Positive Bilinear Mappings Associated with Stochastic Processes

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Summary. On some vector spaces of adapted stochastic processes, we define increasing families of positive bilinear forms, which generalize the usual square brackets $[X, Y]$ and angle brackets $\langle X, Y \rangle$. We study the corresponding Hardy spaces especially for $p = 1$ or 2, and extend to this abstract framework results of Fefferman type from martingale theory.

1 Introduction

The role of the square bracket in martingale theory cannot be overestimated. The starting point of our study was Fefferman's theorem establishing the duality between H^1 and BMO by a positive bilinear form naturally associated to the square bracket. But there also exist other similar forms acting on different spaces leading to similar results (roughly speaking, of Fefferman type); see Pratelli [3], Stein [4] and Yor [5].

Our purpose is to unify all these results of Fefferman type in a common framework. In particular, our main result, Theorem 3.6, simultaneously extends (essentially) Fefferman's theorem and the similar results from [4] and [5]; the relationship with Pratelli's result is slightly different (and simpler).

2 Description of the framework; preliminaries

Throughout this paper $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ is a complete probability space endowed with a filtration $\mathcal{F}_t$ satisfying the usual conditions: it is right continuous and $\mathcal{F}_0$ contains all negligible sets of $\mathcal{F}$. We put $\mathcal{F}_{0-} = \mathcal{F}_0$.

On the vector space of all real valued, adapted processes on $R_+ \times \Omega$ we consider the equivalence relation: $X \sim Y$ iff X and Y are indistinguishable,

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that is, iff the set $\{\omega \in \Omega : \exists t \geqslant 0$ such that $X_t(\omega) \neq Y_t(\omega)\}$ is negligible. We denote by $\mathcal{A}$ the vector space (with induced operations) of equivalence classes with respect to this relation. When no confusion is possible, we identify a process with its equivalence class.

In the sequel we consider a vector subspace $\mathcal{S}$ of $\mathcal{A}$, and a symmetric bilinear mapping $[\,.\,,\,.\,]$ from $\mathcal{S} \times \mathcal{S}$ to $\mathcal{A}$, satisfying the following property: for any $X \in \mathcal{S}$, the process $[X, X]$ is *positive*, *increasing* and *right continuous*. We say that $[\,.\,,\,.\,]$ is a *positive bilinear mapping*. By polarization, it follows that for any $X, Y \in \mathcal{S}$, the process $[X, Y]$ is adapted, right continuous with finite variation on each trajectory.

Example 1. For any local martingales X, Y we consider their square bracket, that is, the unique adapted right continuous process with finite variation, denoted by $[X, Y]$, such that:

1) $XY - [X, Y]$ is a local martingale;
2) $\Delta[X, Y]_t = (\Delta X_t)(\Delta Y_t)$ for any $t > 0$;
3) $[X, Y]_0 = X_0 Y_0$.

It is known that the square bracket extends to semimartingales as a positive bilinear mapping still possessing 2) and 3) (see [1, VII, 44]).

Example 2. For any locally square integrable local martingales X and Y we consider their angle bracket $\langle X, Y\rangle$, the unique predictable right continuous process with finite variation such that $XY - \langle X, Y\rangle$ is a local martingale null at 0. It is in fact the predictable compensator of the square bracket $[X, Y]$ (see [1, VII, 39]).

Example 3. We denote by Λ_0 the space of thin ("minces" in French) optional processes X such that the increasing process

$$A_t^X = \sum_{0 \leqslant s \leqslant t} X_s^2$$

is finite for finite t. We call $\{X, X\}_t$ this process and we define $\{X, Y\}$ for any X and Y in Λ_0 by polarization: $\{X, Y\}_t = \sum_{0 \leqslant s \leqslant t} X_s Y_s$ (see [5]).

Example 4. Given a discrete filtration $\mathcal{F}_n$, we consider the space of sequences of (finite) random variables $(X_n)_{n \geqslant 0}$ such that X_n is $\mathcal{F}_n$-measurable for any $n \in N$. For any such sequences X and Y we define the sequence of random variables (see [4])

$$\{X, Y\}_n = \sum_{m=0}^{n} X_m Y_m.$$

Of course this situation may be "imbedded" in the above by considering the filtration $\mathcal{F}_t = \mathcal{F}_n$ for $n \leqslant t < n + 1$.

The following simple fact (suggested by the end of the proof of [1, VII, 53]) is fundamental for the sequel. We write shortly $[X,Y]_s^t = [X,Y]_t - [X,Y]_s$ for $s < t$, even for $s = 0_-$, which means $[X,Y]_t$.

Proposition 2.1. *For any fixed stopping times S, T such that $S \leqslant T$, and X, $Y \in \mathcal{S}$, we have*

$$\left|[X,Y]_S^T\right| \leqslant \left([X,X]_S^T\right)^{\frac{1}{2}} \left([Y,Y]_S^T\right)^{\frac{1}{2}} \qquad \text{a.s.} \tag{2.1}$$

Proof. For any fixed $r \in \mathbb{R}$, we have

$$0 \leqslant [X+rY, X+rY]_S^T = [X,X]_S^T + 2r[X,Y]_S^T + r^2[Y,Y]_S^T \qquad \text{a.s.} \tag{2.2}$$

The exceptional set on which this inequality does not hold for some rational r is then negligible and hence (2.2) holds on the complement of this set for any $r \in \mathbb{R}$, by continuity; this obviously implies (2.1). □

The proof of the next extension to our framework of the classical inequality of H. Kunita and S. Watanabe is now an easy adaptation of the proof given in [1, VII, 53].

