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The Extension of Grishin's Lemma to Excessive Measures

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Abstract. Given a submarkovian resolvent of positive kernels on a measurable space, we establish an analogous result to Grishin's lemma from classical Potential Theory and we show then how this lemma can be deduced from our result.

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1. Introduction

Many results from classical Potential Theory have been developed in general frameworks based on axiomatic systems like harmonic spaces, H-cones, Markov processes, etc. In the present paper we deal with the phenomenon known as Grishin lemma. Roughly speaking, this result says that for any δ-subharmonic function w defined on an open set $\Omega \subset R^n$ (that is $w = u - v$, where u and v are superharmonic on Ω), if $\mu[w]$ denotes the Riesz charge associated to w, then the following implication is valid

$$w \geqslant 0 \quad \text{on} \quad \Omega \Rightarrow \mu[w] \leqslant 0 \quad \text{on} \quad \{x \in \Omega \colon w(x) = 0\}.$$

Of course, the choice of the set $\{x \in \Omega : w(x) = 0\}$ is a problem, since the function $w = u - v$ is undefined on the polar set $\{u = v = \infty\}$. This was the subject of different versions of Grishin lemma due to De la Vallée Poussin [4], Brelot [3], Grishin [8] and finally Fuglede [6], whose approach is most close to us. Our framework is operatorial, that is we consider the structure generated by a submarkovian resolvent $U = (U_\alpha)_{\alpha>0}$ of positive kernels on a measurable space $(X, \mathcal{X})$. This resolvent may be the resolvent given by a Markov process on X, but we do not need a structure so rich for our purpose. We us the analytic tools, expecially the order- convexity properties of the H-cone of excessive measures, for which we give the analogue of Grishin lemma. Also we refer to some general properties of excessive measures proved by Getoor in his treatise [7], in probabilistic context, but whose proofs are also valid in our general structure. The key in our proof is an extension of the duality formula for balayages due to Beznea and Boboc

[1]. A similar formula is proved in [7], however in probabilistic framework, and so our result may be easily translated in this setting if one wishes.

Let $(X, \mathcal{X})$ be a measurable space and $\mathcal{U} = (U_\alpha)_{\alpha>0}$ be a submarkovian resolvent of positive kernels on $(X, \mathcal{X})$ such that the initial kernel $U = U_0$ is proper and strict, that is there exists a measurable function $f > 0$ such that $0 < Uf < \infty$ on X. Moreover we suppose that the σ-algebra $\mathcal{X}$ *is separable* and $\mathcal{X} = \sigma(S)$, where $\sigma(S)$ denotes the σ-algebra generated by the convex cone of excessive functions S, which is assumed to be *minstable* and *containing the positive constants* $(1 \in S)$. We consider the H-cone of excessive measures $\mathrm{Exc} = \mathrm{Exc}(\mathcal{U})$ consisting of all σ finite positive measures ξ on X such that $\xi \alpha U_\alpha \leqslant \xi$ for any $\alpha > 0$. (It follows then that $\sup_\alpha \xi \alpha U_\alpha = \xi$ since U is strict [5], XII). Following [7], for any $\xi, \eta \in$ Exc we denote by

$$A = \{\xi \leqslant \eta\}$$

any set $[h_\xi \leqslant h_\eta]$ where h_ξ and h_η represent arbitrary $\mathcal{X}$ measurable versions of the Radon– Nikodym derivatives $\mathrm{d}\xi/\mathrm{d}(\xi+\eta)$ and respectively $\mathrm{d}\eta/\mathrm{d}(\xi+\eta)$. Therefore the set A is uniquely determined modulo a subset of X of measure $\xi + \eta$ null. For any set $M \in \mathcal{X}$ such that there exists

$$B^M s = \wedge\{t \in S; t \geqslant s \ \text{ on } \ M\}, \tag{$\wedge$}$$

for any $s \in S$ ($\wedge$ denotes infimum in the ordered convex cone S) we denote by

