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Jumps of the canonical process associated with a Ray semigroup

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Abstract

Given a (submarkovian) Ray semigroup (P_t) on a locally compact metric space E, we consider under some additional hypotheses the nonlocal part of the generator of (P_t), given by a kernel $l(x, \cdot)$, and we establish a relation between $l(x, \cdot)$ and the point process of the jumps of the canonical Ray process with "transition" semigroup (P_t).

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1. INTRODUCTION

It seems to be almost classical that a r.c.l.l. Markov process with transition semigroup (P_t) is in fact continous iff the (infinitesimal) generator of the semigroup (P_t) is a local operator.

In the well known case of process with independent and homogenous increments, the celebrated Lèvy measure is used to "measure" the jumps of the process, leading to an explicit continous representation for the "predictable compensator" of the point process of the jumps of the original process (see [3], [8]).

A generalization of this representation is also given in [2], [4] by mean of Lèvy kernel in the general frame given by a Feller (Ray) semigroup (P_t) on a compact metric space.

We think that the "identification" of the Lèvy kernel as the "nonlocal part" of the generator of the semigroup is not quite clear in this general frame. We consider our main result, theorem 3.3, as a step in direction of establishing this relationship, under some additional hypotheses of course, more general than in [4, XV, 37] which represented the start point for our paper.

2. Preliminaries

On a locally compact space E with countable base we consider a submarkovian Borel semigroup (P_t) with associated resolvent (U_p). We adjoin a point Δ to E, which is the Alexandrov point if E is not compact, we denote $F = E \cup \{\Delta\}$, and we consider on F the markovian canonical extension $(\overline{P}_t)$ (resp. $(\overline{U}_p)$) of (P_t) (resp. (U_p)). We assume that $(\overline{U}_p)$ is a Ray resolvent on F, with associated Ray semigroup $(\overline{P}_t)$. For general properties of Ray resolvents (semigroups, processes) we refer the reader to [4] and [5].

We denote by N the set of nonbranching points for $(\overline{U}_p)$ less Δ, which is known to be a G_δ subset of E, and moreover $E \setminus N$ is negligible for (P_t), that is $P_t(1_{E\setminus N}) \equiv 0$ for any $p \geq 0$. In the sequel we identify the functions on E to the functions on F null at Δ.

It f, g are continous on F, and null at Δ, we say that f belongs to the continous domain $D^c(L)$ of (P_t) and $Lf = g$ if for any $x \in N$ we have

$$f(x) = U_p(pf - g)(x)$$

for any $p > 0$ (using the resolvent equation, it suffices to check this for a single $p > 0$), which is in fact equivalent to saying that $f = U_p(pf - g)$ identically on N if we consider the restriction of (U_p) to the absorbent set N, and also the restrictions of f, g to N.

REMARK 2.1. If $f \in D^c(L)$ and $Lf = g$, then the following relation holds:

$$g(x) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}(P_\varepsilon f(x) - f(x)), \text{ for any } x \in N.$$

Conversely, if f, g are continous on F, null at Δ, if above relation holds, and moreover the following relation is true:

$$P_t f(x) = f(x) + \int_0^t P_s g(x) ds, \text{ for any } x \in N,$$

then $f \in D^c(L)$ and $Lf = g$ (this is the case for example if the mapping $t \to P_t f(x)$ is derivable on $(0, \infty)$ for any $x \in N$ and moreover the (right) derivative at 0 exists in the sense of uniformly bounded pointwise convergence).

We let to the reader the proof, noting for the converse that even if $\frac{d}{dt} P_t f(x) = P_t g(x)$ as right derivative for any $t \geq 0$, the above analogous of Leibniz-Newton formula does not follow necessarily from this.

