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An isomorphic characterization of L^1-spaces ☆

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ABSTRACT

We show that a sequentially (τ)-complete topological vector lattice X_τ is isomorphic to some $L^1(\mu)$, if and only if the positive cone can be written as $X_+ = \mathbb{R}_+ B$ for some convex, (τ)-bounded, and (τ)-closed set $B \subset X_+ \setminus \{0\}$. The same result holds under weaker hypotheses, namely the Riesz decomposition property for X (not assumed to be a vector lattice) and the monotonic σ-completeness (*monotonic* Cauchy sequences converge). The isometric part of the main result implies the well-known representation theorem of Kakutani for (AL)-spaces. As an application we show that on a normed space Y of infinite dimension, the "ball-generated" ordering induced by the cone $Y_+ = \mathbb{R}_+ \overline{B}(u, 1)$ (for $\|u\| > 1$) cannot have the Riesz decomposition property. A second application deals with a pointwise ordering on a space of multivariate polynomials.

1. INTRODUCTION

As for prerequisites, the reader is expected to be familiar with notions, basic properties, and some results on ordered vector spaces from [1,3–5,7]. Following [1], we will use the notation X_τ for an ordered vector space X endowed with a linear topology τ. If two such spaces X_τ and Y_η are isomorphic (algebraically, topologically, and as ordered sets), we write this as $X_\tau \simeq Y_\eta$.

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Any L^1-space is a Banach lattice $(X, \| \|)$ having in particular the following three properties:

(a) X is directed ($X = X_+ - X_+$) and has the Riesz decomposition property (meaning that $[0, x+y] = [0, x] + [0, y]$ for all $x, y \in X_+$, the segments being considered in the order sense).
(b) The topology τ of X is locally solid and σ-monotonically complete (that is, *monotonic* (τ)-Cauchy sequences are (τ)-convergent).
(c) We have $X_+ = \mathbb{R}_+ B$ for some convex, (τ)-bounded, and (τ)-closed set $B \subset X_+ \setminus \{0\}$.

The above property (c) holds in particular for the set $B := \{x \in X_+ \mid \|x\| = 1\}$.

In this paper we show (Theorem 14 and its corollaries) that any Hausdorff topological ordered vector space having these three properties is a Dedekind complete vector lattice, isomorphic as a topological vector lattice to some L^1-space. In particular, so is any sequentially (τ)-complete topological vector lattice X_τ with the property (c).

From now on, X_τ denotes a (real) Hausdorff topological vector space endowed with a directed linear ordering "$\leqslant$", such that the above property (c) *holds.*

2. GENERAL PROPERTIES

Proposition 1. *The topology τ is locally full and the positive cone X_+ is (τ)-closed.*

Proof. Let us first show that τ is a locally full topology. Fix a neighborhood $W \in \mathcal{V}_\tau(0)$ (of 0 in X_τ) and choose a balanced $W_0 \in \mathcal{V}_\tau(0)$, such that $W_0 + W_0 \subset W \cap (X \setminus B)$. As B is (τ)-bounded, we have $\varepsilon B \subset W_0$ for some $\varepsilon \in]0, 1]$. For the full neighborhood

$$V := \varepsilon\,[(W_0 + X_+) \cap (W_0 - X_+)] \in \mathcal{V}_\tau(0),$$

we claim that $V \subset W$. To prove this, fix $x \in V$. Thus, $x = \varepsilon(w_1 + \alpha_1 b_1) = \varepsilon(w_2 - \alpha_2 b_2)$ for some $w_1, w_2 \in W_0$, $b_1, b_2 \in B$, and $\alpha_1, \alpha_2 \in \mathbb{R}_+$. Hence $w_2 - w_1 = \alpha_1 b_1 + \alpha_2 b_2 \in \alpha_1 B + \alpha_2 B = (\alpha_1 + \alpha_2)B$, which yields $(W_0 + W_0) \cap (\alpha_1 + \alpha_2)B \neq \emptyset$. As $(W_0 + W_0) \cap B = \emptyset$ and W_0 is balanced, we must have $\alpha_1 + \alpha_2 < 1$. It follows that $x = \varepsilon(w_1 + \alpha_1 b_1) \in \varepsilon(W_0 + \alpha_1 B) \subset \varepsilon W_0 + \alpha_1 W_0 \subset W_0 + W_0 \subset W$. We thus get $V \subset W$. We conclude that τ is a locally full topology.

Now let us prove that the positive cone X_+ is (τ)-closed. Fix $x \in \overline{X_+} \setminus \{0\}$. There exist two nets $(\alpha_\delta)_{\delta\in\Delta} \subset]0, \infty[$ and $(b_\delta)_{\delta\in\Delta} \subset B$, such that $(\tau)\text{-}\lim_{\delta\in\Delta}(\alpha_\delta b_\delta) = x$. By taking subnets if necessary, we may assume that $\lim_{\delta\in\Delta} \alpha_\delta = \alpha \in [0, \infty]$. We must have $\alpha \in]0, \infty[$. Indeed, for $\alpha = \infty$ we get $0 = (\tau)\text{-}\lim_{\delta\in\Delta} \frac{\alpha_\delta b_\delta}{\alpha_\delta} = (\tau)\text{-}\lim_{\delta\in\Delta} b_\delta \in \overline{B} = B$, a contradiction. For $\alpha = 0$, since B is (τ)-bounded we get $0 = (\tau)\text{-}\lim_{\delta\in\Delta}(\alpha_\delta b_\delta) = x \neq 0$, another contradiction. If $\alpha \in]0, \infty[$, then

$$\frac{x}{\alpha} = (\tau)\text{-}\lim_{\delta\in\Delta} \frac{\alpha_\delta b_\delta}{\alpha_\delta} = (\tau)\text{-}\lim_{\delta\in\Delta} b_\delta \in \overline{B} = B$$

yields $x \in \alpha B \subset X_+$. We thus conclude that X_+ is (τ)-closed. $\square$

The strictly positive functional f from the next lemma will characterize the norm of the L^1-space from our representation theorem.

