\|_h = \|\psi h\|_\infty = \sup_{t \in T}\bigl|\psi(t)h(t)\bigr|.$$

If needed, we can assume $\mathcal{H}$ to be directed upwards. Indeed, we have

$$c_0(T, \mathcal{H}) = c_0\bigl(T, \mathcal{H}^\vee\bigr) = c_0\bigl(T, \mathrm{I}(\mathcal{H})_+\bigr),$$

where $\mathcal{H}^\vee$ denotes the set consisting of all pointwise supremums of nonempty finite subsets of $\mathcal{H}$, and $\mathrm{I}(\mathcal{H}) \subset \mathbb{R}^T$ is the ideal (normal subspace in [2]) generated by $\mathcal{H}$.

Proposition 2.

(i) *$c_0(T, \mathcal{H})$ is a Hausdorff locally convex (M)-lattice and an ideal in $\mathbb{R}^T$ (hence Dedekind complete). Also, $c_0(T, \mathcal{H})$ is an $l^\infty(T)$-module containing $c_{00}(T)$.*

(ii) *The topology θ is complete and order continuous.*[4]

(iii) *For every $\psi \in c_0(T, \mathcal{H})_+$, there exists an increasing net $(\psi_\delta)_{\delta\in\Delta} \subset c_{00}(T)_+$, such that $\bigvee_{\delta\in\Delta} \psi_\delta = \psi$. For every such net, we have (θ)-$\lim_{\delta\in\Delta} \psi_\delta = \psi$.*

(iv) *If $\mathcal{H} \subset c_0(T)_+$, then $\mathbb{R} \subset l^\infty(T) \subset c_0(T, \mathcal{H})$, and $1 \in \mathbb{R}$ is a weak order unit*[5] *in $c_0(T, \mathcal{H})$.*

Proof. The proof is routine. □

In the last section, Example 26 will deal with some particular spaces $c_0(T, \mathcal{H})$.

2. MONOTONIC TOPOLOGICAL COMPLETENESS

In this section we introduce the notion of "monotonic topological completeness". It is worth pointing out that a large amount of known results on TOVS still hold if we replace the usual topological completeness by monotonic completeness.

Definition 3 (*Monotonic completeness*). Let X be an OTVS*. The topology τ of X is said to be *monotonically complete*, if and only if every monotonic (τ)-Cauchy net in X is (τ)-convergent. In the same way we define *monotonic σ-completeness* of the topology, by considering sequences instead of nets.

Example 4. The vector space

$$BV([0,1]) = \{x : [0,1] \to \mathbb{R} \mid x \text{ has bounded variation}\},$$

is a Dedekind complete lattice with respect to the usual ordering defined by the cone

$$BV([0,1])_+ = \{x : [0,1] \to \mathbb{R}_+ \mid x \text{ is increasing}\}.$$

The supremum norm $\| \|_\infty : BV([0,1]) \to \mathbb{R}_+$ is an (L)-norm, is order continuous and monotonically complete, but not complete.

As the previous example shows, monotonic completeness of the topology is not equivalent to, but weaker than its usual completeness.

Proposition 5. *Let X be a metrizable* OTVS*. *The topology of X is monotonically complete, if and only if it is monotonically σ-complete.*

Proof. We only need to prove the implication "$\Leftarrow$". According to the hypothesis, there is a quasinorm $q : X \to \mathbb{R}_+$ defining the topology of X. Let $(x_\delta)_{\delta\in\Delta} \subset X$ be

[4] Order continuous topology: for every increasing net $x_\delta \uparrow x$, we have $x_\delta \to x$ topologically.

[5] Weak order unit: element $u \geqslant 0$ with zero orthocomplement, that is, $u^\perp := \{x \mid u \wedge |x| = 0\} = \{0\}$.

a monotonic (q)-Cauchy net. We can choose an increasing sequence $(\delta_n)_{n\in\mathbb{N}} \subset \Delta$, such that

$$q(x_{\delta'} - x_{\delta''}) < \frac{1}{n} \quad \text{for all } n \geqslant 1 \text{ and } \delta', \delta'' \geqslant \delta_n.$$

Thus, $(x_{\delta_n})_{n\in\mathbb{N}} \subset X$ is a monotonic (q)-Cauchy sequence. As q is monotonically σ-complete, we have $\lim_{n\to\infty} q(x_{\delta_n} - x) = 0$ for some $x \in X$. For all $n \geqslant 1$ and $\delta \geqslant \delta_n$, we have

$$q(x_\delta - x) \leqslant q(x_\delta - x_{\delta_n}) + q(x_{\delta_n} - x) \leqslant \frac{1}{n} + q(x_{\delta_n} - x).$$

This yields $\lim_{\delta\in\Delta} q(x_\delta - x) = 0$. We conclude that q is monotonically σ-complete. □

Proposition 6. *Let X be a* TOVS*, with monotonically complete topology and closed positive cone. The following two statements are equivalent:*

(a) *X is Dedekind σ-complete, with order σ-continuous topology,*
(b) *X is Dedekind complete, with order continuous topology.*

