ON POTENTIALS GENERATED BY INCREASING PROCESSES

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In the frame of a filtered probability space, we consider some processes with finite variation which "approximate" the processes with bounded mean oscillation. Also, in the frame of a Ray process, we consider increasing processes which generate the supermartingales of class (D).

AMS 2000 Subject Classification: 60Gxx, 60Hxx, 60Jxx.

Key words: increasing process, supermartingale, Ray process.

1. INTRODUCTION

The importance of increasing processes in probability is well known. The purpose of this paper is to add some complements to the applications of increasing processes and potentials (resp. the left potentials) that they generate. First, we consider the case of left potentials in the general frame of a filtered probability space, in the study of the space BMO of martingales with bounded mean oscillation, which occurs essentially in establishing the equivalence between the maximal and the quadratic norms on martingales (the inequality of Davis, from which one derives the inequality of Burkholder on H^p for $p > 1$).

Second, we consider the frame of a Markovian Ray process on a compact metric space and give a result concerning to the representation of (positive) supermartingales of class (D) in connection with the well known representation of potentials (excessive functions) by additive functionals. We assume that the reader is familiar with general martingale theory and general theory of Markov processes, as exposed in [1].

1. Let $(\Omega, \mathcal{F}_t, \mathcal{F}, P)$ be a filtered probability space such that the filtration $(\mathcal{F}_t)$ satisfies the usual conditions. We recall that the *potential* (resp. – *left potential*) generated by an increasing and right continuous, predictable and null at 0 (resp. optional, positive but not necessarily null at 0) process (A_t) on $[0, \infty]$, is the process

$$Z_t = E[A_\infty \mid \mathcal{F}_t] - A_t \quad (\text{resp. } Z_t^g = E[A_\infty \mid \mathcal{F}_t] - A_{t-}),$$

where $E[A_\infty \mid \mathcal{F}_t]$ is a r.c.l.l. version of the martingale on $[0,\infty]$ (continuous at ∞ !) taking the value A_∞ at ∞, which dominates the process (A_t).

Of course, the supermartigales (Z_t) and (Z_t^g) of class (D) (the second is not right continuous but is still optional) are properly potentials iff (A_t) does not jump at ∞, and each of them determines the corresponding (A_t).

We define the (vector) spaces $\widetilde{\mathrm{BMO}}_p$ for $p > 0$ as follows: a r.c.l.l. on $[0,\infty]$ and $(\mathcal{F}_t)$ adapted real process (X_t) belongs to $\widetilde{\mathrm{BMO}}_p$ iff there exists some constant $a > 0$ such that

$$E[|X_S - X_{T-}|^P \mid \mathcal{F}_T] \leq a^P$$

for any stopping times T, S such that $T \leq S$. One can see that considering the smallest constant as above, one defines a norm $\| \, \|_{\widetilde{\mathrm{BMO}}_p}$ on (the set of classes up to evanescence of) $\widetilde{\mathrm{BMO}}_p$, and it is known (see [1, VI, (109.7)]) that the spaces $\widetilde{\mathrm{BMO}}_p$ coincide for $p \geq 1$ in the sense that the corresponding norms are equivalent. We write $\widetilde{\mathrm{BMO}}$ instead of $\widetilde{\mathrm{BMO}}_1$.

Theorem 1.1. *Let $(X_t)_{0\leq t\leq\infty}$ be a real r.c.l.l. and $(\mathcal{F}_t)$ adapted process. Then (X_t) belongs to $\widetilde{\mathrm{BMO}}$ iff it admits a decomposition $X_t = U_t + K_t$, where (U_t) is a process (r.c.l.l. on $[0,\infty]$ and $(\mathcal{F}_t)$-adapted) with finite variation such that its variation generates a bounded left potential, and (K_t) is a bounded process. Moreover, if a denotes the $\widetilde{\mathrm{BMO}}_1$ norm of (X_t), one can find a decomposition of X_t as above such that the left potential generated by the variation of U_t is bounded by $6a$, and $|K_t| \leq 2a$ on $[0,\infty]$.*

Proof. The implication "$\Leftarrow$" is almost obvious by using the relations (S, T stopping times, $S \geq T$)

$$|U_S - U_{T-}| \leq \int_{[T,S]} |\mathrm{d}U_s| \leq \int_{[T,\infty]} |\mathrm{d}U_s|.$$

For the converse, we may assume that $X_\infty = X_{\infty-}$ by removing the *bounded* jump of (X_t) at ∞, which will be assigned to (U_t) or (K_t), as we wish, and we work in the rest of the proof with processes on $[0,\infty)$. Denote by a the $\widetilde{\mathrm{BMO}}_1$ norm of (X_t) and fix a constant $c > a$.