Lemma 2. *There is a linear functional $f : X \to \mathbb{R}$, having the following two properties:*[1]

(i) *The restriction $f|_{X_+} : X_+ \to \mathbb{R}$ is continuous at* 0.
(ii) *For every neighborhood $W \in \mathcal{V}_\tau(0)$, there exists $\varepsilon \in]0, \infty[$, such that $x \in W$ whenever $x \in X_+$ and $f(x) < \varepsilon$.*

Any such functional also has the property

(iii) *For every lower (or upper) bounded subset $A \subset X$, we have the equivalence*

$$A \textit{ is } (\tau)\textit{-bounded} \iff f(A) \textit{ is bounded.}$$

Proof. As $X_+ = \mathbb{R}_+ B$ and $0 \notin B$, the set $\{\alpha \in \mathbb{R}_+ \mid x \in \alpha B\}$ is nonempty and bounded in $\mathbb{R}$ for every $x \in X_+$. Hence the map

$$(1) \qquad q : X_+ \to \mathbb{R}_+, \quad q(x) = \sup\{\alpha \in \mathbb{R}_+ \mid x \in \alpha B\},$$

is well-defined. It is easy to check that q is supralinear ($-q$ is sublinear), since B is convex. By the properties of B we deduce that the set $H := \mathrm{co}(B \cup (-B))$ is absorbing, (τ)-bounded, and convex. Therefore, the Minkowski functional

$$(2) \qquad \| \, \|_H : X \to \mathbb{R}_+, \quad \|x\|_H = \inf\{\alpha \in]0, \infty[\mid x \in \alpha H\},$$

is a norm. Let us choose first a balanced neighborhood $W_f \in \mathcal{V}_\tau(0)$ with $W_f \subset X \setminus B$, and then a number $\mu \in]0, \infty[$, such that $H \subset \mu W_f$. We claim that

$$(3) \qquad q(x) \leqslant \mu \|x\|_H \quad \text{for every } x \in X_+.$$

Fix $x \in X_+$ and $\beta > \|x\|_H$. Hence $x \in \alpha H$ for some $\alpha \in]0, \beta[$, by (2). For every $\gamma \geqslant \alpha\mu$, we have

$$\frac{1}{\gamma} x \in \frac{\alpha}{\gamma} H \subset \frac{\alpha\mu}{\gamma} W_f \subset W_f \subset X \setminus B,$$

and so $x \notin \gamma B$. By (1) it follows that $q(x) \leqslant \alpha\mu < \beta\mu$. Since $\beta > \|x\|_H$ was arbitrary, we conclude that $q(x) \leqslant \mu\|x\|_H$. We thus have proved (3). By Bonsall's theorem, there exists a linear functional $f : X \to \mathbb{R}$, such that $q(x) \leqslant f(x)$ for $x \in X_+$ and $f(x) \leqslant \mu\|x\|_H$ for $x \in X$. We next show that f has the required properties.

(i) Fix $x \in W_f \cap X_+$. For every $\gamma \geqslant 1$, we have $\frac{1}{\gamma} x \in W_f \subset X \setminus B$, and so $x \notin \gamma B$. Consequently, there exists $\alpha \in [0, 1[$, such that $x \in \alpha B \subset \alpha H$. It follows that

[1] The property (ii) below yields $f(x) > 0$ for $x \in X_+ \setminus \{0\}$, as well as the (τ)-boundedness of $B_f := f^{-1}(\{1\}) \cap X_+$.

$f(x) \leqslant \mu\|x\|_H \leqslant \mu\alpha < \mu$. Hence $f(W_f \cap X_+) \subset [0, \mu[$. Clearly, the boundedness of $f(W_f \cap X_+)$ yields the continuity of $f|_{X_+}$ at 0.

(ii) Fix a balanced neighborhood $W \in \mathcal{V}_\tau(0)$. As B is (τ)-bounded, we have $\varepsilon B \subset W$ for some $\varepsilon \in]0, \infty[$. Let $x \in X_+$, such that $f(x) < \varepsilon$. Since $q(x) \leqslant f(x) < \varepsilon$, by (1) we deduce that $x \in \alpha B$ for some $\alpha \in [0, \varepsilon[$. We thus get $x \in \alpha B \subset \frac{\alpha}{\varepsilon} W \subset W$.

(iii) After assuming that $A \subset X_+$ (otherwise we apply a translation and, if necessary, we multiply the resulting set by -1), the proof is straightforward. □

Remark 3. For a topological ordered vector space Y_η with locally solid topology and closed positive cone, the property (c) from the Introduction is equivalent to

(c′) There is a functional $f \in Y_\eta^*$ satisfying the requirement (ii) from Lemma 2.