Theorem 2.2. *Let X, $Y \in \mathcal{S}$ and H, K be measurable processes (not necessarily adapted). We have then*

$$\int_0^\infty |H_s||K_s| \left|\mathrm{d}[X,Y]_s\right| \leqslant \left(\int_0^\infty H_s^2 \,\mathrm{d}[X,X]_s\right)^{\frac{1}{2}} \left(\int_0^\infty K_s^2 \,\mathrm{d}[Y,Y]_s\right)^{\frac{1}{2}} \quad \textit{a.s.} \tag{2.3}$$

Proof. We obviously may reduce to the case where H, K are bounded and supported on some interval $[0,N]$. Also, we may replace the left side of (2.3) by

$$\left|\int_0^\infty H_s K_s \,\mathrm{d}[X,Y]_s\right|.$$

Now, using twice the monotone class theorem, we are reduced to the case where

$$H = H_0 I_{\{0\}} + H_1 I_{]0,s_1]} + \cdots + H_m I_{]s_{m-1},s_m]}$$
$$K = K_0 I_{\{0\}} + K_1 I_{]0,t_1]} + \cdots + K_n I_{]t_{n-1},t_n]}$$

with the H_i and K_j measurable and bounded. We obviously may assume then that $m = n$, $s_i = t_i$ for $i = 1, \dots, n$. Putting $s_0 = 0$ and using (2.1), we now get by addition

$$\left|\int_0^\infty H_s K_s \,\mathrm{d}[X,Y]_s\right| \leqslant \left|H_0 K_0 \,[X,Y]_0\right| + \sum_{i=1}^n \left|H_i K_i \,[X,Y]_{s_{i-1}}^{s_i}\right|$$

$$\leqslant |H_0| \left([X,X]_0\right)^{\frac{1}{2}} |K_0| \left([Y,Y]_0\right)^{\frac{1}{2}} + \sum_{i=1}^n \left(H_i^2 \,[X,X]_{s_{i-1}}^{s_i}\right)^{\frac{1}{2}} \left(K_i^2 \,[Y,Y]_{s_{i-1}}^{s_i}\right)^{\frac{1}{2}} \quad \text{a.s.}$$

and (2.3) follows by the Schwarz inequality. □

We are now able to extend the Fefferman inequality to our setting; the proof is the same as in [1, VII, 86], which is itself an adaptation of the proof given by C. Herz in discrete time.

Theorem 2.3. *Let X, $Y \in \mathcal{S}$ and H, K be optional processes. Let $c \in [0,\infty]$ be such that*

$$\mathbb{E}\Big[\int_{[T,\infty]} K_s^2 \,\mathrm{d}[Y,Y]_s \,\Big|\, \mathcal{F}_T\Big] \leqslant c^2 \qquad \text{a.s.} \tag{2.4}$$

for all stopping times T. Then we have

$$\mathbb{E}\Big[\int_{[0,\infty]} |H_s||K_s| \,\big|\mathrm{d}[X,Y]_s\big|\Big] \leqslant c\sqrt{2}\,\mathbb{E}\Big[\Big(\int_{[0,\infty]} H_s^2 \,\mathrm{d}[X,X]_s\Big)^{\frac{1}{2}}\Big]. \tag{2.5}$$

Proof. We may of course suppose $c < \infty$, and H, $K \geqslant 0$. We consider the (positive) increasing processes

$$\alpha_t = \int_{[0,t]} H_s^2 \,\mathrm{d}[X,X]_s \qquad \text{and} \qquad \beta_t = \int_{[0,t]} K_s^2 \,\mathrm{d}[Y,Y]_s,$$

to which we associate the positive optional processes U and V defined by the relations

$$U_s^2 = \begin{cases} \dfrac{H_s^2}{\sqrt{\alpha_s} + \sqrt{\alpha_{s-}}} & \text{for } \alpha_s > 0, \\ 0 & \text{for } \alpha_s = 0; \end{cases} \qquad V_s^2 = K_s^2 \sqrt{\alpha_s}\,.$$

The processes U and V have the following three properties:

$$H_s K_s \leqslant \sqrt{2}\, U_s V_s \tag{2.6}$$

almost surely with respect to the measure $\big|\mathrm{d}[X,Y]_s(\omega)\big|$ for almost all $\omega \in \Omega$,

$$\mathbb{E}\Big[\int_{[0,\infty]} U_s^2 \,\mathrm{d}[X,X]_s\Big] = \mathbb{E}\big[\sqrt{\alpha_\infty}\big], \tag{2.7}$$

$$\mathbb{E}\Big[\int_{[0,\infty]} V_s^2 \,\mathrm{d}[Y,Y]_s\Big] \leqslant c^2\,\mathbb{E}\big[\sqrt{\alpha_\infty}\big]. \tag{2.8}$$

Indeed, one first checks that (2.6) holds almost surely with respect to the measure $\mathrm{d}[X,X]_s(\omega)$ for each ω (an elementary measure-theoretic exercise on the line). One then uses Theorem 2.2 (taking for H the indicator of the optional set $\{HK > \sqrt{2}\,UV\}$ and $K = I_{[0,n]\times\Omega}$, and letting n tend to infinity) to justify relation (2.6). Relation (2.7) follows from the stronger relation

$$\int_{[0,t]} U_s^2 \,\mathrm{d}[X,X]_s = \sqrt{\alpha_t} \qquad \forall\, t \geqslant 0,$$

which stems from the choice of U and from [1, VI, 91 c)]. As to (2.8), we have

$$\begin{aligned}\mathbb{E}\Big[\int V_s^2\,\mathrm{d}[Y,Y]_s\Big] &= \mathbb{E}\Big[\int K_s^2\sqrt{\alpha_s}\,\mathrm{d}[Y,Y]_s\Big] \\ &= \mathbb{E}\Big[\int \sqrt{\alpha_s}\,\mathrm{d}\beta_s\Big] = \mathbb{E}\Big[\int (\beta_\infty - \beta_{s-})\,\mathrm{d}\sqrt{\alpha_s}\Big]\end{aligned} \tag{2.9}$$

from the integration by parts formula [1, VI, 90].

