ON GENERATING H^p SPACES OF MARTINGALES

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Given a Markov process associated with a sufficiently general semigroup (P_t), we consider some stable subspaces of $H^p(P^\mu)$, where $p \geq 1$ and μ is an arbitrary law on the state space. We give a sufficient condition for these spaces to coincide with $H^p(P^\mu)$, and apply this result to the study of the *carré du champ* operator and to the construction of Lévy systems.

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1. **INTRODUCTION**

The well known theorem of Kunita [3] (see also [1, XV, 26]), concerning the existence of the *carré du champ* operator on the extended domain for a general class of semigroups, as well as the theorem of Dellacherie-Meyer [1, XV, 30, 35] concerning the existence of Lévy systems for the processes associated with general Ray semigroups, contain in their proofs an important technical result due to Kunita and Watanabe [4] and extended then in [1, XV, 25]. It deals with stable subspaces of some topological spaces of martingales $H^1(H^1_{\text{loc}})$ associated with a Markov process.

We think that in establishing this last result, some difficulties occur (see the proof in [1]) and, in addition, the manner of its use seems to be not quite clear (because of the topologies involved).

The purpose of this paper is to improve this result. Our second result, Theorem 2.1, deals with stable subspaces of H^p for $p \geq 1$. But we first give in Theorem 1.2 a general result concerning stable subspaces, that is suggested by an interesting result on extremality for probabilities [1, VIII, 57], the stochastic version of a measure theoretic theorem of Douglas [2]. In Section 1 we recall some general facts concerning stable subspaces of martingales and, in addition, consider martingales with respect to signed probabilities, that is, signed measures Q on Ω such that $Q(\Omega) = 1$.

2. MARTINGALES WITH RESPECT TO SIGNED PROBABILITIES

On a probability space $(\Omega, \mathcal{F}, P)$ endowed with a filtration $(\mathcal{F}_t)_{t\geq 0}$, $\mathcal{F}_{0-} = \mathcal{F}_0$, $\mathcal{F} = \mathcal{F}_{\infty-}$, satisfying the usual conditions (that is, $(\mathcal{F}_t)$ is right continuous and $\mathcal{F}_0$ contains all P-null sets), for $1 \leq p \leq \infty$ we consider the spaces H^p (maximal) of all r.c.l.l. martingales (X_t) such that

$$\|X\|_{H_p} := \|X^*\|_{L^p} < \infty,$$

where $X^* = \sup_{t\geq 0} |X_t|$, and we denote by $\| \; \|_{H_p}$ the corresponding norm. We notice that in fact the elements of the above spaces are classes of indistinguishable martingales that are uniformly integrable. We often identify an element of H^p with a representative of its class, when no cofusion is possible. We recall that for $(X_t) \in H^p$, $(Y_t) \in H^q$, with $\frac{1}{p} + \frac{1}{q} = 1$, $1 \leq p \leq \infty$, we say that (X_t) is orthogonal to (Y_t) if the (uniformly integrable) process $(X_t Y_t)$ also is a martingale that is null at 0. We denote by $\times$ this relation. The mapping $(X_t) \to X_\infty = \lim_{t\to\infty} X_t$ defines an imbedding of H^p into L^p, which is an isomorphism for $1 < p < \infty$. If $(X_t) \in H^p$, $(Y_t) \in H^q$, then

$$((X_t), (Y_t)) \to E[X_\infty Y_\infty]$$

defines the duality between the Banach spaces H^p and H^q, which is complete for $1 < p < \infty$. If $E[X_\infty Y_\infty] = 0$, we say that (X_t) is *ordinarily orthogonal* to (Y_t) and we denote by $\perp$ this relation. Of course, $(X_t) \times (Y_t) \Rightarrow (X_t) \perp (Y_t)$. If V is a subset of H^p, we denote by $V^\times$ (resp. $V^\perp$) the set of elements of H^q which are orthogonal (resp. ordinarily orthogonal) to each element of V. A *stable* subspace of H^p is a closed linear subspace G of H^p that is closed under stopping ($X \in G$, T stopping time $\Rightarrow \; X^T \in G$) and under multiplication by I_B, $B \in \mathcal{F}_0 : X \in G \Rightarrow I_B \cdot X \in G$. If V is a subset of H^p, the stable space generated by V is the smallest stable subspace of H^p containing V, and is denoted by $\sigma(V)$.