$$b(M) = \{x \in X; B^M s(x) = s(x), \ \ \forall s \in S\}$$

the *base* of M (a point $x \in b(M)$ is called *regular* for M). It is well-known that the hypothesis $(\wedge)$ as above is satisfied for any $M \subset \mathcal{X}$ in reasonable situations, for example if S is a standard H-cone (or equivalently $\mathcal{U} = (U_\alpha)_{\alpha>0}$ is absolutely continuous with respect to a reference measure) or if there exists a right process on X generating the resolvent $\mathcal{U} = (U_\alpha)_{\alpha>0}$. We say in this case that $\mathcal{U}$ is a right resolvent ([5, 7]). In our framework, which is intended to be more general, we shall consider $b(M)$ only for special sets M for which the condition $(\wedge)$ is checked directly.

If μ_1 and μ_2 are (positive) measures on X and A is a subset of X, we shall write

$$\mu_1 \leqslant \mu_2 \quad \text{on } \ A \quad \text{if } \ \mu_1(B) \leqslant \mu_2(B),$$

for any measurable set $B \subset A$. If $A \subset X$ is measurable, we consider following [1] the set

$$A^* = \{x \in A : \liminf_{n\to\infty} nV_n 1_A(x) = 1\}.$$

From definition A^* is evidently measurable and one can show that $A \backslash A^*$ is of null potential ([5], XII), that is, negligible. For any excessive measure ξ, and any arbitrary subset A or X, we consider the 'reduite' of ξ on A, the excessive measure given by the formula (following [7]):

$$ {}^*R^A_\xi = \wedge\{\eta \in \text{ Exc}; \eta \geqslant \xi \text{ on } A\} $$

where $\wedge$ denotes the infimum in the H-cone Exc, which coincides in fact with the infimum of the set in braces considered in the lattice of all σ finite measures on X. It is known that the mapping

$$ {}^*R^A : \text{ Exc} \to \text{Exc} $$

defined above is a balayage on the H cone Exc, that is ${}^*R^A$ is additive, increasing, continuous in order from below ($\xi_n \nearrow \xi \Rightarrow {}^*R^A_{\xi_n} \nearrow {}^*R^A_\xi$) and idempotent. Also it can be seen directly from the definition that

$$ {}^*R^A_\xi = \xi \quad \text{on } A. $$

In the sequel we make the assumption that for any measurable set $M \subset X$ such that $(\wedge)$ holds the set $b(M)$ belongs to $\mathcal{X}^*$, the universal completion of the σ-algebra $\mathcal{X}$. Also this condition is checked in the examples mentioned above (if S is a standard H-cone and $\mathcal{X}$ is the Borel field with respect to the natural topology on X, then $b(M)$ is even a G_δ set). See [2, 5]. Under the above hypotheses we can now prove our main result.

THEOREM. *Let ξ and η be excessive measures such that $\xi \geqslant \eta$. Fix a decreasing sequence $\{\nu_n\}_{n\in N} \subset$ Exc such that*

$$ \bigwedge_n \nu_n = 0 \quad \text{in Exc}, $$

and consider the set

$$ E = \bigcap_n b(\{\xi \leqslant \eta + \nu_n\}^*). $$

If

$$ \xi = \mu U + \gamma, \qquad \eta = \lambda U + 0 $$

are the Riesz decompositions (see [7]*) in potential part and harmonic part of ξ and η respectively, then the following inequality holds*

$$ \mu \leqslant \gamma \quad \text{on } E. $$

Proof. Let us denote

$$ A_n = \{\xi \leqslant \eta + \nu_n\}, \quad n \in N. $$

Here we can take $m = \xi + \eta + \nu_1$ and

$$A_n = \{h_\xi \leqslant h_\eta + h_{\nu_n}\},$$

where h_ξ, h_η and h_{ν_n} represent arbitrary versions of the Radon–Nikodym derivatives of ξ, η and ν_n respectively, with respect to m, for simplicity. So, A_n is uniquely determined modulo a subset of X of m null measure.