For the sequel we consider a (fixed) algebra $\mathcal{A} \subset D^c(L)$ such that $\mathcal{A}$ is dense in $\mathcal{C}_0(E)$, and moreover the following property of Urâsohn type holds: for any compact $K \subset E$ and for any $x \in (E \setminus K) \cap N$, there exists $h \in \mathcal{A}$, $0 \leq h \leq 1$, such that $h(x) = 0$ and $h = 1$ on K. (for example in the case of Brownian semigroup on $\mathbb{R}^n$ one takes $\mathcal{A} = \mathcal{C}_c^\infty$).

Using the above hypothesis, it follows easily that the mapping $f \to \lim_{t \to 0} \frac{1}{t} P_t f(x)$ defined on $\mathcal{A} \cap \mathcal{C}_0(E \setminus \{x\})$ extends to a (positive) Radon measure on $E \setminus \{x\}$, for any $x \in E$. We denote by $l(x, \cdot)$ this measure, which we consider on E, null on $\{x\}$.

PROPOSITION 2.2. Suppose that E is a group with unit element $e \in N$, and $\mathcal{A}$ is invariant under translation. Then the mapping $x \to l(x, \cdot)$ is a kernel from $(N, \beta(N))$ to $(E, \beta(E))$.

PROOF. Let (K_n) be an increasing sequence of compact sets of E such that $\bigcup_n K_n = E$. For any $n \in N$, let $h_n \in \mathcal{A}$, $0 \leq h_n \leq 1$ such that $h_n(e) = 0$, $h_n = 1$ on K_n, and put $g_n(x, y) = h_n(y - x)$. We can now perform an obvious approximation using the above sequence (g_n), and the definition of $l(x, \cdot)$ as acting on elements of $\mathcal{A}$, to get the desired conclusion. ■

Another situation where the above conclusion holds is given in [4, XV, 37]. The additional hypotheses in above proposition will not be used in the sequel, and in order to avoid a loss of generality, from now on will be in force the assumption that $x \to l(x, \cdot)$ is a kernel from $(\mathrm{N}, \mathcal{E})$ to $(\mathrm{E}, \beta(\mathrm{E}))$, where $\mathcal{E}$ denotes the σ-algebra generated by the (universally measurable) exccesive functions. We denote by $\bar{l}$ the kernel from $(N, \mathcal{E})$ to $(E \times N, \beta(E) \times \mathcal{E})$ defined by $\bar{l}(x, f) = l(x, f(\cdot, x))$

Let $(W, \mathcal{G}, \mathcal{G}_t, Y_t, Q)$ be a right continous Markov process with states space $N \cup \{\Delta\}$ and transition semigroup $(\overline{P}_t)$, such that the filtration $(\mathcal{G}_t)$ is right continous, the left limit Y_{t-} exists in F for any $t > 0$, and Δ is a death point for (Y_t), that is $Y_t(w) = \Delta$ after some first moment $\xi(w)$. We suppose moreover that $Y_0 \in E$, that is $\xi > 0$. The proof of the next result is essentially contained in the proof of [4, XV, 37]. However our frame is more general, and we give only the statement, letting to the reader the adaptation of this proof to our situation (C_c^∞ should be replaced by $\mathcal{A}$, X_t by Y_t, P^x by Q).

For any $x \in F$ and $\varepsilon > 0$ we denote $U_{\varepsilon,x} = \{y \in E, ; d(x,y) > \varepsilon\}$, where d is a distance on F defining the topology. If f is a function on $F \times E$, we put $f_\varepsilon(x,y) = f(x,y)\ \mathrm{I}_{\{(x,y);d(x,y)>2\varepsilon\}}$. Finally we put $l(\Delta,\cdot) = 0$, as measure on E.

LEMMA 2.3. For any positive predictable process (g_t) on W, for any positive Borel function f on $F \times E$, the following relation holds for any $x \in F$, $\varepsilon > 0$, $p > 0$:

$$(2.1) \qquad E_Q\left[g_{T_U}\ \mathrm{I}_{\{0<T_U\}} e^{-pT_U} f_\varepsilon(Y_{T_U-}, Y_{T_U})\right] = E_Q\left[\int_0^{T_U} g_s e^{-ps}\bar{l}(Y_s, f_\varepsilon)ds\right],$$

where we denoted $U = U_{\varepsilon,x}$ for simplicity, and T_U denotes the hitting time of U.