Proof. We only need to prove (a) $\Rightarrow$ (b). Let $(x_\delta)_{\delta\in\Delta} \subset X$ be an upper bounded increasing net. We claim that $(x_\delta)_{\delta\in\Delta}$ is (τ)-Cauchy in X, where τ denotes the topology of X. On the contrary suppose that there is a neighborhood $W \in \mathcal{V}(0)$, such that for every $\delta \in \Delta$, there exist $\delta', \delta'' \geqslant \delta$, with $x_{\delta'} - x_{\delta''} \notin W$. Let us choose a balanced full neighborhood $W_0 \in \mathcal{V}(0)$, such that $W_0 + W_0 + W_0 \subset W$. Fix $\delta \in \Delta$. Choose $\delta', \delta'' \geqslant \delta$ as above and $\gamma(\delta) \geqslant \delta', \delta''$. We must have $x_{\gamma(\delta)} - x_\delta \notin W_0$. Indeed, $x_{\gamma(\delta)} - x_\delta \in W_0$ would lead to

$$x_{\delta'} - x_{\delta''} = (x_{\gamma(\delta)} - x_{\delta''}) + (x_{\delta'} - x_\delta) + (x_\delta - x_{\gamma(\delta)}) \in W_0 + W_0 - W_0 \subset W,$$

a contradiction. We thus get the existence of a function $\gamma : \Delta \to \Delta$, such that $\gamma(\delta) > \delta$ and $x_{\gamma(\delta)} - x_\delta \notin W_0$, for every $\delta \in \Delta$. Consequently, there is an increasing sequence $(\delta_n)_{n\in\mathbb{N}} \subset \Delta$, such that $x_{\delta_{n+1}} - x_{\delta_n} \notin W_0$ for every $n \in \mathbb{N}$. According to the hypothesis (a), the upper bounded increasing sequence $(x_{\delta_n})_{n\in\mathbb{N}} \subset X$ is (τ)-convergent, and hence (τ)-Cauchy, which is impossible. Our claim is proved. Since τ is monotonically complete, we have (τ)-$\lim_{\delta\in\Delta} x_\delta = x$, for some $x \in X$. As X_+ is closed, we have $x = \bigvee_{\delta\in\Delta} x_\delta$. We conclude that (b) holds. □

3. A GENERAL ISOMORPHISM THEOREM

The isomorphism theorem from this section was inspired by some parts of the proofs of several representation theorems for Banach lattices. This theorem together with a result from [6] (restated here as Theorem 17) are the most important ingredients of the representation theorem from Section 4. We first introduce a density notion that fits well with monotonic completeness.

Definition 7 (*Monotonic density*). Let X_τ be a directed OTVS*. A vector subspace $G \subset X$ is said to be *monotonically* (τ)*-dense* (or *top-dense*), if and only if the following two conditions hold:

(i) G_+ is (τ)-dense in X_+.
(ii) If $X_0 \subset X$ is a vector subspace containing G and all limits of (τ)-convergent monotonic nets from X_0, then $X_0 = X$.

In the same way we define σ*-monotonic top-density* of a vector subspace, by considering sequences instead of nets.

Let us observe that a monotonically (τ)-dense subspace G is necessarily (τ)-dense, because $\overline{G} \supset G$ contains all limits of (τ)-convergent nets from $\overline{G}$, and consequently, $\overline{G} = X$. Since our notion of monotonic top-density is somewhat encrypted, we next clarify its relation with some more intuitive conditions, which are stronger but also easier to be checked. Proposition 9 will point out a special case when monotonic top-density reduces to the usual topological density.

Proposition 8. *Let X_τ be a directed* OTVS*. *For a vector subspace $G \subset X$, let us consider the following conditions*:

(a_+) *Every element of X_+ is the (τ)-limit of an increasing net from G_+.*
(a_-) *Every element of X_+ is the (τ)-limit of a decreasing net from G_+.*
(b) *Every element of X_+ is the (τ)-limit of a monotonic net from G_+.*
(c) *Every element of X_+ is the (τ)-limit of a sum $(\xi_\delta + \zeta_\delta)_{\delta\in\Delta}$ of monotonic (τ)-Cauchy nets from G_+.*
(c′) *Every element of X_+ is the (τ)-limit of a "cross-sum" $(\xi_\delta + \zeta_\lambda)_{(\delta,\lambda)\in\Delta\times\Lambda}$ of monotonic (τ)-Cauchy nets from G_+.*
(d) *G is monotonically (τ)-dense in X.*
(e) *G_+ is (τ)-dense in X_+, and there is no proper vector subspace $X_0 \subsetneq X$ containing G and all limits of (τ)-convergent sums $(\xi_\delta + \zeta_\delta)_{\delta\in\Delta}$ of monotonic (τ)-Cauchy nets from X_0.*

We have ($a_\pm$) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Leftrightarrow$ (c′) $\Rightarrow$ (e) *and* (b) $\Rightarrow$ (d) $\Rightarrow$ (e). *If X is monotonically (τ)-complete, then* (d) $\Leftrightarrow$ (e). *The same implications hold between the corresponding sequential conditions.*

Proof. The proof is straightforward. □

Proposition 9. *Let X_τ be a topological lattice.*[6] *For any ideal $G \subset X$, the following statements are equivalent* (*as well as the corresponding sequential conditions*):

[6] According to the usual terminology, this also means that the topology τ is locally solid.

(i) *G is monotonically (τ)-dense in X.*
(ii) *G_+ is (τ)-dense in X_+.*
(iii) *G is (τ)-dense in X.*
(a$_+$) *The condition* (a$_+$) *from Proposition 8 holds.*

Proof. The implications $(a_+) \Rightarrow$ (i) $\Rightarrow$(ii) $\Rightarrow$ (iii) are immediate. In order to prove (iii) $\Rightarrow$ (a$_+$), let us fix $x \in X_+$. By (iii), choose a net $(\xi_\delta)_{\delta\in\Delta} \subset G$, with $(\tau)\text{-}\lim_{\delta\in\Delta} \xi_\delta = x$. Set $x_\delta := (0 \vee \xi_\delta) \wedge x \in G_+$ for every $\delta \in \Delta$. As the lattice operations are (τ)-continuous on X, we have $(\tau)\text{-}\lim_{\delta\in\Delta} x_\delta = x$. Let $\mathcal{F}$ denote the set of all nonempty finite subsets $F \subset \Delta$. Set $x_F := \bigvee_{\delta\in F} x_\delta \in G_+$ for every $F \in \mathcal{F}$. Obviously, the net $(x_F)_{F\in\mathcal{F}} \subset G_+$ is increasing. Since $x_\delta \leqslant x_F \leqslant x$ whenever $\{\delta\} \subset F \in \mathcal{F}$, it follows that $(\tau)\text{-}\lim_{F\in\mathcal{F}} x_F = x$. We thus conclude that the condition (a$_+$) holds. $\square$

We can now give some concrete examples of monotonically (τ)-dense subspaces.