A large amount of known results on topological ordered vector spaces still hold if we replace the usual topological completeness by the monotonic completeness defined in [6]:

Definition 4 (Monotonic completeness). The topology τ of X is said to be *monotonically complete*, if and only if every monotonic (τ)-Cauchy net in X is (τ)-convergent. In the same way we define *monotonic σ-completeness*, by considering sequences instead of nets.

Monotonic completeness is strictly weaker than the usual completeness of the topology:

Example 5. The vector space $X = BV([0,1]) = \{x : [0,1] \to \mathbb{R} \mid x \text{ has bounded variation}\}$ is a Dedekind complete vector lattice with respect to the usual ordering "$\preceq$" defined by the cone

$$X_+ = \{x : [0,1] \to \mathbb{R}_+ \mid x \text{ is increasing}\}.$$

The L^∞ norm $\| \ \|_\infty$ considered on X is monotonically complete and additive on X_+, but neither complete nor solid. Nonetheless, Theorem 14 (stated with the weaker monotonic σ-completeness) may be applied for the space $(X, \preceq, \| \ \|_\infty)$.

Proposition 6. *We have the following*:

(i) *Any (τ)-bounded monotonic net from X is (τ)-Cauchy.*
(ii) *For τ, monotonic completeness is equivalent to monotonic σ-completeness.*

Proof. Let us choose a linear functional $f : X \to \mathbb{R}$ as in Lemma 2 and a neighborhood $W_f \in \mathcal{V}_\tau(0)$, such that $f(W_f \cap X_+)$ is bounded.

(i) Let $(x_\delta)_{\delta \in \Delta} \subset X$ be a (τ)-bounded increasing net and let $W \in \mathcal{V}_\tau(0)$. Choose a balanced neighborhood $W_0 \in \mathcal{V}_\tau(0)$, such that $W_0 + W_0 \subset W$. By Lemma 2(ii) we can find $\varepsilon \in]0, \infty[$, such that $x \in W_0$ whenever $x \in X_+$ and $f(x) < \varepsilon$. Choose

$\delta_0 \in \Delta$. Since $(x_\delta)_{\delta \geqslant \delta_0} \subset X_+$ is a (τ)-bounded increasing net, by Lemma 2(iii) we deduce that $(f(x_\delta))_{\delta \geqslant \delta_0} \subset \mathbb{R}$ is an upper bounded increasing net. Let $\alpha := \bigvee_{\delta \geqslant \delta_0} f(x_\delta)$ and $\delta_W \geqslant \delta_0$, such that $\alpha - \varepsilon < f(x_{\delta_W}) \leqslant \alpha$. For every $\delta \geqslant \delta_W$ we have $f(x_\delta - x_{\delta_W}) < \varepsilon$, and so $x_\delta - x_{\delta_W} \in W_0$. Consequently, for arbitrary $\delta', \delta'' \geqslant \delta_W$ we have $x_{\delta'} - x_{\delta''} \in W_0 - W_0 \subset W$. We conclude that $(x_\delta)_{\delta \in \Delta}$ is a (τ)-Cauchy net.

(ii) Assume τ is monotonically σ-complete and let $(x_\delta)_{\delta \in \Delta} \subset X$ be an increasing (τ)-Cauchy net. Thus, there exists $\delta' \in \Delta$, such that $(x_\delta - x_{\delta'})_{\delta \geqslant \delta'} \subset W_f \cap X_+$. According to Lemma 2(iii), the net $(x_\delta)_{\delta \geqslant \delta'} \subset X$ is (τ)-bounded. Also, $(f(x_\delta))_{\delta \geqslant \delta'} \subset \mathbb{R}$ is an upper bounded increasing net. For $\alpha := \bigvee_{\delta \geqslant \delta'} f(x_\delta)$, let us choose an increasing sequence $(\delta_n)_{n \in \mathbb{N}} \subset \Delta$, such that $\delta_0 \geqslant \delta'$ and $\lim_{n \to \infty} f(x_{\delta_n}) = \alpha$. The sequence $(x_{\delta_n})_{n \in \mathbb{N}} \subset X_+$ is increasing and (τ)-bounded, and hence (τ)-Cauchy, by (i). As τ is monotonically σ-complete, the limit $(\tau)\text{-}\lim_{n \to \infty} x_{\delta_n} =: x \in X$ exists. We claim that

$$(4) \qquad (\tau)\text{-}\lim_{\delta \in \Delta} x_\delta = x.$$

To prove this, fix $W \in \mathcal{V}_\tau(0)$ and choose $W_0 \in \mathcal{V}_\tau(0)$, such that $W_0 + W_0 \subset W$. By Lemma 2(ii), there exists $\varepsilon \in]0, \infty[$, such that $z \in W_0$ whenever $z \in X_+$ and $f(z) < \varepsilon$. Now choose $m \in \mathbb{N}$, such that $f(x_{\delta_m}) > \alpha - \varepsilon$ and $x_{\delta_m} - x \in W_0$. Let $\delta \geqslant \delta_m$. We have $\alpha - \varepsilon < f(x_{\delta_m}) \leqslant f(x_\delta) \leqslant \alpha$, hence $f(x_\delta - x_{\delta_m}) < \varepsilon$, which forces $x_\delta - x_{\delta_m} \in W_0$. It follows that

$$x_\delta - x = (x_\delta - x_{\delta_m}) + (x_{\delta_m} - x) \in W_0 + W_0 \subset W.$$

We thus have proved (4). We conclude that (τ) is monotonically complete. $\square$

The following proposition is a useful shortcut for proving Dedekind completeness, since directed upwards/downwards sets may be viewed as monotonic nets. Such nets are more convenient and fit well with monotonic completeness in topological ordered vector spaces.