Since α_s, and hence $\sqrt{\alpha_s}$ too, is optional, we may replace the process $(\beta_\infty - \beta_{s-})_s$ by its optional projection, which we know is dominated by c^2 by hypothesis. Finally, using (2.6), (2.3) (applied to U and V), the Hölder inequality, (2.7) and (2.8), we have

$$\begin{aligned}\mathbb{E}\Big[\int H_s K_s\,\big|\mathrm{d}[X,Y]_s\big|\Big] &\leqslant \sqrt{2}\,\mathbb{E}\Big[\int U_s V_s\,\big|\mathrm{d}[X,Y]_s\big|\Big] \\ &\leqslant \sqrt{2}\left(\mathbb{E}\Big[\int U_s^2\,\mathrm{d}[X,X]_s\Big]\right)^{\frac{1}{2}}\left(\mathbb{E}\Big[\int V_s^2\,\mathrm{d}[Y,Y]_s\Big]\right)^{\frac{1}{2}} \leqslant \sqrt{2}\,c\,\mathbb{E}[\sqrt{\alpha_\infty}]\end{aligned}$$

and the proof is over. □

Remark 2.4. Suppose in addition that the processes $[X,X]$ *and* H *are predictable and condition (2.4) is replaced by the weaker condition*

$$\mathbb{E}\Big[\int_{[T,\infty]} K_s^2\,\mathrm{d}[Y,Y]_s \;\Big|\; \mathcal{F}_{T-}\Big] \leqslant c^2 \qquad \text{a.s.} \tag{2.4$'$}$$

for all predictable *stopping times* T. *Then the same conclusion (2.5) holds.*

The proof is the same, except that one passes from (2.9) to (2.8) by considering the predictable projection of the process $(\beta_\infty - \beta_{s-})$, the measure $\mathrm{d}(\sqrt{\alpha_s})$ now being *predictable.*

For example, (2.4') is implied by the condition

$$K_0^2\,[Y,Y]_0 + \mathbb{E}\Big[\int_{(T,\infty]} K_s^2\,\mathrm{d}[Y,Y]_s \;\Big|\; \mathcal{F}_T\Big] \leqslant c^2 \qquad \text{a.s.}$$

for all stopping times T. Conversely (2.4') implies that:

a) $K_0^2\,[Y,Y]_0 \leqslant c^2$;

b) $\mathbb{E}\big[\int_{(T,\infty]} K_s^2\,\mathrm{d}[Y,Y]_s \mid \mathcal{F}_T\big] \leqslant c^2$ a.s. for all stopping times T. (Approximate T by the predictable stopping time $T + 1/n$.)

3 The main result

First, remark that (2.1) leads immediately to the Minkowski-type inequality: for all X, $Y \in \mathcal{S}$ and $s < t$, one has

$$([X+Y, X+Y]_s^t)^{\frac{1}{2}} \leqslant ([X, X]_s^t)^{\frac{1}{2}} + ([Y, Y]_s^t)^{\frac{1}{2}}; \tag{3.1}$$

by taking $s = 0_-$, $t = \infty$, this enables us to consider for any $1 \leqslant p \leqslant \infty$ the seminorms on $\mathcal{S}$:

$$\|X\|_{H^p} = \big(\mathbb{E}\big[([X, X]_\infty)^{\frac{p}{2}}\big]\big)^{\frac{1}{p}} = \big\|[X, X]_\infty^{\frac{1}{2}}\big\|_{L^p}$$

and to define $H^p = \{X \in \mathcal{S} : \|X\|_{H^p} < \infty\}$ as subspaces of $\mathcal{S}$.

Throughout the rest of the paper, we assume that the following implication holds for all $X \in \mathcal{S}$:

$$[X, X]_\infty \equiv 0 \qquad \Longrightarrow \qquad X \equiv 0 \tag{S}$$

(that is, all above seminorms are in fact norms on the corresponding spaces), and in addition H^2 is *complete* with respect to $\| \,.\, \|_{H^2}$ (that is, H^2 is a Hilbert space). On the vector space H^2 we also consider the restriction of the norm $\| \,.\, \|_{H^1}$, and we denote it by $(H^2, \| \,.\, \|_{H^1})$. Finally, we consider on $\mathcal{S}$ the norm $\| \,.\, \|_{BMO}$ so defined: $\|X\|_{BMO}$ is the smallest (possibly infinite) constant $c \geqslant 0$ such that, for all stopping times T, one has $\mathbb{E}\big[[X, X]_\infty - [X, X]_{T_-} \bigm| \mathcal{F}_T\big] \leqslant c^2$. Homogeneity of $\| \,.\, \|_{BMO}$ is obvious. In order to check its subadditivity, we note the inequality