Example 10 (*Top-dense subspaces*). On the following vector spaces we consider the usual ordering.

(i) In $(C([0, 1]), \|\ \|_\infty)$, the vector subspace of all polynomial functions restricted to $[0, 1]$ is σ-monotonically top-dense.
(ii) If T is a locally compact space, then the ideal $C_c(T)$ is σ-monotonically top-dense in $(C_\infty(T), \|\ \|_\infty)$. In particular, so is $c_{00}(T)$ in $(c_0(T), \|\ \|_\infty)$ for any set $T \neq \emptyset$.
(iii) Let $(T, \mathcal{T}, \mu)$ be a measurable space and let $p \in [1, \infty[$. Then in $(L^p(\mu), \|\ \|_p)$, the vector sublattice of all classes of μ-integrable step functions is σ-monotonically top-dense. In particular, so is the ideal $c_{00}(T)$ in $(l^p(T), \|\ \|_p)$ for any set $T \neq \emptyset$.
(iv) For a measurable space $(T, \mathcal{T}, \mu)$ with $\mu(T) < \infty$, let us consider the space $M(\mu)$ of all classes of μ-measurable functions, endowed with the quasinorm $q(\widehat{x}) = \int_T \frac{|x(t)|}{1+|x(t)|}\, d\mu(t)$. Then the vector sublattice of all classes (with respect to the equality a.e.) of μ-measurable step functions is σ-monotonically top-dense in $(M(\mu), q)$.
(v) For any set $T \neq \emptyset$, the ideal $c_{00}(T) \subset \mathbb{R}^T$ is monotonically top-dense with respect to the product topology on $\mathbb{R}^T$.

For all five above monotonic top-densities at least one of the conditions (a$_\pm$) (or the sequential versions) from Proposition 8 holds. Nonetheless, even for a vector sublattice of a normed lattice it is possible that both conditions (a$_\pm$) fail, but the sequential version of (b) holds:

Example 11. On the vector lattice $X = C([0, 1])$, let us consider the topology τ defined by the solid (M)-norm $\|x\| = \max_{t\in[0,1]} |tx(t)|$. Then the vector sublattice $G := \{x \in X \mid x(0) = x(1)\}$ is σ-monotonically top-dense in $(X, \|\ \|)$. More precisely, for every $x \in X$ we have:

(i) If $x(1) < x(0)$, then no decreasing net in G is (τ)-convergent to x, but $(\tau)\text{-}\lim_{n\to\infty} x_n = x$ for the increasing sequence $(x_n)_{n\in\mathbb{N}} \subset G$ defined by $x_n(t) = x(t) \wedge (x(1) + nt)$.

(ii) If $x(1) > x(0)$, then no increasing net in G is (τ)-convergent to x, but $(\tau)\text{-}\lim_{n\to\infty} x_n = x$ for the decreasing sequence $(x_n)_{n\in\mathbb{N}} \subset G$ defined by $x_n(t) = x(t) \vee (x(1) - nt)$.

Theorem 12 (Isomorphism). *Let X_τ and Y_η be directed* OTVS**, with closed positive cones and monotonically complete topologies. If $G_\tau \simeq H_\eta$ for some monotonically top-dense vector subspaces $G \subset X$ and $H \subset Y$, then $X_\tau \simeq Y_\eta$.*

Proof. According to the hypothesis, there exists an isomorphism $U : G_\tau \to H_\eta$ of OTVS*. Let $\widetilde{Y}_\eta$ denote the completion of the topological vector space Y_η. As G is (τ)-dense in X, the isomorphism U extends uniquely to a continuous linear operator $\widetilde{U} : X \to \widetilde{Y}$. We claim that $\widetilde{U}(X) \subset Y$, and that $\widetilde{U} : X \to Y$ is a positive operator. We have

$$\widetilde{U}(X_+) = \widetilde{U}\big(\overline{G_+}\big) \subset \overline{\widetilde{U}(G_+)} = \overline{U(G_+)} = \overline{H_+},$$

where the last three closures are considered in $\widetilde{Y}_\eta$. Since H_+ is (η)-dense in Y_+ and Y has a closed positive cone, it follows that

$$\widetilde{U}(X_+) \cap Y \subset \overline{H_+} \cap Y = Y_+. \tag{2}$$

Obviously, $G \subset \widetilde{U}^{-1}(Y) \subset X$. Let $(x_\delta)_{\delta\in\Delta} \subset \widetilde{U}^{-1}(Y)$ be a (τ)-convergent monotonic net, and let $x := (\tau)\text{-}\lim_{\delta\in\Delta} x_\delta \in X$. By (2) we see that $(\widetilde{U}x_\delta)_{\delta\in\Delta}$ is a monotonic (η)-Cauchy net in Y. As η is monotonically complete, this net (η)-converges to some $y \in Y$. It follows that

$$\widetilde{U}x = (\eta)\text{-}\lim_{\delta\in\Delta} \widetilde{U}x_\delta = y \in Y,$$

hence that $x \in \widetilde{U}^{-1}(Y)$. Since the subspace G is top-dense in X, we conclude that $\widetilde{U}^{-1}(Y) = X$, that is, $\widetilde{U}(X) \subset Y$. Hence $\widetilde{U}(X_+) \subset Y_+$, by (2). Our claim is proved. Applying all these arguments again for $V = U^{-1} : H_\eta \to G_\tau$ shows that V extends uniquely to a positive continuous linear operator $\widetilde{V} : Y \to X$. For $\xi \in G$ we have $\widetilde{V}\widetilde{U}\xi = VU\xi = \xi$, that is, $(I_X - \widetilde{V}\widetilde{U})|_G \equiv 0$. But G is (τ)-dense in X, and so $\widetilde{V}\widetilde{U} = I_X$. Since in exactly the same way we get $\widetilde{U}\widetilde{V} = I_Y$, we conclude that $\widetilde{U}$ is an isomorphism of ordered topological vector spaces. □