Let the sequence (T_n) of stopping times defined as

$$T_0 = 0, \quad T_{n+1} = \inf\{t > T_n : |X_t - X_{T_n}| > c\} \tag{1.1}$$

(of course we put $\inf \emptyset = \infty$), the increasing process

$$A_t = \sum_{n\geq 0} I_{\{T_n \leq t\}}, \quad A_\infty = \lim_{t\to\infty} A_t \tag{1.2}$$

and, finally,

$$U_t = X_0 + \sum_{n\geq 1}(X_{T_n} - X_{T_{n-1}})I_{\{T_n\leq t\}}, \quad U_\infty = \lim_{t\to\infty} U_t. \tag{1.3}$$

We show first that, with $A_{0-} = 0$ by definition,

$$E[A_\infty - A_{T-} \mid \mathcal{F}_T] \leq c/(c-a) \tag{1.4}$$

for any stopping time T (which means that (A_t) is an integrable increasing process with bounded left potential). Hence it will then follow that

$$E\Big[\int_{[T,\infty]} |\mathrm{d}U_s| \mid \mathcal{F}_T\Big] \leq c(c+a)/(c-a) \tag{1.5}$$

for any stopping time T because of the relation

$$|\mathrm{d}U_s(\omega)| \leq (c+a)\mathrm{d}A_s(\omega), \quad \omega\in\Omega, \tag{1.6}$$

obviously implied by the inequality

$$|X_{T_n} - X_{T_{n+1}}| \leq c+a, \quad n\in\mathbf{N}^*, \tag{1.7}$$

consequence of the construction of (T_n) and the fact that $|\Delta X_t| \leq a$ by hypothesis ($|X_0| \leq a$, too!).

So, returning to show (1.4), we fix a stopping time T, remark that $T_n \to \infty$ since (X_t) has (finite) left limits by hypothesis, and consider the random variable $n_T = \inf\{n : T \leq T_n\}$. If we put $S_n = T_{n_T+n}(T_\infty = \infty)$, one can easily see that (S_n) is an increasing sequence of stopping times. From the construction of (T_n) we deduce

$$cI_{\{S_{n+1}<\infty\}} \leq |X_{S_{n+1}} - X_{S_n}|(\leq c+a), \quad n\in\mathbf{N}^*, \tag{1.8}$$

and since the hypothesis implies the relation

$$E[|X_V - X_U| \mid \mathcal{F}_U] \leq a \tag{1.9}$$

for any stopping times U, V such that $U \leq V$ (we apply the hypothesis to $U+\frac{1}{n}$, $V+\frac{1}{n}$ and let $n\to\infty$) we can write

$$cE[S_{n+1}<\infty \mid \mathcal{F}_{S_n}] \leq E[|X_{S_{n+1}} - X_{S_n}| \mid \mathcal{F}_{S_n}] \leq aI_{\{S_n<\infty\}} \tag{1.10}$$

for any $n\in\mathbf{N}^*$. By inverse recurrence we get

$$E[S_{n+1}<\infty \mid \mathcal{F}_{S_0}] \leq (a/c)^n, \quad n\in\mathbf{N}^*. \tag{1.11}$$

We remark that $A_\infty - A_{T-} = \sum_{n\geq 0} I_{\{S_n<\infty\}}$, hence

$$E[A_\infty - A_{T-} \mid \mathcal{F}_T] = \sum_{n\geq -} E[S_n<\infty \mid \mathcal{F}_T] \leq c/(c-a). \tag{1.12}$$

Finally, it follows by the construction of (U_t) that $|X_t - U_t|$ is a process bounded by c, which completes the proof by taking $c = 2a$. □

Corollary ([1, VI, 109]). *The norms* $\| \|_{\widetilde{\mathrm{BMO}}_p}$ *on* $\widetilde{\mathrm{BMO}}$ *are all equivalent for* $p \geq 1$.

New proof. Since clearly $\| \|_{\widetilde{\mathrm{BMO}}_p} \geq \| \|_{\widetilde{\mathrm{BMO}}_q}$ for $p > q$ by Jensen's inequality, it suffices to show that for any $n \in \mathbf{N}$ there exists some constant $C_n > 0$ such that $\| \|_{\widetilde{\mathrm{BMO}}_n} \leq C_n \| \|_{\widetilde{\mathrm{BMO}}_1}$.