(3.2)

$$\big|[X, Y]_\infty - [X, Y]_{T-}\big| \leqslant \big([X, X]_\infty - [X, X]_{T-}\big)^{\frac{1}{2}} \big([Y, Y]_\infty - [Y, Y]_{T-}\big)^{\frac{1}{2}} \quad \text{a.s.}$$

which can be proved exactly as (2.1), or deduced from it. Let now $X, Y \in \mathcal{S}$ and $c_1, c_2 \in [0, \infty]$ be such that

$$\mathbb{E}\big[[X, X]_\infty - [X, X]_{T-} \bigm| \mathcal{F}_T\big] \leqslant c_1^2 \quad \text{a.s.}$$
$$\mathbb{E}\big[[Y, Y]_\infty - [Y, Y]_{T-} \bigm| \mathcal{F}_T\big] \leqslant c_2^2 \quad \text{a.s.}$$

for some fixed stopping time T. We have

$$\begin{aligned} &\mathbb{E}\big[[X+Y, X+Y]_\infty - [X+Y, X+Y]_{T-} \bigm| \mathcal{F}_T\big] \\ &\qquad \leqslant c_1^2 + c_2^2 + 2\,\mathbb{E}\big[[X, Y]_\infty - [X, Y]_{T-} \bigm| \mathcal{F}_T\big] \\ &\qquad \leqslant c_1^2 + c_2^2 + 2\,\mathbb{E}\big[\big([X, X]_\infty - [X, X]_{T-}\big)^{\frac{1}{2}} \big([Y, Y]_\infty - [Y, Y]_{T-}\big)^{\frac{1}{2}} \bigm| \mathcal{F}_T\big] \quad \text{a.s.} \end{aligned}$$

For any $A \in \mathcal{F}_T$ the Hölder inequality gives

$$\begin{aligned} &\int_A \big([X, X]_\infty - [X, X]_{T-}\big)^{\frac{1}{2}} \big([Y, Y]_\infty - [Y, Y]_{T-}\big)^{\frac{1}{2}} \,\mathrm{d}\mathbb{P} \\ &\qquad \leqslant \Big(\int_A \big([X, X]_\infty - [X, X]_{T-}\big) \,\mathrm{d}\mathbb{P}\Big)^{\frac{1}{2}} \Big(\int_A \big([Y, Y]_\infty - [Y, Y]_{T-}\big) \,\mathrm{d}\mathbb{P}\Big)^{\frac{1}{2}} \\ &\qquad \leqslant c_1 c_2 \,\mathbb{P}[A] \end{aligned}$$

and we conclude that

$$\mathbb{E}\big[[X+Y,X+Y]_\infty - [X+Y,X+Y]_{T-}) \mid \mathcal{F}_T\big] \leqslant (c_1+c_2)^2 \qquad \text{a.s.}$$

We can now define $BMO = \{X \in \mathcal{S} : \|X\|_{BMO} < \infty\}$ as a subspace of $\mathcal{S}$; taking $T = 0$ in its definition we see that $BMO \subset H^2$.

Using the above language, we remark that th. 2.3 (with $H \equiv K \equiv 1$) implies that the mapping

$$BMO \ni Y \mapsto \mathbb{E}\big[[\,.\,,Y]_\infty\big] \in (H^2, \|\,.\,\|_{H^1})^*$$

is a linear continuous injection with norm $\leqslant \sqrt{2}$; in fact we have more, namely, $\mathbb{E}\big[[\,.\,,Y]_\infty\big]$ defines an element of $(H^1)^*$ with norm at most $\sqrt{2}\,\|Y\|_{BMO}$). The aim of this section is to show that this mapping admits a continuous inverse (as a consequence BMO is a Banach space). To this end we impose some additional conditions on $\mathcal{S}$ and $[\,.\,,\,.\,]$, following the intuitive idea that the r.v. $[X,X]_t$ represents the accumulation up to time t of the information about some property which depends quadratically upon the behaviour of X on $[0,t]$.

A1. For all $X \in \mathcal{S}$ and for all $t \in [0,\infty)$ and $H \in \mathcal{F}_t$, the following implication holds: $X = 0$ on $([0,t] \times \Omega) \cup ([t,\infty) \times H) \Rightarrow [X,X] = 0$ on $([0,t] \times \Omega) \cup ([t,\infty) \times H)$. (This expresses the fact that $[X,X]$ marks nothing as long as X remains identically null, and still continues to mark nothing from the last moment when X is identically null, on some subset of Ω on which X continues to be null up to ∞.)

A1′. For all $X \in \mathcal{S}$, $t \in [0,\infty)$, and $H \in \mathcal{F}_t$, if $X = 0$ on $[0,t] \times \Omega$, the process $I_H X$ belongs to $\mathcal{S}$.

A2. For all $X \in \mathcal{S}$ and $t \in [0,\infty)$, there exists some $\widetilde{X} \in \mathcal{S}$ such that $\widetilde{X} = X$ on $[0,t] \times \Omega$ and $\big[\widetilde{X},\widetilde{X}\big]_\infty = [X,X]_t$. (Note that $\big[\widetilde{X},\widetilde{X}\big] = [X,X]$ on $[0,t] \times \Omega$, from A1 and prop. 2.1. Roughly speaking, we can modify X from any moment such that the property marked by $[\,.\,,\,.\,]$ is stopped.)

Definition 3.1. Given a fixed stopping time T, an element $X \in H^2$ is called a T*-atom* if $[X,X]_\infty = \Delta[X,X]_T$. (We put $[X,X]_\infty = [X,X]_{\infty-} = \lim_{t\to\infty} [X,X]_t$, so that $\Delta[X,X]_T = 0$ on the set $\{T = \infty\}$, which may be big.)