Remark 13. The above theorem still holds if we replace monotonic completeness and top-density by the corresponding sequential conditions.

Remark 14. Let X_τ be a directed OTVS*, and let $G \subset X$ be a vector subspace. If X is a lattice and G a sublattice of X, or if X satisfies the Riesz decomposition property and G_+ is a full subset of X, then the set $G_x := G \cap [0, x]$ is directed upwards for every $x \in X_+$. In this case, G_x can be viewed as a net in X. Thus, G_x may be (τ)-convergent to x.

Corollary 15. *Let X_τ and Y_η be* OTVS**, which are lattices with closed positive cones and monotonically complete topologies. Assume that $G_\tau \simeq H_\eta$ for some vector sublattices $G \subset X$ and $H \subset Y$, such that*

$$x = (\tau)\text{-}\lim G_x, \quad y = (\eta)\text{-}\lim H_y, \quad \text{for all } x \in X_+,\ y \in Y_+.$$

Then $X_\tau \simeq Y_\eta$.

Proof. The proof is immediate, by Proposition 8(a), Theorem 12, and Remark 14. □

Remark 16. The above corollary still holds if for one/both of the inclusions $G \subset X$ and $H \subset Y$, say $M \subset Z$, we replace the condition "*M sublattice of Z*" by the alternative condition "*M is an ideal in Z, which is directed upwards and has the Riesz decomposition property*".

4. REPRESENTATION THEOREM

For the convenience of the reader we first repeat a needed theorem from [6] (a variant of it may be found in [1]).

Theorem 17. *Let $C_c(T)$ denote the vector lattice of all real continuous functions with compact support, defined on a locally compact space T. Let us consider the sets*

$$\begin{aligned} \mathcal{S}_T &:= \{a : T \to \mathbb{R}_+ \mid a \text{ is upper semicontinuous}\}, \\ \mathcal{M}_T &:= \big\{p : C_c(T) \to \mathbb{R}_+ \mid p \text{ is a solid } (M)\text{-seminorm}\big\}, \end{aligned}$$

endowed with the pointwise ordering. For every $a \in \mathcal{S}_T$, define on $C_c(T)$ the solid (M)-seminorm

$$p_a \in \mathcal{M}_T, \quad p_a(f) = \|af\|_\infty = \sup_{t \in T} |a(t) f(t)|.$$

We have the isomorphism of ordered sets

$$\Phi : \mathcal{S}_T \to \mathcal{M}_T, \quad \Phi(a) = p_a.$$

We can now state and prove our main results. The following two theorems characterize a class of locally convex (M)-lattices, with or without weak order unit. The known representation of Banach (M)-lattices with order continuous norm will follow as a particular case (Corollary 23).

Theorem 18 (Representation, the "unit" case). *Let X_τ be a Hausdorff locally convex (M)-lattice. Assume that*

(a) *the lattice X is Dedekind σ-complete, with weak order unit,*
(b) *the topology τ is order σ-continuous and monotonically complete.*

Then $X_\tau \simeq c_0(T, \mathcal{H})_\theta$, for some set T and some $\mathcal{H} \subset c_0(T)_+$ satisfying (1).

Proof. By Proposition 6, the lattice X is Dedekind complete, with order continuous topology. Let $\mathcal{P}$ be a family of solid (M)-seminorms defining the topology τ of X, and let $u \in X$ be a weak order unit. Let us consider the ideal (hence, Dedekind complete sublattice of X)

$$G := \mathrm{I}\big(\{u\}\big) = \big\{x \in X \mid |x| \leqslant \lambda u \text{ for some } \lambda \in \mathbb{R}_+\big\},$$

endowed with the strong order unit norm $\|x\|_u = \inf\{\lambda \in \mathbb{R}_+ \mid |x| \leqslant \lambda u\}$. Applying the well known representation theorem of Kakutani (see [3, Theorem 4, p. 59], or [5, Theorem 7.4, p. 104]) to the Banach lattice $(G, \|\ \|_u)$, shows the existence of an isomorphism $U : (G, \|\ \|_u) \to (C(K), \|\ \|_\infty)$ of normed lattices, for some compact topological space K. In particular, $C(K)$ is Dedekind complete and $Uu = 1 \in C(K)$. For every $p \in \mathcal{P}$, let us define

$$\widetilde{p} = p \circ U^{-1} : C(K) \to \mathbb{R}_+.$$

Each $\widetilde{p}$ is a solid (M)-seminorm. By Theorem 17, there exists an upper semicontinuous function $g_p \in \mathcal{S}_K$, such that $\widetilde{p}(\varphi) = \|\varphi g_p\|_\infty$ for every $\varphi \in C(K)$. We thus get

$$p(x) = \widetilde{p}(Ux) = \|Ux \cdot g_p\|_\infty \quad \text{for every } x \in G. \tag{3}$$

Let us consider

$$\widetilde{\mathcal{P}} := (\widetilde{p})_{p\in\mathcal{P}}, \quad \mathcal{G} := \{g_p \mid p \in \mathcal{P}\}, \quad T := K \setminus \bigcap_{p\in\mathcal{P}} g_p^{-1}\big(\{0\}\big),$$

and the locally convex topology $\widetilde{\tau}$ defined on $C(K)$ by the family $\widetilde{\mathcal{P}}$ of seminorms. Since by (3), $U : G_\tau \to C(K)_{\widetilde{\tau}}$ is an isomorphism of locally convex lattices, $\widetilde{\tau}$ is order continuous.