Let $(X_t) \in \widetilde{\mathrm{BMO}}$, denote $a = \|X\|_{\widetilde{\mathrm{BMO}}_1}$ and consider a decomposition $X_t = K_t + U_t$ such that $|K_t| \leq 2a$ and $E[\int_{[T,\infty]} |\mathrm{d}U_s| \mid \mathcal{F}_T] \leq 6a$ for any stopping time T. Denote for simplicity $B_t = \int_{[0,t]} |\mathrm{d}U_s|$, which is an increasing process with bounded left potential, and fix T as above. For any $n \in \mathbf{N}$ we have

$$E\left[\left(\sup_{s\geq T} |X_s - X_{T-}|\right)^n \mid \mathcal{F}_T\right] \leq$$

$$\leq E\left[\left(\sup_{s\geq T} |K_s - K_{T-}| + \int_{[T,\infty]} |\mathrm{d}U_s|\right)^n \mid \mathcal{F}_T\right] \leq \tag{1.13}$$

$$\leq E\left[\sum_{k=0}^{n} C_n^k (4a)^k (B_\infty - B_{T-})^{n-k} \mid \mathcal{F}_T\right].$$

But it is well known and "classical" (see [1, VI, 106 c)]) that relatively to the *increasing* process (B_t) we have

$$E[(B_\infty - B_{T-})^m \mid \mathcal{F}_T] \leq m!(6a)^m, \quad m \in \mathbf{N}. \tag{1.14}$$

Returning to (1.13), we see that there exists some constant C'_n (for each n) such that

$$E\Big[\Big(\sup_{s\geq T} |X_s - X_{T-}|\Big)^n \mid \mathcal{F}_T\Big] \leq C'_n a^n, \tag{1.15}$$

which implies of course the desired conclusion. In addition, for $n = 1$, the above relation implies that the increasing process $A_t = \sup_{s\leq t} |X_s|$ generates a bounded left potential. □

Remarks. We recall that the subspace of $\widetilde{\mathrm{BMO}}_p$ consisting of uniformly integrable martingales continuous at ∞ is known as BMO_p, and BMO_1 is isomorphic to the dual of the Banach space H^1_* (with maximal norm), subject to a theory essentially due to Herz and Lepingle, developped in [1, VII, 70–80]. It follows from this theory that the set $\{X_\infty : (X_t) \in \mathrm{BMO}\}$ coincides with the set $\{A_\infty : (A_t)$ is a process with finite variation on $[0,\infty]$ (r.c.l.l. and $(\mathcal{F}_t)$ adapted) whose variation generates a bounded left potential$\}$, which coincides according to our result to the set $\{X_\infty : (X_t) \in \widetilde{\mathrm{BMO}}\}$.

We note that the martingale property of elements of BMO occurs "discretely" in establishing the duality between H^1_* and BMO by above quoted "maximal" theory, where the set $\{X_\infty : (X_t) \in \mathrm{BMO}\}$ is of interest. But when establishing the equivalence between maximal and quadratic norms on H^1 (the inequality of Davis) one considers BMO_2 (which is isomorphic to BMO_1) as imbeded in the space H^2 of square integrable martingales, where the square bracket $[,]$ is defined and the equations below hold (for any $(X_t) \in H^2$):

$$E[(X_\infty - X_{T-})^2 \mid \mathcal{F}_T] = E[[X,X]_\infty - [X,X]_{T-} \mid \mathcal{F}_T] \tag{1.16}$$

for any stopping time T, and

$$E[M_\infty X_\infty] = E[[M,X]_\infty] \tag{1.17}$$

for any $(M_t) \in H^2$. In fact, the second equation follows from the first by taking $T = 0$ and polarization. The reader could check the role of above relations for $(X_t) \in \mathrm{BMO}$ in different proofs of the inequality of Davis (see for example [1, VII, 90]). The right hand of 1.16 suggests to consider the "quadratic" BMO norm. See, for example, [3] for an extension of the "quadratic" theory.

2. On a compact metric space F we let $(P_t)_{t\geq 0}$ be a Ray semigroup which is supposed to be Markovian ($P_t 1 = 1$). We consider the canonical Ray Markovian process $(\Omega, \mathcal{F}_t, \mathcal{F}, \Theta_t, X_t, P^x)$ on F with transition semigroup (P_t). We refer the reader for definition and theory of Ray semigroups and Ray processes to [1, XIV] and [2]. Here, we recall that $P_0 \neq \mathrm{Id}$ in general ($P_0 = \mathrm{Id}$ characterizes the Feller semigroups) and if N denotes the set of nonbranch points of F, then the "path space" Ω is the set of right continuous mappings $\omega : [0,\infty) \to N$ such that $\omega(t-)$ exists in F for any $t > 0$; we put $X_t(\omega) = \omega(t)$ for $t \geq 0$. In most situations, there exists a distinguished absorbing point (relatively to (P_t)) $\Delta \in F$ called "death point" or "cemetery". Clearly, $\Delta \in N$, and in this case the subset of Ω consisting of paths with "life time" carries all probabilities $P^x_{x\in N}$ and can be also considered a path space.