Proposition 7. *For a directed ordered vector space Y with the Riesz decomposition property, the following three statements are equivalent*:

- (D) *Y is a Dedekind complete vector lattice.*
- (D_+) $\sup A$ *exists for any nonempty upper bounded $A \subset Y_+$ with a directed downwards set of upper bounds $M_A := \bigcap_{a \in A}(a + Y_+)$.*
- (D_-) $\inf A$ *exists for any nonempty lower bounded $A \subset Y_+$ with a directed upwards set of lower bounds $m_A := \bigcap_{a \in A}(a - Y_+)$.*

Proof. (D) $\Rightarrow$ (D_+) $\Leftrightarrow$ (D_-) are obvious. In order to show the implication (D_+) $\Rightarrow$ (D), let us first consider a nonempty upper bounded finite subset $A_0 \subset Y_+$. As Y has the Riesz decomposition property, the set M_{A_0} is directed downwards. Hence $\sup A_0$ exists, by (D_+). It follows that Y is a vector lattice. Now fix a nonempty upper bounded $A \subset Y_+$. Since Y is a vector lattice, the set M_A is directed downwards. Hence $\sup A$ exists, by (D_+). We conclude that Y is Dedekind complete. $\square$

In the presence of topological monotonic completeness, the Riesz decomposition property turns X_τ into a Dedekind complete vector lattice with order continuous topology:[2]

Proposition 8. *If X has the Riesz decomposition property and (τ) is monotonically σ-complete, then X is a Dedekind complete vector lattice and τ is order continuous.*

Proof. Let us first prove that X is a Dedekind complete vector lattice. Fix a nonempty lower bounded $A \subset X_+$ with a directed upwards set of lower bounds $m_A = \bigcap_{a \in A}(a - X_+)$. Thus, we may consider $m_A \supset m_A \cap X_+$ as an increasing upper bounded (by any element of A) net in X. By Proposition 6, we deduce that $m_A \cap X_+$ is (τ)-Cauchy, and that τ is monotonically complete. Hence the limit

$$(\tau)\text{-}\lim(m_A \cap X_+) = (\tau)\text{-}\lim m_A =: x \in X$$

exists. Since X_+ is (τ)-closed, for arbitrary $a \in A$ and $b \in m_A$ we have $b \leqslant (\tau)\text{-}\lim m_A \leqslant a$, that is, $b \leqslant x \leqslant a$. It follows that $x = \max m_A = \inf A$. By Proposition 7 we conclude that X is a Dedekind complete vector lattice.

Let us show that τ is order continuous. Fix a decreasing net $(x_\delta)_{\delta \in \Delta} \subset X$, such that $\bigwedge_{\delta \in \Delta} x_\delta = 0$. Since the positive cone X_+ is (τ)-closed, as for m_A we deduce that the limit $z = (\tau)\text{-}\lim_{\delta \in \Delta} x_\delta$ exists and $\bigwedge_{\delta \in \Delta} x_\delta = z$. We conclude that $(\tau)\text{-}\lim_{\delta \in \Delta} x_\delta = 0$. $\square$

3. THE ASSOCIATED LOCALLY SOLID TOPOLOGY $\check{\tau}$

Let us recall that a subset A of a directed ordered vector space Y is called *solid*, if and only if $A = \bigcup_{y \in A \cap Y_+} [-y, y]$. Then a linear topology η on Y is said to be *locally solid*, if and only if every neighborhood $W \in \mathcal{V}_\eta(0)$ contains a solid $V \in \mathcal{V}_\eta(0)$. A seminorm $p : Y \to \mathbb{R}_+$ is called solid, if and only if $p(y) = \inf_{x \geqslant \pm y} p(x)$ for every $y \in Y$.

The following theorem associates to the original topology τ a locally solid stronger topology $\check{\tau}$, defined by a solid norm which is additive on the positive cone. Proposition 12 will show that several monotonicity-related properties are the same for $\check{\tau}$ and τ.

Theorem 9. *For linear $f : X \to \mathbb{R}$ as in Lemma 2, let us consider $B_f = \{x \in X_+ \mid f(x) = 1\}$ and $H_f = \mathrm{co}(B_f \cup (-B_f))$. Then*

(5) $\qquad \| \, \|_f : X \to \mathbb{R}_+, \quad \|x\|_f = \inf\{\alpha \in]0, \infty[\mid x \in \alpha H_f\},$

[2] That is, for every increasing net $(x_\delta)_{\delta \in \Delta} \subset X$ with $\bigvee_{\delta \in \Delta} x_\delta = x$, we have $(\tau)\text{-}\lim_{\delta \in \Delta} x_\delta = x$.

is a solid norm and $\|x\|_f = f(x)$ for every $x \in X_+$. This norm defines on X the weakest locally solid topology $\check{\tau} \geqslant \tau$. We also have $|f(x)| \leqslant \|x\|_f$ for every $x \in X$, and hence $f \in X^*_{\check{\tau}}$.