Claim 1. T is a discrete dense subspace of K, and $\mathcal{G} \subset c_0(K)_+$.

We first prove that T is discrete. For fixed upper semicontinuous $\omega \in \mathcal{S}_K$, choose a decreasing net $(\varphi_\delta)_{\delta\in\Delta} \subset C(K)_+$, such that $\omega(t) = \inf_{\delta\in\Delta} \varphi_\delta(t)$ for every $t \in K$ (pointwise infimum). Thus,

$$\lim_{\delta\in\Delta} (\varphi_\delta g_p)(t) = (\omega g_p)(t) \quad \text{for all } p \in \mathcal{P}, t \in K. \tag{4}$$

Since $C(K)_{\widetilde{\tau}}$ is a Dedekind complete vector lattice with order continuous topology, for $\varphi := \bigwedge_{\delta\in\Delta} \varphi_\delta \in C(K)$ we have $(\widetilde{\tau})$-$\lim_{\delta\in\Delta} \varphi_\delta = \varphi$, that is,

$$\varphi_\delta g_p \xrightarrow{\text{unif.}} \varphi g_p \quad \text{for every } p \in \mathcal{P}.$$

This and (4) yield $\omega g_p = \varphi g_p$ for every $p \in \mathcal{P}$, that is, $\omega|_T = \varphi|_T$. We also have $0 \leqslant \varphi \leqslant \omega$ in $\mathbb{R}^K$. In particular, if $\omega|_{K\setminus T} \equiv 0$, then $\omega = \varphi \in C(K)$. We thus have proved that

$$\mathcal{S}_{K,T} := \big\{\omega \in \mathcal{S}_K \mid \omega(t) = 0 \text{ for every } t \in K \setminus T\big\} \subset C(K),$$

hence that $\mathcal{G} \subset \mathcal{S}_{K,T} \subset C(K)$. For the characteristic function of any $t \in T$, we have $\chi_{\{t\}} \in \mathcal{S}_{K,T} \subset C(K)$, and so $\{t\} = \chi_{\{t\}}^{-1}(]0, \infty[)$ is an open subset of K. Therefore, T is a discrete subspace of K.

We next show that $\mathcal{G} \subset c_0(K)_+$. For all $g_p \in \mathcal{G}$ and $\varepsilon > 0$, the subset $g_p^{-1}([\varepsilon, \infty[) \subset T \subset K$ is compact (closed in K), and hence finite, since T is discrete. Therefore, $\mathcal{G} \subset c_0(K)_+$.

From our claim, it remains to prove that T is dense in K. On the contrary, suppose that $\overline{T} \neq K$. By Urysohn's theorem, choose $\varphi \in C(K)_+ \setminus \{0\}$, such that $\varphi|_{\overline{T}} \equiv 0$. As $\widetilde{p}(\varphi) = \|\varphi g_p\|_\infty = 0$ for every $p \in \mathcal{P}$, we get $\varphi = 0$, a contradiction. Hence, $\overline{T} = K$. Our claim is proved.

Let us consider

$$h_p := g_p|_T \text{ for every } p \in \mathcal{P}, \qquad \mathcal{H} := \{h_p \mid p \in \mathcal{P}\}.$$

Obviously, $\mathcal{H}$ satisfies (1) and $\mathcal{H} \subset c_0(T)_+$.

Claim 2. We have $X_\tau \simeq c_0(T, \mathcal{H})_\theta$. We will prove this by applying Theorem 12.

Let us consider the vector space $H := C(K)|_T$ consisting of all restrictions to T of functions from $C(K)$. We have $H \cdot \mathcal{H} \subset l^\infty(T) \cdot c_0(T) \subset c_0(T)$, and so $H \subset c_0(T, \mathcal{H})$. Define the linear operator

$$U_0 : G_\tau \to H_\theta, \quad U_0 x = (Ux)|_T.$$

That U_0 is onto follows from $U(G) = C(K)$. Since for all $p \in \mathcal{P}$ and $x, y \in G$, we have

$$\begin{aligned} \|U_0 x\|_{h_p} &= \|U_0 x \cdot h_p\|_\infty = \|Ux \cdot g_p\|_\infty = p(x), \\ U_0(x \vee y) &= \big(U(x \vee y)\big)\big|_T = (Ux \vee Uy)|_T = U_0 x \vee U_0 y, \end{aligned}$$

we deduce that U_0 is an isomorphism of locally convex lattices. As T is a discrete topological space, we see that $c_{00}(T) \subset C(K)|_T = H$. It follows that H is monotonically (θ)-dense in $c_0(T, \mathcal{H})$, since so is $c_{00}(T)$, by Proposition 2(iii).