Proposition 3.2. *Let X be a T-atom and let $Y \in \mathcal{S}$ be arbitrary. Then the process $[X,Y]$ verifies $[X,Y]_t = \Delta[X,Y]_T I_{\{T \leqslant t\}}$; it is null on $[0,T)$ and pathwise constant on $[T,\infty)$.*

Proof. For $Y = X$ this follows directly from above definition. For arbitrary Y, use prop. 2.1 and conclude by the optional section theorem. □

Definition 3.3. The positive bilinear mapping $[\,.\,,\,.\,]$ is said to have *square linear jumps* if for all stopping times T there exists a linear application $\Psi_T : H^2 \to L^2$ such that

$$\Delta[X,X]_T = \big(\Psi_T(X)\big)^2 \qquad \text{a.s.} \tag{3.5}$$

for any $X \in H^2$.

By polarization and linearity of Ψ_T we then have $\Delta[X,Y]_T = \Psi_T(X)\Psi_T(Y)$ a.s. for all $X, Y \in H^2$. Note also that $\Psi_T(X) = 0$ a.s. on $\{T = \infty\}$, so that Ψ_T may be considered as taking values in $L^2(\{T < \infty\})$.

Fix now a family $\mathcal{T}$ of stopping times such that for any stopping time T, the graph of T is contained in a countable union of graphs of stopping times belonging to $\mathcal{T}$.

Denote by $\mathcal{A}_T$ the set of T-atoms, which is a linear subspace of H^2 by prop. 3.2.

Definition 3.4. We say that the set of atoms is *$\mathcal{T}$-full* if for any $T \in \mathcal{T}$ we have $\Psi_T(H^2) = \Psi_T(\mathcal{A}_T)$ as identical subspaces of L^2.

A3. The set of atoms is $\mathcal{T}$-full. One can see that $\mathcal{A}_T$ is closed in H^2 (complete), and hence $\Psi_T(H^2)$ is closed in L^2 for any $T \in \mathcal{T}$ by isometry.

Proposition 3.5. *Suppose that $[\,.\,,\,.\,]$ has square linear jumps, and A3 holds. Then for any $U \in L^2$ and $T \in \mathcal{T}$, there exists a unique atom $X \in \mathcal{A}_T$ that satisfies for all $Y \in H^2$ the following relations:*

$$\mathbb{E}\big[[X,Y]_\infty\big] = \mathbb{E}\big[\Delta[X,Y]_T\big] = \mathbb{E}[\Psi_T(X)\Psi_T(Y)] = \mathbb{E}[U\Psi_T(Y)]\,. \tag{3.6}$$

Proof. Denote by U' the orthogonal projection of U onto the closed space $\Psi_T(H^2)$, and pick $X \in \mathcal{A}_T$ such that $U' = \Psi_T(X)$. The first equality follows from prop. 3.2, the other ones are obvious now; uniqueness of X is also clear. □

In the sequel we refer to the correspondence $U \mapsto X$ as the *atomic map* for fixed $T \in \mathcal{T}$, and we denote this map by a_T.

Theorem 3.6. *a) Suppose that A1 and A2 hold, and let $\varphi \in (H^2, \|\,.\,\|_{H^1})^*$. Then there exists a unique $Y \in H^2$ such that $\varphi(\,.\,) = \mathbb{E}\big[[\,.\,,Y]_\infty\big]$. Moreover*

$$\mathbb{E}\big[[Y,Y]_\infty - [Y,Y]_T \;\big|\; \mathcal{F}_T\big] \leqslant \|\varphi\|^2 \tag{3.7}$$

for all stopping times T.

b) If in addition $[\,.\,,\,.\,]$ has square linear jumps, if A3 holds too and if the family of atomic maps $(a_T)_{T\in\mathcal{T}}$ is uniformly bounded in norms from $(L^2, \|\,.\,\|_{L^1})$ to $(H^2, \|\,.\,\|_{H^1})$ by some $M > 0$, then $Y \in BMO$ and moreover $\|Y\|_{BMO} \leqslant (M^2+1)^{\frac{1}{2}}\|\varphi\|$.

Proof. a) Remark first that T may be replaced by t (a constant stopping time) in (3.7); this follows from the right continuity of $[\,.\,,\,.\,]$ and from a classical approximation of T by a decreasing sequence of discrete stopping times.

Since the norm $\|\,.\,\|_{H^1}$ is obviously dominated by $\|\,.\,\|_{H^2}$, the existence and uniqueness of $Y \in H^2$ representing φ as desired is assured by the Riesz representation theorem. To prove (3.7), fix $t \in [0,\infty)$ and $H \in \mathcal{F}_t$, and consider the element X of $\mathcal{S}$ defined by $X = I_H\big(Y - \widetilde{Y}\big)$ (use A2 and A1′). Let us show that

$$[X,Y]_\infty = [X,X]_\infty = \big([Y,Y]_\infty - [Y,Y]_t\big)\, I_H. \tag{3.8}$$

We have for any $s \geqslant t$ the relations

$$[X,Y]_s = \big[I_H\big(Y-\widetilde{Y}\big),Y\big]_s = \big[I_H\big(Y-\widetilde{Y}\big),Y-\widetilde{Y}\big]_s + \big[I_H\big(Y-\widetilde{Y}\big),\widetilde{Y}\big]_s. \tag{3.9}$$

But since $\big[\widetilde{Y},\widetilde{Y}\big]_s = \big[\widetilde{Y},\widetilde{Y}\big]_t$ (by A2), it follows from prop. 2.1 that

$$\big[I_H\big(Y-\widetilde{Y}\big),\widetilde{Y}\big]_s = \big[I_H\big(Y-\widetilde{Y}\big),\widetilde{Y}\big]_t = 0 \qquad \text{a.s.}, \tag{3.10}$$

because $I_H(Y-\widetilde{Y})$ is null on $[0,t]\times\Omega$, hence $\big[I_H\big(Y-\widetilde{Y}\big),I_H\big(Y-\widetilde{Y}\big)\big]$ is null on $[0,t]\times\Omega$ by A1, and one more application of prop. 2.1 suffices.