Proof. Since $f(B_f) = \{1\}$, by Lemma 2(i,iii) it follows that $B_f \subset X_+$ is (τ)-bounded and $0 \notin \overline{B_f}$. Hence H_f is (τ)-bounded. As X is directed and $X_+ = \mathbb{R}_+ B_f$, the set H_f is absorbing. Consequently, the Minkowski functional $\| \ \|_f$ defined in (5) is a norm. It is easily seen that $\|x\|_f = f(x)$ for every $x \in X_+$. We claim that

$$H_f = \bigcup_{x \in B_f} [-x, x] = \bigcup_{x \in H_f \cap X_+} [-x, x]. \tag{6}$$

As $H_f \cap X_+ = [0, 1] \cdot B_f$, the second above equality holds. We next prove the first above equality.

"$\subset$". For $y \in H_f$, we have $y = (1-\lambda)u - \lambda v$ for some $u, v \in B_f$ and $\lambda \in [0, 1]$. Since B_f is convex, it follows that $x := (1-\lambda)u + \lambda v \in B_f$ and $y \in [-x, x]$.

"$\supset$". If $x \in B_f$ and $y \in [-x, x] \setminus \{-x, x\}$, then $x \pm y \in X_+ \setminus \{0\}$, and so $f(y) \in]-1, 1[$, by Lemma 2(ii). We thus get $u := \frac{x+y}{1+f(y)} \in B_f$ and $v := \frac{y-x}{1-f(y)} \in -B_f$, and hence $y = \frac{1+f(y)}{2}u + \frac{1-f(y)}{2}v \in H_f$.

We thus have proved (6). It follows that $H_f \subset X$ is a solid set, and hence that $\| \ \|_f$ is a solid norm. Therefore, the topology $\check{\tau}$ defined by $\| \ \|_f$ is locally solid. As H_f is (τ)-bounded, we have $\tau \leqslant \check{\tau}$. Now let χ be a locally solid topology on X, such that $\tau \leqslant \chi$. We need to show that $\check{\tau} \leqslant \chi$. Since $\check{\tau}$ is defined by the norm $\| \ \|_f$, it suffices to show that $H_f \supset V$ for some $V \in \mathcal{V}_\chi(0)$. Choose a balanced neighborhood $W \in \mathcal{V}_\tau(0)$, such that $W \subset X \setminus B_f$. Thus, $W \cap X_+ \subset [0, 1[\cdot B_f \subset H_f \cap X_+$. As χ is locally solid and $\tau \leqslant \chi$, there is a solid neighborhood $V \in \mathcal{V}_\chi(0)$, such that $V \subset W$. We thus get

$$V = \bigcup_{x \in V \cap X_+} [-x, x] \subset \bigcup_{x \in W \cap X_+} [-x, x] \subset \bigcup_{x \in H_f \cap X_+} [-x, x] = H_f.$$

Hence $\check{\tau} \leqslant \chi$. The last conclusion on f follows from $f(H_f) = \mathrm{co}(f(B_f) \cup f(-B_f)) = [-1, 1]$. $\square$

Example 10. Let $(X, \preceq, \| \ \|_\infty)$ be the space from Example 5. The requirements (i) and (ii) from Lemma 2 are fulfilled by the linear functional $f(x) = x(1)$. The topology $\check{\tau}$ is defined by the norm[3] $\|x\|_f = f(|x|) = |x(0)| + \mathrm{Var}_{[0,1]}x$, which is the standard solid norm on $BV([0, 1])$.

Remark 11. For any directed ordered normed space $(Y, \| \ \|)$ whose norm is solid and additive on Y_+ (hence also for (AL)-spaces), we have $\| \ \| = \| \ \|_f$ for the linear functional defined by

$$f : Y \to \mathbb{R}, \quad f(u - v) = \|u\| - \|v\| \quad \text{for all } u, v \in Y_+.$$

[3] For every $x \in BV([0, 1])$, we have $|x|(t) = |x(0)| + \mathrm{Var}_{[0,t]}x$ for $t \in [0, 1]$.

Several monotonicity-related properties transmit from the original topology τ to the stronger $\check{\tau}$.

Proposition 12. *We have the following:*

(i) *A monotonic net in X is $(\check{\tau})$-Cauchy, if and only if it is (τ)-Cauchy.*
(ii) *A monotonic net in X is $(\check{\tau})$-convergent, if and only if it is (τ)-convergent.*
(iii) *The topology $\check{\tau}$ is order continuous, if and only if so is τ.*
(iv) *The topology $\check{\tau}$ is monotonically complete, if and only if so is τ.*
(v) *A lower (or upper) bounded subset of X is $(\check{\tau})$-bounded, if and only if it is (τ)-bounded.*

Proof. Let $\| \|_f$ be the norm defined by (5). (v) follows easily from Lemma 2(iii), since $\|x\|_f = f(x)$ for every $x \in X_+$. Also, (iii) and (iv) will follow if we prove (i) and (ii). Since $\tau \leqslant \check{\tau}$, the implications "$\Rightarrow$" hold for both (i) and (ii). We next prove the remaining converse implications "$\Leftarrow$".