It remains to prove that G is monotonically (τ)-dense in X. For fixed $x \in X_+$, the sequence $(x \wedge nu)_{n \in \mathbb{N}} \subset G_+$ is increasing. As u is a weak order unit in X (which is Dedekind (σ)-complete), we have $\bigvee_{n \in \mathbb{N}} (x \wedge nu) = x$. This yields (τ)-$\lim_{n \to \infty} (x \wedge nu) = x$, since τ is order σ-continuous. Hence G is monotonically (τ)-dense in X. Applying Theorem 12 finally shows that $X_\tau \simeq c_0(T, \mathcal{H})_\theta$. $\square$

Remark 19. In the above theorem, for a family $\mathcal{P}$ of solid (M)-seminorms defining the topology of X, we obtain $\mathcal{H} = \{h_p \mid p \in \mathcal{P}\} \subset c_0(T)_+$ and an isomorphism $U : X_\tau \to c_0(T, \mathcal{H})_\theta$, such that

$$p(x) = \|Ux\|_{h_p} = \|Ux \cdot h_p\|_\infty \quad \text{for all } x \in X, \ p \in \mathcal{P}.$$

Similar comments will apply to Theorem 20, with $\mathcal{H} \subset \mathbb{R}_+^T$.

Theorem 20 (Representation, general case). *Let X_τ be a Hausdorff locally convex (M)-lattice. Assume that*

(a) *the lattice X is Dedekind σ-complete,*
(b) *the topology τ is order σ-continuous and monotonically complete.*

Then $X_\tau \simeq c_0(T, \mathcal{H})_\theta$, for some set T and some $\mathcal{H} \subset \mathbb{R}^T_+$ satisfying (1).

Proof. By Proposition 6, the lattice X is Dedekind complete, with order continuous topology. Let $\mathcal{P}$ be a family of solid (M)-seminorms defining the topology τ of X. By Zorn's lemma, choose a maximal orthogonal subset $E \subset X_+ \setminus \{0\}$. For each $e \in E$, the projection band $X_e := e^{\perp\perp}$ is a closed subspace of X, since the topology τ is locally solid. Thus, X_e satisfies the hypothesis of Theorem 18. Hence there is an isomorphism

$$U_e : X_e \to c_0(T_e, \mathcal{H}_e),$$

for some set T_e and some $\mathcal{H}_e = \{h_{p,e} \mid p \in \mathcal{P}\} \subset c_0(T_e)_+$ satisfying (1), such that

$$p(x) = \|U_e x \cdot h_{p,e}\|_\infty \quad \text{for all } x \in X_e,\ p \in \mathcal{P}.$$

(see also Remark 19). We can assume that all T_e $(e \in E)$ are mutually disjoint sets (otherwise, we may replace each T_e by $T_e \times \{e\}$). Set

$$T = \bigcup_{e \in E} T_e, \qquad \mathcal{H} = \left\{h_p \in \mathbb{R}^T_+ \mid p \in \mathcal{P}\right\},$$

where each function h_p is defined by its restrictions: $h_p|_{T_e} = h_{p,e}$ for every $e \in E$.

Claim. We have $X_\tau \simeq c_0(T, \mathcal{H})_\theta$. We will prove this by applying Theorem 12.

For function $g \in \mathbb{R}^{T_e}$, we may define the extension $\bar{g} \in \mathbb{R}^T$, by $\bar{g}|_{T_e} = g$ and $\bar{g}|_{T \setminus T_e} \equiv 0$. Set

$$\begin{aligned}
&c_0(T, \mathcal{H})_e := \left\{\overline{\psi} \mid \psi \in c_0(T_e, \mathcal{H}_e)\right\} = \left\{\psi \in c_0(T, \mathcal{H}) \mid \psi|_{T \setminus T_e} \equiv 0\right\} \\
&\quad (e \in E), \\
&G := \bigoplus_{e \in E} X_e, \qquad H := \bigoplus_{e \in E} c_0(T, \mathcal{H})_e.
\end{aligned}$$

Note that G and H are sublattices of X and $c_0(T, \mathcal{H})$, respectively. Define the linear operator

$$U : G_\tau \to H_\theta, \quad Ux = \overline{U_e x} \quad \text{for all } e \in E,\ x \in X_e.$$

It is easily seen that U is an isomorphism of ordered vector spaces. Fix $x \in G$. We have $x = \sum_{e \in F} x_e$ for some finite subset $F \subset E$ and some $x_e \in X_e$ $(e \in F)$. Thus,

$$|x| = \sum_{e \in F} |x_e| = \bigvee_{e \in F} |x_e|, \qquad Ux = \sum_{e \in F} \overline{U_e x_e}, \qquad |Ux| = \bigvee_{e \in F} \overline{U_e(|x_e|)}.$$

Consequently, for every $p \in \mathcal{P}$ we have

$$\begin{aligned}\|Ux\|_{h_p} &= \|Ux \cdot h_p\|_\infty = \||Ux| \cdot h_p\|_\infty = \left\| h_p \cdot \bigvee_{e\in F} \overline{U_e(|x_e|)} \right\|_\infty \\ &= \bigvee_{e\in F} \|h_p \cdot \overline{U_e(|x_e|)}\|_\infty = \bigvee_{e\in F} \|h_{p,e} \cdot U_e(|x_e|)\|_\infty \\ &= \bigvee_{e\in F} p(|x_e|) = p\left(\bigvee_{e\in F} |x_e|\right) = p(|x|) = p(x).\end{aligned}$$

It follows that U is an isomorphism of locally convex lattices. As $c_{00}(T_e) \subset c_0(T, \mathcal{H})_e$ for every $e \in E$, we have $c_{00}(T) \subset H$. Hence, H is monotonically (θ)-dense in $c_0(T, \mathcal{H})$, since so is $c_{00}(T)$.