Putting $Z = Y - \widetilde{Y}$, to show that $[X,Y]_s = [X,X]_s$ (and s will then go to infinity), it now suffices to check that

$$[I_H Z, Z]_s = [I_H Z, I_H Z]_s = I_H [Z,Z]_s \qquad \text{a.s.} \tag{3.11}$$

The first equality is equivalent to $[I_H Z, I_{H^c} Z]_s = 0$, which is a consequence of prop. 2.1, by using A1.

Since obviously $[I_H Z, Z] + [I_{H^c} Z, Z] = [Z,Z]$, it suffices for the second one to see that

$$[I_H Z, Z]_s = 0 \qquad \text{a.s. on } H^c$$

$$[I_{H^c} Z, Z]_s = 0 \qquad \text{a.s. on } H,$$

one more application of prop 2.1, by using A1.

To show now the second half of (3.8), take $s \geqslant t$. We have from (3.11)

$$[X,X]_s = [I_H Z, I_H Z]_s = I_H [Z,Z]_s \qquad \text{a.s.}$$

and therefore we may suppose that $H = \Omega$, $X = Z$. We have finally

$$[X,X]_s = [Y-\widetilde{Y}, Y-\widetilde{Y}]_s = [Y,Y]_s - 2[Y,\widetilde{Y}]_s + [\widetilde{Y},\widetilde{Y}]_s,$$

and the desired relation follows from (3.10) by letting s tend to infinity.

We can now write

$$\begin{aligned} \mathbb{E}\big[[X,Y]_\infty\big] = \varphi(X) &\leqslant \|\varphi\|\, \mathbb{E}\big[[X,X]_\infty^{\frac12}\big] \\ &= \|\varphi\|\, \mathbb{E}\big[I_H\big([Y,Y]_\infty - [Y,Y]_t\big)^{\frac12}\big]. \end{aligned} \tag{3.12}$$

On the other hand it follows from the Hölder inequality that

$$\mathbb{E}\big[I_H\big([Y,Y]_\infty - [Y,Y]_t\big)^{\frac12}\big] \leqslant \mathbb{P}[H]^{\frac12}\, \big(\mathbb{E}\big[I_H\big([Y,Y]_\infty - [Y,Y]_t\big)\big]\big)^{\frac12}. \tag{3.13}$$

From (3.8), (3.12) and (3.13) we conclude that

$$\mathbb{E}\big[I_H\big([Y,Y]_\infty - [Y,Y]_t\big)\big] \leqslant \|\varphi\|\, \mathbb{P}[H]^{\frac12}\, \big(\mathbb{E}\big[I_H([Y,Y]_\infty - [Y,Y]_t)\big]\big)^{\frac12}$$

and this implies (3.7) since H is an arbitrary element of $\mathcal{F}_t$.

b) For any any $T \in \mathcal{T}$ and $U \in L^2$, putting $X = a_T(U)$ and using prop. 3.5, we have

$$\mathbb{E}[U\Psi_T(Y)] = E\big[[X,Y]_\infty\big] \leqslant \|\varphi\|\,\|a_T(U)\|_{H^1} \leqslant M\|\varphi\|\,\|U\|_{L^1}\,. \tag{3.14}$$

As L^2 is a dense subspace of L^1 it follows:

$$\big\|(\Delta[Y,Y]_T)^{\frac{1}{2}}\big\|_{L^\infty} = \|\Psi_T(Y)\|_{L^\infty} \leqslant M\,\|\varphi\|. \tag{3.15}$$

Since the graph of any stopping time is contained in a countable union of graphs of stopping times from $\mathcal{T}$, (3.15) holds in fact for all stopping times T.

Finally, summing (3.7) and the square of (3.15) we get

$$\mathbb{E}\big[[Y,Y]_\infty - [Y,Y]_{T-} \;\big|\; \mathcal{F}_T\big] \leqslant (M^2+1)\,\|\varphi\|^2 \qquad \text{a.s.};$$

as T is arbitrary, the desired conclusion follows:

$$\|Y\|_{BMO} \leqslant (M^2+1)^{\frac{1}{2}}\|\varphi\|. \qquad \square$$

Remark 3.7. The uniform boundedness of the family of atomic maps from $(L^2, \|\,.\,\|_{L^1})$ *to* $(H^2, \|\,.\,\|_{H^1})$ *may seem strange and strong. However, it is also necessary for the validity of b) (supposing that all other conditions hold).* Indeed, suppose that for some constant $C > 0$, every $\varphi \in (H^2, \|\,.\,\|_{H^1})^*$ is represented by a (unique) $Y \in BMO$ such that $\|Y\|_{BMO} \leqslant C\|\varphi\|$. Then, by a well known consequence of the Hahn–Banach theorem and by (3.6), for any $T \in \mathcal{T}$ and for any $U \in L^2$, puting $X = a_T(U)$, one has