We note that in fact $T_U < \xi$ if $T_U < \infty$, and we do not exclude the possibility that $Y_{T_U-} = \Delta$ in this case.

3. THE MAIN RESULT

From now on we consider the canonical Ray process $(\Omega, \mathcal{F}, \mathcal{F}_t, \Theta_t, X_t, P^x)$ with transition semigroup $(\overline{P}_t)$ on the states space $N \cup \{\Delta\}$, such that X_{t-} exists in F, for any $t > 0$ (see [4]). If T is a fixed stopping time, we denote $Y_t = X_{T+t}$ as defined on $\{T < \infty\}$, where we also consider the filtrations $\mathcal{F}'^0_t = \sigma(Y_s; s \le t)$ for $0 \le t \le \infty$, and finally $\mathcal{F}'_t$ the completion of $\mathcal{F}'^0_t$ in $\mathcal{F}^0_\infty$ with respect to all P^μ, where μ is a law on N. If $\{S_y\}_{y \in N \cup [\Delta]}$ and S are stopping times with respect to $(\mathcal{F}'_t)$ such that $S_y = S$ on $\{Y_0 = y\}$ for any $y \in N \cup [\Delta]$, we say that $\{S_y\}_{y \in N \cup [\Delta]}$ is a collection of stopping times represented by S.

We remark now that for any $0 \le t \le \infty$, the mapping

$$G \to G \circ \Theta_T \mid_{\{T<\infty\}}$$

is a linear bijection from $b(\mathcal{F}_t^0)$ to $b(\mathcal{F'}_t^0)$. This follows from the fact that $\Theta_T^{-1}\{\mathcal{F}_t^0\} = \mathcal{F'}_t^0$, a consequence of the monotone class theorem (starting from the products $f_1 \circ Y_{t_1} \dots f_n \circ Y_{t_n}$), and using the hypothetis that $\Theta_T\{T < \infty\} = \Omega$. For any $x \in N \cup [\Delta]$, the probability P'^x on $\{T < \infty\}$ endowed with $\mathcal{F'}_\infty^0$, defined by

$$E'^x[G \circ \Theta_T] = E^x[G] \tag{3.1}$$

is well defined, and taking in particular $G = f_1 \circ X_{t_1} \dots f_n \circ X_{t_n}$, one can see that the process (Y_t) is Markovian with respect to P'^x, and with transition semigroup $(\overline{P}_t)$. Moreover $P'^x\{Y_0 = x\} = 1$.

PROPOSITION 3.1. Let $(H_\cdot^1)$, $(H_\cdot^2)$ be two positive processes on $\{T < \infty\}$, measurable with respect to $\beta(R_+) \times \mathcal{F}'_\infty$, and let $\{S_y\}_{y \in N \cup [\Delta]}$ be a collection of stopping times represented by S. Suppose that $\Theta_T\{T < \infty\} = \Omega$, and

$$E'^x[H^1_{S_x}] = E'^x[H^2_{S_x}] \tag{3.2}$$

for any $x \in N \cup [\Delta]$. Then we have

$$E^x[H_S^1 U; T < \infty] = E^x[H_S^2 U; T < \infty] \tag{3.3}$$

for any positive random variable U measurable with respect to $\mathcal{F}_T$, and for any $x \in N \cup [\Delta]$.