Theorem 2.1. *Let (Y_t) be a positive and finite r.c.l.l. and $(\mathcal{F}_t)$-adapted process on Ω such that (Y_t) is a supermartingale of class (D) with respect to P^x for any $x \in N$. Then there exists an increasing r.c. and $(\mathcal{F}_t)$-adapted process (A_t), predictable and null at 0, unique up to evanescence such that for any $x \in N$ it is integrable with respect to P^x and we have*

$$Y_t = E^x[A_\infty - A_t \mid \mathcal{F}_t] + E^x[Y_\infty \mid \mathcal{F}_t] \quad P^x\text{-a.s.}, \tag{2.1}$$

where $Y_\infty = \liminf_{t\to\infty} Y_t$.

Proof. We first note that Y_∞ is integrable with respect to any P^x because (Y_t) is a positive supermartingale and, moreover, $Y_\infty = \lim_{t\to\infty} Y_t$ P^x-a.s. (see [1, VI, 6]).

Fix $x \in N$. The process $Y'_t = Y_t - E^x[Y_\infty \mid \mathcal{F}^x_t]$, where $(E^x[Y_\infty \mid \mathcal{F}^x_t])$ is considered as a r.c.l.l. martingale dominated by (Y_t), is a potential of class (D) with respect to P^x. It is then well known (see [1, VII, 8]) that there exists a unique P^x integrable *increasing predictable* process (B^x_t) null at 0, r.c., $(\mathcal{F}^x_t)$-adapted, such that

$$Y_t = E^x[B^x_\infty - B^x_t \mid \mathcal{F}^x_t] + E^x[Y_\infty \mid \mathcal{F}^x_t] \quad P^x\text{-a.s.} \tag{2.2}$$

We look to find a common version of all processes (B^x_t) over $x \in N$, which should be increasing, null at 0, r.c., $(\mathcal{F}_t)$ adapted. For this purpose we first use the standard approach (see [1, VII, 22]) by "close Laplacians"

$$\Delta^n_t = nE^x[Y'_t - Y'_{t+\frac{1}{n}} \mid \mathcal{F}^x_t] = nE^x[Y_t - Y_{t+\frac{1}{n}} \mid \mathcal{F}^x_t]. \tag{2.3}$$

In fact, (Δ^n_t) is the difference between the supermartingale (nY_t) and a r.c.l.l. version of the supermartingale $(nE^x[Y_{t+\frac{1}{n}} \mid \mathcal{F}^x_t])$ dominated by (nY_t), so that (Δ^n_t) is a positive, r.c.l.l. and $(\mathcal{F}^x_t)$ adapted process. As in [1, VII, 22], we take

$$B^n_t = \int_{[0,t]} \Delta^n_s \mathrm{d}s \tag{2.4}$$

which converges to B^x_t in the weak topology $\sigma(L^1, L^\infty)$ with respect to P^x. Using now a slight modification of the selection lemma [1, XV, 2b)] applied to $B^x_t (x \in N)$ for fixed t, we can find a $(\mathcal{F}_t)$-measurable random variable B_t such that $B_t = B^x_t$ P^x-a.s. for any $x \in N$, if we are able to show that the mapping

$$x \to E^x[\varphi B^n_t]$$

is universally measurable for any (fixed) bounded $\mathcal{F}^\circ_t$-measurable function φ and for any $n \in N$. We have

$$\begin{gathered} E^x[B^n_t\varphi] = E^x\left[\Big(\int_{[0,t]} \Delta^n_s \mathrm{d}s\Big)\varphi\right] = \int_{[0,t]} E^x[\Delta^n_s\varphi]\mathrm{d}s \\ n\int_{[0,t]} E^x[E^x[Y_s - Y_{s+\frac{1}{n}} \mid \mathcal{F}^\circ_s]\varphi]\mathrm{d}s = n\int_{[0,t]} E^x[(Y_s - Y_{s+\frac{1}{n}})E^x[\varphi|\mathcal{F}^\circ_s]]\mathrm{d}s. \end{gathered} \tag{2.5}$$