$$\begin{aligned}\|X\|_{H^1} &\leqslant \sup_{\|Y\|_{BMO}\leqslant C} \big|\mathbb{E}\big[[X,Y]_\infty\big]\big| = C \sup_{\|Y\|_{BMO}\leqslant 1} \big|\mathbb{E}\big[[X,Y]_\infty\big]\big| \\ &= C \sup_{\|Y\|_{BMO}\leqslant 1} \big|E[U\Psi_T(Y)]\big| \leqslant C \sup_{\Delta[Y,Y]_T\leqslant 1} \big|E[U\Psi_T(Y)]\big| \leqslant C\,\|U\|_{L^1}\,,\end{aligned}$$

since $|\Psi_T(Y)| = (\Delta[Y,Y]_T)^{\frac{1}{2}}$ and obviously $\Delta[Y,Y]_T \leqslant \|Y\|^2_{BMO}$. a.s.

Remark 3.8. For $X \in H^2$ and fixed $t \geqslant 0$, consider the element $\widetilde{X} \in \mathcal{S}$ given by A2. Looking at relation (3.10) (with X instead of Y, and $H = \Omega$), $\widetilde{X}$ appears as the orthogonal projection of X onto the orthogonal complement (in H^2) of the linear space $F = \{X \in H^2 : X = 0 \text{ on } [0,t] \times \Omega\}$. It suffices to check that F is closed in H^2; this follows from the fact that X belongs to F if and only if X is in H^2 and $[X,X]_t = 0$ (consequence of A1 and A2). Indeed, if $X^n \in F$, $X \in H^2$ and $X^n \to X$ in H^2, then taking $T = t$ and $S = 0_-$ in (2.1), one has

$$\mathbb{E}\big[[X,X]_t\big] = \mathbb{E}\big[[X-X^n, X-X^n]_t\big] \leqslant \mathbb{E}\big[[X-X^n, X-X^n]_\infty\big] \longrightarrow 0.$$

4 Applications

We now illustrate the theory of section 3 by applying it to the four examples listed in section 2.

Example 1 (square bracket). The implication $[X,X]_\infty \equiv 0 \Rightarrow X \equiv 0$ is well known; see for example [1, VII, 52]. In fact we need this, and the axioms Ai, for $X \in H^2$ only. To check Ai, recall three basic properties of $[\,.\,,\,.\,]$:

a) $[X^T,Y]=[X,Y]^T$ a.s. for any stopping time T, where $X^T_t = X_{T\wedge t}$.

b) Let T be an (arbitrary) fixed stopping time, and let X be a local martingale. Then the process

$$X'_s = (X_{T+s} - X_T) \qquad (s \in [0,\infty))$$

is a local martingale with respect to the filtration $\mathcal{F}'_s = \mathcal{F}_{T+s}$, and we have $[X',X']_s = [X,X]_{T+s} - [X,X]_T$ for any $s \geqslant 0$. (We may consider as well X_{T-} (resp. $[X,X]_{T-}$) instead of X_T (resp. $[X,X]_T$).

c) If X is a local martingale and $H \in \mathcal{F}_0$, then $I_H X$ is a local martingale, and $[I_H X, I_H X] = I_H[X,X]$.

These three properties easily imply A1 and A2 (take $\widetilde{X} = X^t$). Axiom A1$'$, which is also an extension of the first part of *c*), is satisfied because of the definition of local martingales.

We now pass to the second set of conditions, dealing with the jumps of $[\,.\,,\,.\,]$. Of course we take $\Psi_T(X) = \Delta X_T$ in definition 3.3. Further, we take $\mathcal{T} = \{T : T \text{ is predictable or totally inaccessible}\}$; it is well known that the graph of any stopping time can be covered by a countable union of graphs of predictable s.t. and the graph of a totally inaccessible s.t. (see [1, IV, 81]). We then have $\Psi_T(H^2) = \Psi_T(\mathcal{A}_T) = L^2(\mathcal{F}_T \,|\, \{T<\infty\})$ for T totally inaccessible and $\Psi_T(H^2) = \Psi_T(\mathcal{A}_T) = \{U - \mathbb{E}[U \,|\, \mathcal{F}_{T-}] : U \in L^2(\mathcal{F}_T \,|\, \{T<\infty\})\} = \{U \in L^2(\mathcal{F}_T \,|\, \{T<\infty\}) : \mathbb{E}[U \,|\, \mathcal{F}_{T-}] = 0\}$ for T predictable. To justify these pleasant relations, we invoque (with slight modifications) the discussion from the proof of [1, VII, 74] (due to Lépingle): for an arbitrary stopping time T and $U \in L^2(\mathcal{F}_T)$, consider the process $A_t = U I_{\{T \leqslant t\}}$; it has *finite variation.* This A is obviously integrable, and if $\widetilde{A}$ denotes its dual predictable projection (compensator), consider $X = A - \widetilde{A}$, which is a T-atom such that $\Delta X_T = U$ on $\{T<\infty\}$ (hence $\Delta[X,X]_T = U^2$ on $\{T<\infty\}$) if T is totally inaccessible and $\Delta X_T = U - \mathbb{E}[U \,|\, \mathcal{F}_{T-}]$ on $\{T<\infty\}$ (hence $\Delta[X,X]_T = (U - \mathbb{E}[U \,|\, \mathcal{F}_{T-}])^2$ on $\{T<\infty\}$) if T is predictable.