PROOF. Suppose first that U is $\mathcal{F}_T$ measurable and bounded on $\{T < \infty\}$. We remark that for any law γ on $N \cup [\Delta]$, we have

$$E^\gamma[G'U; T < \infty] = E^\gamma\left[E'^{X_T}[G']U; T < \infty\right] \tag{3.4}$$

(we assume that G' is extended somehow on $\{T = \infty\}$ to the left side). Indeed, if $G' = G \circ \Theta_T$, where $G \in b(\mathcal{F}_\infty^0)$, the above relation expresses the strong Markov property, by using (3.1). Moreover (3.4) holds if G' is $\mathcal{F}'_\infty$ measurable by using a standard close framing from above and from below (consider first the case where U is positive). Finally (3.4) still holds if U and G' are taken positive (instead of bounded) by monotone convergence. We can write now:

$$E^x[H^1_S U; T<\infty] = E^x\left[E'^{X_T}[H^1_S]U; T<\infty\right] =$$

$$= E^x\left[E'^{X_T}[H^1_{S_{X_T}}]U; T<\infty\right] = E^x\left[E'^{X_T}[H^2_{S_{X_T}}]U; T<\infty\right] = \tag{3.5}$$

$$= E^x\left[E'^{X_T}[H^2_S]U; T<\infty\right] = E^x[H^2_S U; T<\infty],$$

using the fact that H^i_L is $\mathcal{F}'_\infty$ measurable for $i=1,2$ and for any stopping time L with respect to $(\mathcal{F}'_t)$, (3.4), and hypothesis (3.2) to the second line. The second and the fourth equality follow since $P'^y\{Y_0=y\}=1$ for any $y\in N\cup[\Delta]$ and $S_y=S$ on $\{Y_0=y\}$ by hypothetis on these stopping times. ■

REMARK 3.2. Relation (3.4) from above represents a slight extension and precisation of the well known strong Markov property, concerning to the measurability of G': if G is $\mathcal{F}=\mathcal{F}_\infty$ measurable, then $G'=G\circ\Theta_T$ is $\mathcal{F}'_\infty$ measurable. The price paid was to consider the additional hypothesis $\Theta_T\{T<\infty\}=\Omega$, which enables us to consider the probabilities P'^x.

THEOREM 3.3. For any positive Borel function $a(t)$ on $[0,\infty)$, for any positive Borel function f on $F\times E$, and for any $x\in N$, the following relation holds:

$$E^x\left[\sum_{s<\xi} a(s)f(X_{s-},X_s)\,\mathrm{I}_{\{X_{s-}\neq X_s\}}\right] = E^x\left[\int_0^\xi a(s)\bar{l}(X_s,f)ds\right]. \tag{3.6}$$

Proof. We fix $x \in N$, and we consider first a stopping time T with respect to $(\mathcal{F}_t)$ such that $P^x\{T < \xi\} > 0$, It is well known from the strong Markov property that the process $(\{T < \xi\}, \mathcal{F}, \mathcal{F}_{T+t}, X_{T+t}, P^x)$ satisfies the conditions prior to lemma 2.3; of course we consider the traces of $\mathcal{F}$ and $\mathcal{F}_{T+t}$ on $\{T < \xi\}$, and P^x should be normalized if $P^x\{T < \xi\} \neq 1$. We keep the notation $Y_t = X_{T+t}$, and for any $\varepsilon > 0$ we define $S_\varepsilon = \inf\{t > 0; Y_t \in E, d(Y_t, Y_0) > \varepsilon\}$ on $\{T < \xi\}$, which is a stopping time with respect to $(\mathcal{F}'^0_t)$. If we assume in addition that $\Theta_T\{T < \infty\} = \Omega$, we claim that the following relation holds:
(3.7)

$$E^x\left[a(T+S_\varepsilon)e^{-pS_\varepsilon} f_\varepsilon(Y_{S_\varepsilon-}, Y_{S_\varepsilon}); T < \xi\right] = E^x\left[\int_0^{S_\varepsilon} a(T+s)e^{-ps}\bar{l}(Y_s, f_\varepsilon)ds; T < \xi\right].$$