(i) Let us fix an increasing (τ)-Cauchy net $(x_\delta)_{\delta\in\Delta} \subset X$ and $\varepsilon \in]0,\infty[$. By Lemma 2(i), choose a neighborhood $W \in \mathcal{V}_\tau(0)$, such that $f(W \cap X_+) \subset [0, \frac{\varepsilon}{2}[$. As $(x_\delta)_{\delta\in\Delta}$ is (τ)-Cauchy, there exists $\delta_0 \in \Delta$, such that $x_\delta - x_{\delta_0} \in W \cap X_+$ for every $\delta \geqslant \delta_0$. For arbitrary $\delta', \delta'' \geqslant \delta_0$, we have

$$\begin{aligned}\|x_{\delta'} - x_{\delta''}\|_f &\leqslant \|x_{\delta'} - x_{\delta_0}\|_f + \|x_{\delta''} - x_{\delta_0}\|_f \\ &= f(x_{\delta'} - x_{\delta_0}) + f(x_{\delta''} - x_{\delta_0}) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

Hence $(x_\delta)_{\delta\in\Delta}$ is $(\check{\tau})$-Cauchy.

(ii) Let us fix an increasing net $(x_\delta)_{\delta\in\Delta} \subset X$, such that $(\tau)\text{-}\lim_{\delta\in\Delta} x_\delta = x$. As the positive cone X_+ is (τ)-closed, we have $\bigwedge_{\delta\in\Delta} x_\delta = x$, and so $(x - x_\delta)_{\delta\in\Delta} \subset X_+$. By Lemma 2(i) it follows that $\lim_{\delta\in\Delta} \|x - x_\delta\|_f = \lim_{\delta\in\Delta} f(x - x_\delta) = f(0) = 0$, hence that $(\check{\tau})\text{-}\lim_{\delta\in\Delta} x_\delta = x$. □

Since our representation theorem will deal with the space $X_{\check{\tau}}$, we next give a coincidence criterion for the topologies τ and $\check{\tau}$.

Proposition 13. *The topology τ is locally solid ($\tau = \check{\tau}$), if and only if X_τ is a bornological locally convex space with the property*

(S) *For every (τ)-bounded subset $A \subset X$, there is another (τ)-bounded $A_0 \subset X_+$, such that $A \subset A_0 - A_0$.*

Proof. "$\Rightarrow$". By Theorem 9, the topology $\tau = \check{\tau}$ is normable, hence also bornological. Let us fix a (τ)-bounded $A \subset X$. As $\tau = \check{\tau}$, we have $A \subset \varepsilon H_f$ for some $\varepsilon \in]0,\infty[$. It follows that

$$A \subset \varepsilon \cdot \mathrm{co}(B_f \cup (-B_f)) \subset \varepsilon([0,1] \cdot B_f - [0,1] \cdot B_f).$$

Since $[0, 1] \cdot B_f$ is (τ)-bounded, we conclude that the property (S) holds.

"$\Leftarrow$". It suffices to show that $H_f \in \mathcal{V}_\tau(0)$. Clearly, H_f is a balanced convex set. We claim that H_f absorbs any (τ)-bounded subset of X. To prove this, fix a (τ)-bounded $A \subset X$. According to the property (S), we have $A \subset A_0 - A_0$ for some (τ)-bounded $A_0 \subset X_+$. By Proposition 12(v) we deduce that $A \subset A_0 - A_0$ is $(\check{\tau})$-bounded, since so is A_0. Hence H_f absorbs A. Our claim is proved. Since X_τ is a bornological space, it follows that $H_f \in \mathcal{V}_\tau(0)$. We thus conclude that the topology $\tau = \check{\tau}$ is locally solid. $\square$

4. THE REPRESENTATION THEOREM

Let us recall that a (KB)*-space* is a Dedekind complete normed lattice with order continuous norm (topology), such that every normly bounded increasing sequence is upper bounded. Any (KB)-space is also a Banach space (see for instance [1], Chapter VII, §2, Proposition 1).

We can now prove our main result providing an isomorphic characterization of L^1-spaces. Let us recall that the property (c) from the Introduction is assumed to hold for X_τ.

Theorem 14 (Representation). *Assume that X_τ has the Riesz decomposition property, and that τ is monotonically σ-complete. Then $(X, \|\,\|_f) \simeq L^1(\mu)$ as normed lattices for some measure space $(T, \mathcal{T}, \mu)$. If in addition the topology τ is locally solid, then $X_\tau \simeq L^1(\mu)$ as topological vector lattices.*

Proof. We claim that $(X, \|\,\|_f)$ is a (KB)-space. By Theorem 9, $\|\,\|_f$ is a solid norm. According to Propositions 6(ii) and 8, X is a Dedekind complete vector lattice, and the topology τ is order continuous and monotonically complete. By Proposition 12(iii), $\check{\tau}$ is order continuous. Let us fix an increasing $(\check{\tau})$-bounded sequence $(x_n)_{n\in\mathbb{N}} \subset X_+$. By Propositions 6(i) and 12(v), the sequence is (τ)-Cauchy. Hence the limit $(\tau)\text{-}\lim_{n\to\infty} x_n =: x \in X$ exists, since τ is monotonically complete. As X_+ is (τ)-closed, we have $\bigvee_{n\in\mathbb{N}} x_n = x$. Our claim is proved. Consequently, $(X, \|\,\|_f)$ is a Banach lattice. Since $\|\,\|_f$ is additive on X_+, by the representation theorem of Kakutani for (AL)-spaces (see [2]) we conclude that $(X, \|\,\|_f) \simeq L^1(\mu)$ as normed lattices for some measure space $(T, \mathcal{T}, \mu)$. $\square$