We state without proof Theorem 1.1 below. It is proved in [1, VIII, 49] for $p = 2$, a result due to Kunita-Watanabe, and for $p = 1$. For $1 < p < \infty$, the proof is exactly the same as for $p = 2$.

THEOREM 1.1. *Let V be an arbitrary subset of H^p, $1 \leq p < \infty$. Then $\sigma(V) = (V^\times)^\perp = V^{\times\times}$.*

If Q is a signed measure on $\mathcal{F}$, then we say that a real, $(\mathcal{F}_t)$ adapted process $(X_t)_{t\geq 0}$ on Ω is a martingale with respect to Q, relatively to $(\mathcal{F}_t)$, if X_t is integrable with respect to $|Q|$ (the variation of Q) for any $t \geq 0$, and the martingale property holds in the obvious manner with respect to Q. The situation we consider in this paper is as follows: P is a fixed probability on $\mathcal{F}$, $Q = Y_\infty \cdot P$, where $Y_\infty \in L^q(P)$, and (X_t) is a process which initially is an element of $H^p(P)$. The following remark is fundamental for the sequel: (X_t) is a martingale with respect to Q iff the process (X_tY_t) is a martingale with respect to P (relatively to $((\mathcal{F}_t))$, where $Y_t = E[Y_\infty|\mathcal{F}_t]$.

If V is a subset of H^p, we denote by $\bar{M}(V)$ the set of all signed measures Q absolutely continuous with respect to P on $\mathcal{F}$, such that $\mathrm{d}Q/\mathrm{d}P \in L^q(P)$ and

i) $Q = P$ on $\mathcal{F}_0$;

ii) any element (X_t) of V also is a martingale with respect to Q.

THEOREM 1.2. *Let V be a subset of H^p, $1 \leq p < \infty$, such that $1 \in V$. Then $\sigma(V) = H^p$ iff the set $\bar{M}(V)$ reduces to the element P.*

Proof. We consider $V^\times$ as a linear subspace of H^q ($H^\infty = L^\infty$). If we also consider the set $\bar{N}(V) = \{Q : Q$ is a signed measure absolutely continuous with respect to P on $\mathcal{F}$, $\mathrm{d}Q/\mathrm{d}P \in L^q(P)$, $Q = 0$ on $\mathcal{F}_0$ and any element of V is a martingale with respect to $Q\}$ then, obviously, $\bar{N}(V)$ is a linear space consisting of signed measures. Moreover, we have $\bar{M}(V) = P + \bar{N}(V)$, therefore $\bar{M}(V)$ reduces to the element P iff $\bar{N}(V)$ is the null space. Next, remark that $V^\times$ consists of uniformly integrable martingales. If we identify such a martingale (Y_t) with Y_∞, and any signed measure from $\bar{N}(V)$ with its derivative with respect to P, then using the fact that $1 \in V$ and the remark above, it follows that $\bar{N}(V)$ is isomorphic to $V^\times$ in an obvious manner, since any element of $V^\times$ is a martingale null at 0. Using then Theorem 1.1, the proof is complete. □

Remark 1.3. In the case $p = 1$, the fact that $\bar{M}(V)$ reduces to the element P is equivalent to the fact that P is extremal in the convex set $M(V)$ consisting of the positive elements of $\bar{M}(V)$ (that is, of genuine probabilities), and we recover [1, VIII, 57]. Indeed, if $\bar{M}(V)$ does not reduce to $\{P\}$, then we can pick an element $L \neq 0$ belonging to the linear space $\bar{N}(V)$ defined in the above proof. Then because L_∞ is bounded (excepting on a P-null set), if c is some constant exceeding $|L_\infty|$, then the elements

$$Q = (1 - L_\infty/c) \cdot P, \quad R = (1 + L_\infty/c) \cdot P$$

belong to $M(V)$, so that P is not extremal in $M(V)$. Conversely, if P is not extremal in $M(V)$, it is not alone in $M(V)$ and so neither in $\bar{M}(V)$.

However, we note that in the rest of our paper the case $p = 2$ will be of special interest for us.