It remains to prove that G is monotonically (τ)-dense in X. Fix $x \in X_+$. Let us consider the set $\mathcal{F}$ consisting of all nonempty finite subsets of E. For every $F \in \mathcal{F}$, set $x_F := \bigvee_{e\in F}[e]x \in G_+$. We thus get the increasing net $(x_F)_{F\in\mathcal{F}} \subset G_+$. As X is Dedekind complete and the orthogonal subset E is maximal, we have $\bigvee_{F\in\mathcal{F}} x_F = x$. This yields (τ)-$\lim_{F\in\mathcal{F}} x_F = x$, since τ is order continuous. Hence, G is monotonically (τ)-dense in X. Applying Theorem 12 finally shows that $X_\tau \simeq c_0(T, \mathcal{H})_\theta$. □

Corollary 21 (*Representation, normed case*). *Let $(X, \|\,\|)$ be a normed (M)-lattice. Assume that*

(a) *the lattice X is Dedekind σ-complete,*
(b) *the norm of X is order σ-continuous and monotonically σ-complete.*

Then $(X, \|\,\|) \simeq (c_0(T), \|\,\|_\infty)$ as normed lattices, for some set T and some $\mathcal{H} \subset \mathbb{R}^T_+$ satisfying (1).

Proof. By Proposition 5, the norm of X is monotonically complete. By Theorem 20 and Remark 19, we have the isomorphism of normed lattices $(X, \|\,\|) \simeq (c_0(T, \{h\}), \|\,\|_h)$, for some set T and some $h \in \mathbb{R}^T_+$, with $h^{-1}(\{0\}) = \emptyset$. Since

$$U_h : \big(c_0(T, \{h\}), \|\,\|_h\big) \to \big(c_0(T), \|\,\|_\infty\big), \quad U_h\psi = \psi h$$

is an isomorphism of normed lattices, the conclusion follows. □

Proposition 22. *Let X_τ be an OTVS* with closed positive cone. Assume the topology τ is order continuous and monotonically complete. If a nonempty subset $A \subset X$ is upper bounded and directed upwards, and if the set of its upper bounds is directed downwards, then A converges and*

$$(\tau)\text{-}\lim A = \sup A.$$

In particular, if X is a vector lattice, then it is Dedekind complete.

Proof. For subsets $E, F \subset X$, we may write $E \leqslant F$, if $x \leqslant y$ for all $x \in E$ and $y \in F$. We shall use notations such as $z \leqslant E$ or $E \leqslant z$ (for $z \in X$), with an obvious meaning.

For $A \subset X$ as in the hypothesis, let B denote the set of its upper bounds. Recall that upwards directed subsets of X may be considered as increasing nets. We claim that A is a (τ)-Cauchy net.

Fix a neighborhood $W \in \mathcal{V}(0)$, and choose $W_0 \in \mathcal{V}(0)$, such that $W_0 - W_0 \subset W$. Let $z \geqslant A - B$. We have $A \leqslant B + z$, which yields $B + z \subset B$. An obvious induction shows that for every $n \in \mathbb{N}$ we have $B + nz \subset B$, that is, $A \leqslant B + nz$, or equivalent, $n(-z) \leqslant B - A \subset X_+$. Since X is Archimedean, we must have $z \geqslant 0$. We thus have proved that $\sup(A - B) = 0$. The set $A - B$ is directed upwards. As τ is order continuous, we have $(\tau)\text{-}\lim(A - B) = 0$. Therefore, there exist $a \in A,\ b \in B$, such that $(A - B) \cap [a - b, 0] \subset W_0$. For all $a_1, a_2 \in A$, with $a_1, a_2 \geqslant a$, we have $a_1 - b, a_2 - b \in (A - B) \cap [a - b, 0] \subset W_0$, and so $a_1 - a_2 \in W_0 - W_0 \subset W$. Our claim is proved. Since the net A is increasing and (τ)-Cauchy, and τ is monotonically complete, there exists $x = (\tau)\text{-}\lim A \in \overline{A}$. As X_+ is closed, we conclude that $x = \sup A$. $\square$

The following corollary is a slight improvement of the known representation theorem for Banach (M)-lattices with order continuous norm, since monotonic σ-completeness is required instead of the stronger classical (Banach) completeness.

Corollary 23. *Let $(X, \|\,\|)$ be a normed (M)-lattice. Assume the norm of X is order continuous and monotonically σ-complete. Then $(X, \|\,\|) \simeq (c_0(T), \|\,\|_\infty)$ as normed lattices, for some set T.*

Proof. The proof is immediate, by Propositions 5 and 22, and Corollary 21. $\square$

5. THE DUAL OF $C_0(T, \mathcal{H})_\theta$

Let us consider $T \neq \emptyset$ and an upwards directed subset $\mathcal{H} \subset \mathbb{R}_+^T$ satisfying (1). Set

$$l^1(T, \mathcal{H}) := l^1(T) \cdot \mathcal{H} = \{vh \mid v \in l^1(T),\ h \in \mathcal{H}\}.$$

Remark 24. $l^1(T, \mathcal{H})$ is an ideal in $\mathbb{R}^T$, and a $l^\infty(T)$-module containing $c_{00}(T)$.

It is less obvious that $l^1(T, \mathcal{H})$ is a vector space (the other properties are immediate). Let us show that $\gamma := \alpha uh + \beta vk \in l^1(T, \mathcal{H})$ for $\alpha, \beta \in \mathbb{R},\ u, v \in l^1(T),\ h, k \in \mathcal{H}$. Choose $g \in \mathcal{H}$, with $h, k \leqslant g$. There exist $r, s \in l^\infty(T)$, such that $h = rg,\ k = sg$. Hence, $\gamma = (\alpha ur + \beta vs)g \in l^1(T) \cdot \mathcal{H} = l^1(T, \mathcal{H})$.

Theorem 25. *We have the lattice isomorphism $U : l^1(T, \mathcal{H}) \to c_0(T, \mathcal{H})^*_\theta$, defined by*

$$(U\gamma)\psi = \int_T (\gamma\psi)\, \mathrm{d}\mu_\mathrm{c} \quad \textit{for all } \gamma \in l^1(T, \mathcal{H}), \psi \in c_0(T, \mathcal{H}),$$

where $\mu_c : 2^T \to [0,\infty]$ *denotes the cardinal measure on* T.