Remark 15. The first (isometric) part of Theorem 14 implies the well-known representation theorem of Kakutani for (AL)-spaces. Indeed, for such a space $(Y, \|\,\|)$ we have $\|\,\| = \|\,\|_f$ for the linear functional $f(y) = \|y_+\| - \|y_-\|$, according to Remark 11. By Theorem 14, the Banach lattice $(Y, \|\,\|_f)$ is isomorphic to some $L^1(\mu)$. By Theorem 14 we also see that in Kakutani's theorem the completeness hypothesis may be replaced by that (weaker) of monotonic σ-completeness.

The following two corollaries are not consequences of Kakutani's result, but follow easily from Theorem 14.

Corollary 16. *Assume that X_τ is a sequentially (τ)-complete topological vector lattice. Then $X_\tau \simeq L^1(\mu)$ as topological vector lattices for some measure space $(T, \mathcal{T}, \mu)$.*

Proof. By definition, the topology of any topological vector lattice is assumed to be locally solid. As $\tau = \check{\tau}$, the conclusion follows by Theorem 14. $\square$

Corollary 17. *Assume that X_τ is a vector lattice and a Fréchet space (metrizable and complete). Then $X_\tau \simeq L^1(\mu)$ as topological vector lattices for some measure space $(T, \mathcal{T}, \mu)$.*

Proof. By Theorem 14, we have the conclusion for $X_{\check{\tau}}$. As $\tau \leqslant \check{\tau}$ and both τ and $\check{\tau}$ are metrizable and complete, we must have $\tau = \check{\tau}$. The conclusion follows by Theorem 14 or by Corollary 16. $\square$

Example 18. Let $(T, \mathcal{T}, \mu)$ be a measure space. On the Dedekind complete vector lattice $L^\infty(\mu)$, the L^1 norm $\|x\|_1 = \int_T |x(t)| \mathrm{d}\mu(t)$ is solid, but not monotonically σ-complete. Hence the σ-completeness condition from Theorem 14 cannot be removed.

5. APPLICATIONS

For any abstract real normed space $(Y, \| \ \|)$ without a "built-in" (natural) ordering, there is a standard method for obtaining one: by "projecting" from the origin of the space a translated closed unit ball. More precisely, for fixed $u \in Y$ with $\|u\| > 1$, let us consider the closed convex cone

$$(7) \qquad Y_+ := \mathbb{R}_+ \overline{B}_Y(u, 1),$$

and the associated linear ordering: $x \leqslant y \iff y - x \in Y_+$. Then Y becomes an ordered normed space. A natural question arises: how "good" can such an ordering be? Is it possible to obtain a vector lattice in this way? In infinite dimension, the answer is negative.

Theorem 19. *Let Y be a real normed space. If the ordering induced by the cone (7) has the Riesz decomposition property, then Y has finite dimension.*

Proof. Suppose that $(Y, \leqslant)$ has the Riesz decomposition property. Set $B := \overline{B}_Y(u, 1)$. Let $(\widetilde{Y}, \| \ \|)$ denote the completion of the normed space $(Y, \| \ \|)$. For any subset $A \subset \widetilde{Y}$, let $\widetilde{A}$ denote the closure of A in $\widetilde{Y}$. Clearly, $\widetilde{B}$ is a convex bounded closed subset of $\widetilde{Y}$ and $\widetilde{Y_+} = \mathbb{R}_+\widetilde{B}$. Let "$\preceq$" denote the linear ordering defined on $\widetilde{Y}$ by the cone $P := \widetilde{Y_+}$. We have divided our proof into two steps.

Step 1. We show that $(\widetilde{Y}, \preceq)$ *is a vector lattice.* Fix $\bar{x}_1, \bar{x}_2 \in \widetilde{Y}$ and set $M_0 := (\bar{x}_1 + P) \cap (\bar{x}_2 + P) \neq \emptyset$.

Claim 1. The set $M :=]0, \infty[\cdot u + M_0 = \bigcup_{\varepsilon \in]0,\infty[} (\varepsilon u + M_0)$ *is directed downwards.* To prove this, fix $\bar{y}_1, \bar{y}_2 \in M$. Choose $\varepsilon \in]0, \infty[$, such that $\bar{y}_j \in 3\varepsilon u + M_0$, that is,

$$\bar{x}_i + 3\varepsilon u \preceq \bar{y}_j \quad \text{for } i, j = 1, 2.$$

Let us observe that in $\widetilde{Y}$ any order segment $[\bar{a}, \bar{a} + u]$ contains the open ball $B_{\widetilde{Y}}(\bar{a} + \frac{u}{2}, \frac{1}{2})$, thus having nonempty interior. Therefore, for $i, j = 1, 2$ we have

$$\bar{x}_i + \varepsilon u \preceq x_i \preceq \bar{x}_i + 2\varepsilon u, \qquad \bar{y}_j - \varepsilon u \preceq y_j \preceq \bar{y}_j,$$

for some $x_i, y_j \in Y$. It follows that $y_j - x_i \succeq \bar{y}_j - \bar{x}_i - 3\varepsilon u \succeq 0$, hence that $y_j - x_i \in P \cap Y = Y_+$ for $i, j = 1, 2$. As Y has the Riesz decomposition property, we have $x_i \leqslant y \leqslant y_j$ $(i, j = 1, 2)$ for some $y \in Y$. We thus get $y \in \varepsilon u + M_0 \subset M$ and $y \preceq \bar{y}_1, \bar{y}_2$. Our first claim is proved.