3. GENERATORS FOR SOME H^P SPACES

On a compact metric space F we consider a Ray Markovian semigroup $(P_t)_{t\geq 0}$, and denote by N the set of nonbranching points for the Ray resolvent

$(U_p)_{p>0}$ associated with (P_t). We note that in general $P_0 \neq \mathrm{Id}$ (or equivalently $N \neq F$), the case $P_0 = \mathrm{Id}$ characterizing the Feller semigroups. Let $(\Omega, \mathcal{F}, \mathcal{F}_t, \Theta_t, X_t, P^\times)$ be the canonical Ray process with transition semigroup (P_t). Here, Ω consists of all mappings $\omega : [0, \infty) \to N$ which are right continuous, $\omega(t_-)$ exists in F for any $t > 0$, and $X_t(\omega) = \omega(t)$. In most cases there exists some distinguished point Δ absorbent relatively to (P_t), called cemetery, and we may reduce Ω to the paths with lifetime, that is, $\omega(t) = \Delta$ after some time t.

We can consider (U_p) as a resolvent of bounded operators (p-contractions) on the Banach space $b(\beta_u(F)))$ of all bounded universally measurable functions on F. We recall that given $f, g \in b(\beta_u(F))$, one says that f belongs to the domain $D(L)$ of (P_t) and $Lf = g$ if $f = U_p(pf - g)$ for some $p > 0$ or, equivalently, for any $p > 0$ by the resolvent equation. Obviously, $D(L)$ coincides with the image of the resolvent, which generates a σ-field denoted by $\mathcal{E}$. Recalling that a universally measurable set A is said to be negligible if $U_p(1_A) = 0$, it follows from a result of Mokobodzki [1, XII, 56] that, given f, Lf is unique up to a negligible set and, moreover, we can take $Lf = \liminf_{n\to\infty} n(nU_n f - f)$.

It is well known that the processes

$$M_t^{p,f} = \mathrm{e}^{-pt} f \circ X_t + \int_0^t \mathrm{e}^{-ps}(pf - g) \circ X_s \mathrm{d}s$$

defined on Ω, for $f \in D(L)$ and $p > 0$, are *bounded* r.c.l.l. martingales with respect to any P^μ, where μ is a law on F.

We say that a subset B of $b(\beta_u(F))$ is *almost total* if the measure $\eta = 0$ is the only signed measure η on F not charging the negligible sets such that $\eta(h) = 0$ for any $h \in B$.

We say that a subset P of $(0, \infty)$ *inverts the Laplace transform* if

$$\int_0^\infty \mathrm{e}^{-ps} u(s)\mathrm{d}s = 0 \quad \text{for } p \in P \Rightarrow u = 0,$$

for any bounded r.c. function u on $[0, \infty)$.

Finally, if P is as above, a subset A of $D(L)$ is said to be P-*full* if for any $p \in P$ the sets $B_p = \{pf - g; f \in A\}$ are almost total.

Example 1. If Λ is a subset of $D(L)$ such that $\Lambda \cup L(\Lambda)$ is almost total in $b((\beta_u(F))$ (this is the case in [1, XV, 24.1]), then the set $A = \Lambda \cup \{U_p f;\ f \in \Lambda,\ p \in P\}$ is P-full in our sense for any given P as above.

Example 2. If B is almost total in $b((\beta_u(F))$, then the set $A = \bigcup_{p\in P} U_p(B)$ is P-full in our sense for any given P as above.

THEOREM 2.1. *Let $A \subset D(L)$ be a P-full set such that $1 \in A$. Then for any law μ on F, and for any $a \geq 1$, $r > 0$, the stable space generated in $H^a(P^\mu)$ by the set $V = \{(M^{r,f}),\ f \in A\}$ coincides with $H^a(P^\mu)$.*

Proof. Fix $a \geq 1$, $r > 0$ and μ. First, we enlarge the set V by taking an *arbitrary* $r > 0$, thanks to the equation

$$M_t^{p,f} = \int_0^t \mathrm{e}^{-(p-r)s} \mathrm{d}M_s^{r,f}, \quad p > 0,$$

and using an obvious extension of [1, VIII, 47] for any $a \geq 1$. According to Theorem 1.2, applied in the frame of $(\Omega, \mathcal{F}^\mu, \mathcal{F}_t^\mu, P^\mu)$, where $(\mathcal{F}_t^\mu)$ satisfies the usual conditions, we have to check that any signed measure $Q \in \bar{M}(P^\mu)$ is equal to P^μ. We claim that this is equivalent to