Indeed, we may first reduce to the case where a and f are bounded, and then using the monotone class theorem we may reduce to the case where $a(t) = e^{-qt}$, $q > 0$. In this case, relation (3.7) becomes
(3.8)

$$E^x\left[e^{-(p+q)S_\varepsilon} f_\varepsilon(Y_{S_\varepsilon-}, Y_{S_\varepsilon})e^{-qT}; T < \xi\right] = E^x\left[\left(\int_0^{S_\varepsilon} e^{-(p+q)s}\bar{l}(Y_S, f_\varepsilon)ds\right) e^{-qT}; T < \xi\right].$$

If we take into account lemma 2.3, we can apply proposition 3.1, by taking $S_y = T_{U_{\varepsilon,y}}$, $S = S_\varepsilon$, $U = e^{-qT}\, \mathrm{I}_{\{T<\xi\}}$, and finally

$$H_t^1 = e^{-(p+q)t} f_\varepsilon(Y_{t-}, Y_t), H_t^2 = \int_0^t e^{-(p+q)s}\bar{l}(Y_s, f_\varepsilon)ds.$$

Therefore (3.7) is now established. We put $S_\varepsilon^0 = 0$, we define S_ε^1 to be S_ε as above in the case $T \equiv 0$, and we define recurrently (as in [4]) for $n \geq 1$:

$$S_\varepsilon^{n+1} = S_\varepsilon^n + S_\varepsilon^1 \circ \Theta_{S_\varepsilon^n}.$$

We remark that in fact $S_\varepsilon^n < \xi$ if $S_\varepsilon < \infty$, and then $S_\varepsilon^{n+1} =$ $= \inf\{t > S_n; X_t \in E, d(X_t, X_{S_\varepsilon^n}) > \varepsilon\}$. If we apply now (3.7) for $T = S_\varepsilon^n$

and we let $p \to 0$, it follows that for any $n \geq 1$ the following relation is valid:
(3.9)

$$E^x\left[a(S_\varepsilon^{n+1})f_\varepsilon(X_{S_{\varepsilon-}^{n+1}}, X_{S_\varepsilon^{n+1}}); S_\varepsilon^{n+1} < \xi\right] = E^x\left[\int_{S_\varepsilon^n}^{S_\varepsilon^{n+1}} a(s)\bar{l}(X_s, f_\varepsilon)ds; S_\varepsilon^n < \xi\right]$$

and the desired result follows by summation over $n \geq 0$ and letting then $\varepsilon \to 0$. We note that $\bar{l}(X_s, f) = 0$ for $s \geq \varepsilon$ and $\lim_n S_\varepsilon^n \geq \xi$ by construction, because of the existence of the left limits (X_{t-}). ■

REMARKS AND COMMENTS

We make express reference to [4, XV, 37],. Apparently our result theorem 3.3 looks weaker than refered (37.2) because we considered only the case where (g_t) is a Borel function on $[0, \infty)$, letting aside the general case, for (g_t) predictable with respect to $(\mathcal{F}_t)$. However, a careful examination of the proof of refered (37.2) shows that it is not clear how the passing from refered (37.3) to refered (37.6) is done, because of the difficulty which arises: $X_{S_\varepsilon^n}$ is no longer equal to x almost surely P^x if the starting point is fixed, unless if $n = 0$!.

We think that the problem is how to exploit our lemma 2.3 for $Y_t = X_{S_\varepsilon^n + t}$, and suitable choice of the other variables which occur in (2.1), namely Q and g. Considering the case of a Borel function $a(t)$, we can reduce to the case where the transported process $a(S_\varepsilon^n + t)$ splits in a product of a $\mathcal{F}_{S_\varepsilon^n}$ measurable factor and a $(\mathcal{F}'_t)$ predictable factor, making possible to apply proposition 3.1 (together of course with lemma 2.3). We consider an open question if relation (3.6) still holds if a is replaced by a general predictable process g (with respect to $(\mathcal{F}_t)$). It is not difficult to see that this is equivalent to establish the analogous for (3.9) with a replaced by g (replace a by $g\ \mathrm{I}_{(S_\varepsilon^n, S_\varepsilon^{n+1}]}$ and f by f_ε in (3.6)).

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