Proof. We have divided the proof into two steps.

Step 1. We first show that U is well defined and injective. Fix $\gamma = vh \in l^1(T,\mathcal{H})$, with $v \in l^1(T)$ and $h \in \mathcal{H}$. For every $\psi \in c_0(T,\mathcal{H})$, we have $\gamma\psi = v(h\psi) \in l^1(T) \cdot c_0(T) \subset l^1(T)$. Hence the integral exists and

$$\begin{aligned}\left|(U\gamma)\psi\right| &= \left|\int\limits_T (\gamma\psi)\,\mathrm{d}\mu_c\right| = \left|\int\limits_T (vh\psi)\,\mathrm{d}\mu_c\right| \leqslant \|v\|_{l^1(T)}\|\psi h\|_\infty \\ &= \|v\|_{l^1(T)}\|\psi\|_h.\end{aligned}$$

It follows that $U\gamma \in c_0(T,\mathcal{H})^*_\theta$ for every $\gamma \in l^1(T,\mathcal{H})$, that is, U is well defined. We see that U is linear. As $c_{00}(T) \subset c_0(T,\mathcal{H})$, we have the equivalence

$$(5) \qquad \gamma \geqslant 0 \quad \text{in } l^1(T,\mathcal{H}) \quad \Longleftrightarrow \quad U\gamma \geqslant 0 \quad \text{in } c_0(T,\mathcal{H})^*_\theta.$$

This shows that U is injective.

Step 2. We next show that U is onto. To prove this, fix $f \in c_0(T,\mathcal{H})^*_\theta$. There exist $\alpha \in \mathbb{R}_+$ and $h \in \mathcal{H}$, such that $|f(\psi)| \leqslant \alpha\|\psi\|_h$ for every $\psi \in c_0(T,\mathcal{H})$. Define the map

$$\gamma : T \to \mathbb{R}, \quad \gamma(t) = f(\chi_{\{t\}}).$$

Let us observe that

$$(6) \qquad f(\varphi) = \int\limits_T (\gamma\varphi)\,\mathrm{d}\mu_c \quad \text{for every } \varphi \in c_{00}(T),$$

since for $F = \operatorname{supp}\varphi$, we have $f(\varphi) = f(\sum_{s\in F}\varphi(s)\chi_{\{s\}}) = \sum_{s\in F}\gamma(s)\varphi(s) = \int_T(\gamma\varphi)\,\mathrm{d}\mu_c$.

We now claim that $\gamma \in l^1(T,\mathcal{H})$ and $U\gamma = f$. On T we have $|\gamma(t)| = |f(\chi_{\{t\}})| \leqslant \alpha\|\chi_{\{t\}}\|_h = \alpha h(t)$, and so $\operatorname{supp}\gamma \subset \operatorname{supp}h$. Therefore, there exists $v \in \mathbb{R}^T$, such that $\gamma = vh$ and $\operatorname{supp}v = \operatorname{supp}\gamma$. Our claim will follow by (6), if we prove that $v \in l^1(T)$. As $\operatorname{supp}v = \operatorname{supp}\gamma$, we have $|v| = \gamma w$ for some $w \in \mathbb{R}^T$. For every nonempty finite subset $F \subset \operatorname{supp}v \subset \operatorname{supp}h$, by (6) we get

$$\begin{aligned}\int\limits_F |v|\,\mathrm{d}\mu_c &= \int\limits_T (\gamma w\chi_F)\,\mathrm{d}\mu_c = f(w\chi_F) \leqslant \alpha\|w\chi_F\|_h \\ &= \alpha\|(h|w|)\chi_F\|_\infty = \alpha\|\chi_F\|_\infty = \alpha.\end{aligned}$$

This yields $\int_{\operatorname{supp}v}|v|\,\mathrm{d}\mu_c \leqslant \alpha$, that is, $v \in l^1(T)$. We thus get $\gamma = vh \in l^1(T,\mathcal{H})$.

As by (6), the functionals $f, U\gamma \in c_0(T,\mathcal{H})^*_\theta$ coincide on $c_{00}(T)$, which is (θ)-dense in $c_0(T,\mathcal{H})$, we conclude that $f = U\gamma$. That U is also an isomorphism of ordered vector spaces follows from (5). $\square$

Example 26. Let $T \neq \emptyset$. Then

(i) For $\mathcal{H} = c_{00}(T)_+$, we have

$$c_0(T, \mathcal{H}) = \mathbb{R}^T, \quad l^1(T, \mathcal{H}) = c_{00}(T),$$

and θ is the pointwise convergence topology on $\mathbb{R}^T$.

(ii) For $\mathcal{H} = \mathbb{R}^T_+$, we have

$$c_0(T, \mathcal{H}) = c_{00}(T),$$
$$l^1(T, \mathcal{H}) = \left\{\varphi \in \mathbb{R}^T \mid \varphi^{-1}(\mathbb{R} \setminus \{0\}) \text{ is at most countable}\right\}.$$

(iii) For $\mathcal{H} = c_0(T)_+$, we have

$$c_0(T, \mathcal{H}) = l^\infty(T), \quad l^1(T, \mathcal{H}) = l^1(T).$$

(iv) For $\mathcal{H} = l^\infty(T)_+$, we have

$$c_0(T, \mathcal{H}) = c_0(T), \quad l^1(T, \mathcal{H}) = l^1(T),$$

and θ is the uniform convergence topology on $c_0(T)$.

(v) For $\mathcal{H} = l^p(T)_+$ ($p > 0$), we have

$$c_0(T, \mathcal{H}) = l^\infty(T), \quad l^1(T, \mathcal{H}) = l^{\frac{p}{p+1}}(T).$$

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