$$E_Q[1_A \cdot h \circ X_{s+t}] = E_Q[1_A \cdot P_t h \circ X_s] \tag{2.1}$$

for any $s \geq 0$, $t > 0$, $A \in \mathcal{F}_s^\circ$, and any *bounded* Borel function h on F. Indeed, if (2.1) holds, let $0 = t_0 < t_1 < \cdots < t_n$ and let $h_0, h_1, \ldots, h_n$ be bounded Borel functions on F. Since (X_t) is Markovian with respect to P^μ, by induction, using the fact that $Q = P^\mu$ on $\mathcal{F}_0^\circ \subset \mathcal{F}_0^\mu$, we have

$$E_Q[h_0 \circ X_0 \ldots h_n \circ X_{t_n}] = E_{P^\mu}[h_0 \circ X_0 \ldots h_n \circ X_{t_n}], \tag{2.2}$$

and the monotone class theorem implies that $Q = P^\mu$, first on $\mathcal{F}^\circ$ and then on $\mathcal{F}^\mu$, because $Q \ll P^\mu$.

Next, remark that both sides of (2.1) are signed measures depending on h, so we may only consider continuous functions. As usual, by appling the Laplace transform, (2.1) becomes

$$\eta_s^p(h) \stackrel{\text{def}}{=} E_Q\left[1_A\Big(\int_{[s,\infty)} \mathrm{e}^{-pu} h \circ X_u \mathrm{d}u - \mathrm{e}^{-ps} U_p(h) \circ X_s\Big)\right] = 0 \tag{2.3}$$

for any $p \in P$. Now, remark that all measures η_s^p do not charge the negligible sets, because $Q \ll P^\mu$. Since A is P-full by hypothesis, to show that $\eta_s^p = 0$ it suffices to show that $\eta_s^p(h) = 0$ for any $h \in B_p$ (such an h is not necessarilly continuous), which is equivalent to prove that $(M^{p,f})_t$ are martingales with respect to Q for any $p \in P$ and any $f \in A$. But this property holds for $p > 0$ since $Q \in \bar{M}(P^\mu)$. The proof is complete. □

4. REMARKS AND APPLICATIONS

We recall that the processes

$$C_t^{p,f} = \mathrm{e}^{-pt} f \circ X_t - f \circ X_0 + \int_0^t \mathrm{e}^{-ps}(pf - g) \circ X_s \mathrm{d}s,$$

$$C_t^f = f \circ X_t - f \circ X_0 - \int_0^t g \circ X_s \mathrm{d}s, \quad p > 0,$$

define the “fundamental” martingales (in addition, (C_t^f) is an additive functional in a larger sense, which is its main quality).

Referring now to [1, XV, 26, 30], let us show how the proofs of these results can be adapted to our approach. For a given P-full subset A of $D(L)$, let $p > 0$ be fixed. We pass from (C_t^f) to $(M_t^{p,f})$ through the stochastic integral transform $C_t^{p,f} = \int_0^t \mathrm{e}^{-ps} \mathrm{d}C_s^f$ followed by the addition of the constant process $f \circ X_0$, and this way may be reversed. Next, we remark that both linear spaces

$$G = \{M \in H^2(P^\mu) : \mathrm{d}\langle M, M\rangle_t \ll \mathrm{d}t \text{ on } (0,\infty)\}$$

and

$$G' = \{M \in H^2(P^\mu) : \mathrm{d}\langle M, M\rangle_t^c \ll \mathrm{d}H_t \text{ on } [0,\infty)\}$$

are stable subspaces of $H^2(P^\mu)$, where $\langle\ ,\ \rangle$ denotes the angle bracket, and $\langle\ ,\ \rangle^c$ its continuous part as an increasing process. We then can apply Theorem 2.1 for $a = 2$, which implies that G and G' coincide with $H^2(P^\mu)$. This suffices since by the above transform $\mathrm{d}\langle C^f, C^f\rangle_t$ is equivalent to $\mathrm{d}\langle M^{p,f}, M^{p,f}\rangle_t$ on $(0,\infty)$ while $\mathrm{d}\langle C^f, c^f\rangle_t^c$ is equivalent to $\mathrm{d}\langle M^{p,f}, M^{p,f}\rangle_t^c$ on $[0,\infty)$. These are consequences of [1, VIII, 22] and general properties of the angle bracket.

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