Therefore A3 holds ($\Delta X_T = \mathbb{E}[X_\infty \,|\, \mathcal{F}_T] - \mathbb{E}[X_\infty \,|\, \mathcal{F}_{T-}]$ if T is predictable, generally $X_T \in L^2(\mathcal{F}_T)$ if T is arbitrary), and moreover one can see that the family of atomic maps $(a_T)_{T\in\mathcal{T}}$ is uniformly bounded in norm from $(L^2, \|\,.\,\|_{L^1})$ to $(H^2, \|\,.\,\|_{H^1})$ by the constant $M=2$. Summing up, we see that the hypotheses of th. 3.6. *a*) and *b*) hold, so that our result extends Fefferman's theorem ([1, VII, 88]) *up to the assumption that* H^2 *is dense in* H^1, which makes it possible to identify $(H^1)^*$ with $(H^2, \|\,.\,\|_{H^1})^*$. Looking at the proof of this fact in [1, VII, 85], we see that it has nothing in common with Fefferman's inequality; this explains why we consider here $(H^2, \|\,.\,\|_{H^1})$ instead of H^1, and we prefer not to assume that H^2 be dense in H^1 and that $(H^2, \|\,.\,\|_{H^1})^*$ be equal to $(H^1)^*$.

Example 2 (angle bracket). Remark first that in this case H^2 is dense in H^1 by definition of $\mathcal{S}$ (to make possible the definition of the angle bracket $\langle\,.\,,\,.\,\rangle$).

In this case, following Pratelli [3], one defines $\|X\|_{BMO_2}$ as the smallest (possibly infinite) constant c such that

$$\mathbb{E}\big[\langle X,X\rangle_\infty - \langle X,X\rangle_T \;\big|\; \mathcal{F}_T\big] \leqslant c^2 \qquad \text{for all stopping times } T,$$

so that we are interested only in the statement of th. 3.6. a), whereas b) is uninteresting because of remark 2.4.

The validity of A1, A1′, A2 are again consequences of properties a), b) and c), which hold for the angle bracket too (of course we must take care to consider only locally square integrable martingales, for which $\langle\,.\,,\,.\,\rangle$ exists).

To complete the description of the dual of H^1 (suggested by remark 2.4) in this example, remark the following simple fact: if $\varphi \in (H^2, \|\,.\,\|_{H^1})^*$ and $Y \in H^2$ are such that $\varphi(\,.\,) = \mathbb{E}\big[\langle\,.\,,Y\rangle_\infty\big]$, then it follows (in addition to (3.7)) that $\|Y_0\|_{L^\infty} \leqslant \|\varphi\|$. Indeed, if X_0 denotes the constant process equal to the r.v. $X_0 \in L^2(\mathcal{F}_0)$, we may write

$$\mathbb{E}[X_0Y_0] = \mathbb{E}[X_0Y_\infty] = \mathbb{E}\big[\langle X_0, Y\rangle_\infty\big] \leqslant \|\varphi\|\,\|X_0\|_{H^1} = \|\varphi\|\,\|X_0\|_{L^1}\,,$$

and since $L^2(\mathcal{F}^0)$ is dense in $L^1(\mathcal{F}^0)$, the desired relation follows. Summing up, we see that the dual of H^1 may in this case be identified with the linear space $\big\{X \in \mathcal{S} : \|X\|_{BMO_2} < \infty,\ \|X_0\|_{L^\infty} < \infty\big\}$ endowed with the norm $\|X\| = \|X\|_{BMO_2} + \|X_0\|_{L^\infty}$.

Example 3. The matter is considerably simpler in this case. The validity of A1 and A1′ are obvious, and to A2 we take brutally $\widetilde{X} = X \cdot I_{[0,t]\times\Omega}$. The significance of the notion of T-atom has a strong intuitive support here. Naturally, we take $\Psi_T(X) = X_T \cdot I_{\{T<\infty\}}$ and $\mathcal{T}$ is the family of all stopping times; $\Psi_T(H^2) = \Psi_T(\mathcal{A}_T) = L^2(\mathcal{F}_T \,|\, \{T<\infty\})$ for any T, and the atomic maps are isometries from $(L^2(\mathcal{F}_T \,|\, \{T<\infty\}), \|\,.\,\|_{L^p})$ to $(H^2, \|\,.\,\|_{H^p})$ for all $1 \leqslant p \leqslant \infty$.

Example 4. It is quite similar to above; the discrete processes carry over to "mince" processes null outside the set $\mathbb{Z}_+ \times \Omega$, and $\{\,.\,,\,.\,\}$ extends in the obvious way to whole $\mathbb{R}_+$. Here we take $\Psi_T(X) = X_T \cdot I_{\{T\in\mathbb{Z}_+\}}$ and we have $\Psi_T(H^2) = \Psi_T(\mathcal{A}_T) = L^2(\mathcal{F}_T \,|\, \{T \in \mathbb{Z}_+\})$ for any stopping time T.

Remark. Axioms A1 and A1′ express some local properties of $\mathcal{S}$ and $[\,.\,,\,.\,]$. As to A1′, it refers only to $\mathcal{S}$, whereas A1 suggests that the trajectory $[X,X](\omega)$ depends only on the trajectory $X(\omega)$.

For example, in the case of the square bracket, recall that for all $X \in H^2$, putting $t_i^n = i \cdot 2^{-n}t$, one has

$$[X,X]_t = \lim_n \sum_{i=0}^{2^n-1} \left(X_{t_{i+1}^n} - X_{t_i^n}\right)^2$$

strongly in L^1 (see [2]), which implies a property stronger than A1.

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