Claim 2. $\inf M$ *exists in* $(\widetilde{Y}, \preceq)$. For fixed $a \in M$, set $M_a := M \cap (a - P)$. As $M \neq \emptyset$ is directed downwards, we can consider $-M \supset -M_a$ as increasing nets in $\widetilde{Y}$. The set M_a is normly bounded, since it is order bounded and the topology of $\widetilde{Y}$ is locally full, according to Proposition 1. By Proposition 6(i) we deduce that $-M_a$ in a Cauchy net, hence that the limit

$$\lim(-M_a) = \lim(-M) =: \xi \in \widetilde{Y}$$

exists in the Banach space $(\widetilde{Y}, \|\ \|)$. Since the positive cone P is normly closed, it follows that $\xi = \sup(-M) = -\inf M$. We thus get $-\xi = \inf M$, which proves our second claim.

Claim 3. We have $\bar{x}_1 \vee \bar{x}_2 = -\xi$ *in* $(\widetilde{Y}, \preceq)$. Let $z \in M_0$. For every $\varepsilon \in]0, \infty[$ we have $z + \varepsilon u \in M$, and so $-\xi \preceq z + \varepsilon u \xrightarrow{\|\ \|} z$ as $\varepsilon \downarrow 0$. As P is normly closed, we have $-\xi \preceq z$. We conclude that $-\xi = \min M_0 = \bar{x}_1 \vee \bar{x}_2$. This proves our third claim, as well as the statement from Step 1.

Step 2. We show that Y *has finite dimension.* According to Corollary 17, $(\widetilde{Y}, \|\ \|)$ is isomorphic as topological vector lattice to some $L^1(\mu)$. Hence the positive cone $L^1(\mu)_+$ has nonempty interior, since $P = \widetilde{Y_+}$ contains the open ball $B_{\widetilde{Y}}(u, 1)$. It follows that the measure μ is purely atomic and has finitely many atoms. Consequently, $\widetilde{Y}$ has finite dimension and $Y = \widetilde{Y}$. $\square$

Our second application deals with a pointwise ordering on a space of homogeneous polynomials.

Theorem 20. *Let $\mathcal{P}_n^d$ denote the vector space of all real polynomials of degree at most $d \geqslant 2$ in $n \geqslant 1$ indeterminates, endowed with the ordering "$\leqslant$" induced by the cone*

$$\left(\mathcal{P}_n^d\right)_+ := \left\{p \in \mathcal{P}_n^d \mid p(x) \geqslant 0 \text{ for every } x \in \mathbb{R}_+^n\right\}.$$

Then the ordered vector space $(\mathcal{P}_n^d, \leqslant)$ does not have the Riesz decomposition property, and hence is not a vector lattice.

Proof. Suppose the conclusion false. For abbreviation, set $X := \mathcal{P}_n^d$ and $q := \dim X$. We claim that X is a vector lattice isomorphic to $\mathbb{R}^q$. Let us consider on X the linear functional f and the norm $\| \ \|$ defined by

$$f(p) = \int\limits_{[0,1]^n} p(x)\,\mathrm{d}x, \quad \|p\| = \int\limits_{[0,1]^n} |p(x)|\,\mathrm{d}x.$$

As X has finite dimension, the above norm is complete and defines the unique Hausdorff linear topology τ of X. Clearly, the positive cone X_+ is (τ)-closed, since τ coincides with the pointwise convergence topology. Hence the convex (τ)-bounded set $B_f := \{p \in X_+ \mid f(p) = 1\}$ is (τ)-closed. We see that $0 \notin B_f$ and $X_+ = \mathbb{R}_+ B_f$. By Theorem 14 we deduce that $(X, \| \ \|_f) \simeq (\mathbb{R}^q, \| \ \|_1)$ as Banach lattices. Our claim is proved. Consequently, the set $E := \mathrm{ext}(B_f)$ of all extremal points of the convex set B_f is finite, since the same is true for $\{y \in \mathbb{R}_+^q \mid \|y\|_1 = 1\}$. We have $B_f = \mathrm{co}(E)$. We next show that

$$\mathbb{R}_+^n \subset \bigcup_{e \in E} e^{-1}(\{0\}). \tag{8}$$

For fixed $\xi \in \mathbb{R}_+^n$, consider $p \in X_+ \setminus \{0\}$ defined by $p(x) := \sum_{i=1}^n (x_i - \xi_i)^2$. Since $p(\xi) = 0$ and $\frac{1}{f(p)} \cdot p \in B_f = \mathrm{co}(E)$, we must have $e(\xi) = 0$ for some $e \in E$. This proves the claimed inclusion. But (8) is contradictory, since E is finite and every $e^{-1}(\{0\})$ from the union is a negligible set. $\square$

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