Using the monotone class theorem, in order to check the (universal) measurability in x of the last expression, we may consider just the case where $\varphi = f_1 \circ X_{t_1} f_2 \circ X_{t_2} \cdots f_n \circ X_{t_n}$, $0 \le t_1 < t_2 \cdots < t_n = t$, with f_i Borel and bounded on N for $i = 1, 2, \dots, n$. Using now the Markov property, we have

$$\int_{[0,t]} E^x[(Y_s - Y_{s+\frac{1}{n}})E^x[\varphi \mid \mathcal{F}^\circ_s]]\mathrm{d}s = \tag{2.6}$$

$$= \sum_{k=0}^{n-1} \int_{[t_k, t_{k+1})} E^x[(Y_s - Y_{s+\frac{1}{n}})E^x[\varphi \mid \mathcal{F}^\circ_s]]\mathrm{d}s =$$

$$= \sum_{k=0}^{n-1} \int_{[t_k, t_{k+1})} E^x[(Y_s - Y_{s+\frac{1}{n}}) f_1 \circ X_{t_1} \dots f_k \circ X_{t_k} P_{t_{k+1}-s} g_k(X_s)] \mathrm{d}s,$$

where $g_{n-1} = f_n$, and for $k = 0, 1, \dots, n-2$ we put

$$g_k(\cdot) = f_{k+1}(\cdot) P_{t_{k+2}-t_{k+1}}(\cdot, dx_{k+2}) \dots P_{t_n - t_{n-1}}(x_{n-1}, dx_n) f_{k+2}(x_{k+2}) \dots f_n(x_n).$$

Of course, all g_k are Borel bounded functions on N. Next, we remark that for any bounded, Borel function g on N and $u \geq 0$, the function $P_{u-s}g(X_s(\omega))$ on $[0, u] \times \Omega$ is $\beta[0, u] \times \mathcal{F}_u^\circ$ measurable. Indeed, using the monotone class theorem, we may consider just the case where g is the trace on N of a continuous function on F, and using the right continuity of the semigroup (P_t) we have the approximation

$$P_{u-s}g(X_s(\omega)) = \lim_n h^n(s, \omega),$$

where $h_n(s, \omega) = P_{u-\frac{ku}{2^n}} g(X_s(\omega))$ for $\frac{ku}{2^n} \leq s < \frac{(k+1)u}{2^n}$.

The measurability claimed above being now clear, we return to (2.6). For each $k = 0, \dots, n-1$, the function

$$1_{[t_k, t_{k+1})}(s)(Y_s - Y_{s+\frac{1}{n}})(\omega) f_1 \circ X_{t_1}(\omega) \dots f_k \circ X_{t_k}(\omega) P_{t_{k+1}-s} g_k(X_s(\omega))$$

is $\beta[0, \infty) \times \mathcal{F}$-(in fact $\beta[0, \infty) \times \mathcal{F}_{t_{k+1}+\frac{1}{n}}$-) measurable.

Taking their sum, by Fubini theorem and the well known fact that the mapping $x \to E^x\lceil H \rceil$ is universally measurable for any $\mathcal{F}$-measurable (bounded or positive) random variable H, the universal measurability of the mappings $x \to E^x[\varphi B_t^n]$ considered above is now clear. Finally, it remains to make a standard regularization of the "process" (B_t) by considering

$$a_r^1 = \sup_{\substack{s \text{ rational} \\ s \leq r}} B_s$$

for rational $r \geq 0$, and

$$a_t^2 = a_{t+}^1 \text{ (right limit along the rationals } > t),$$

$A_t = a_t^2$ for any $t \geq 0$ if $a_0^2 = 0$, $A_t = 0$ for any $t \geq 0$ if $a_0^2 \neq 0$.

The process (A_t) is r.c., $(\mathcal{F}_t)$-adapted (we used the fact that $(\mathcal{F}_t)$ is right continuous) and P^x-indistinguable from (B_t^x) for any $x \in N$, which completes the proof. □

Remarks. Our result implies the first half of the well known result concerning the representation of finite p-potentials of class (D) by p-additive functionals (see [1, XV, 2]). It points out that the p-additivity of the process (A_t) is exclusively a consequence of the p-homogeneity of the process (Y_t), namely that $Y_t = \mathrm{e}^{-pt} f \circ X_t$, where f is a p-excessive finite function on F. Another application of our result is the extension of [1, XV, 7b)] to the case of Ray semigroups, by the possibility to represent supermartingales $Y_t = f \circ X_t^T$, where f is a finite excessive function and T is a stopping time relative to $(\mathcal{F}_t)$ such that (Y_t) is a supermartingale of class (D) (relative to P^x for any $x \in N$).

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Received 20 October 2006

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