For each $n \in N$, let us consider

$$\mu_n' = \mu \,|\, b(A_n^*), \qquad \mu_n'' = \mu \,|\, X \backslash b(A_n^*), \tag{1}$$

the restrictions of the measure μ to the universally measurable sets $b(A_n^*)$ and $X \backslash b(A_n^*)$ respectively, which form a partition of the space X. If we consider now the potential measures admitting as charges the above measures, the hypothesis $\eta \leqslant \xi$ can be written like below

$$\eta \leqslant \mu_n' U + \mu_n'' U + \gamma = \xi, \quad n \in N. \tag{2}$$

According to Riesz decomposition property in the H-cone Exc, if we denote

$$\gamma_n' = \mu_n'' U + \gamma,$$

(this component in the decomposition of ξ is not interesting for our purpose, from now on), there exists excessive measures $\eta_1^n, \eta_2^n \in$ Exc such that we have

$$\eta = \eta_1^n + \eta_2^n, \qquad \eta_1^n \leqslant \mu_n' U, \qquad \eta_2^n \leqslant \gamma_n'. \tag{3}$$

More precisely, we take

$$\eta_1^n = R(\eta - \gamma_n'), \qquad \eta_2^n = \eta - \eta_1^n, \tag{3$'$}$$

where $R(\eta - \gamma_n')$ means the reduite of $\eta - \gamma_n$ in the H-cone Exc, that is

$$R(\eta - \gamma_n') = \wedge\{\eta' \in \text{Exc}; \;\; \eta' \geqslant \eta - \gamma_n'\}.$$

Since the sequence of sets $b(A_n^*)_{n \in N}$ is clearly decreasing (like the sequence $(A_n^*)_{n \in N}$ itself) it follows that the sequence $(\gamma_n')_{n \in N}$ is increasing and hence, the sequence $(\eta_1^n)_{n \in N}$ is decreasing. We keep n fixed. From the definition of the set A_n and from the definition of the ‘reduite’ of an excessive measure on a set, we get

$$^*R_\xi^{A_n} \leqslant {}^*R_\eta^{A_n} + \nu_n. \tag{4}$$

Using (2) and (3), from (4) follows

$$\begin{aligned} {}^*R_{\mu_n' U}^{A_n} + {}^*R_{\gamma_n'}^{A_n} &\leqslant {}^*R_{\eta_1^n}^{A_n} + {}^*R_{\eta_2^n}^{A_n} + \nu_n \\ &\leqslant {}^*R_{\eta_1^n}^{A_n} + {}^*R_{\gamma_n'}^{A_n} + \nu_n \end{aligned}$$

and hence, if we subtract ${}^*R^{A_n}_{\gamma'_n}$ from the first and from the last expression, we get

$$ {}^*R^{A_n}_{\mu'_n U} \leqslant {}^*R^{A_n}_{\eta_1^n} + \nu_n \leqslant \eta_1^n + \nu_n. \tag{5} $$

Now, we show that

$$ {}^*R^{A_n}_{\mu'_n U} = \mu'_n U. \tag{6} $$

For this purpose, we invoque the general formula of duality for balayages, according to [1], that is

$$ L({}^*R^M_m, s) = L(m, B^{M^*}_s), \tag{7} $$

for any $m \in$ Exc, $s \in S$ and $M \subset X$ measurable, where $L\colon$ Exc $\times\, S \to \overline{R}_+$ is the energy functional. Particularly for $A = A_n, m = \mu^1_n U$, we get for any $s \in S$

$$ L(\mu'_n U, B^{A^*_n} s) = \mu'_n (B^{A^*_n} s) = \mu'_n(s) = L(\mu'_n U, s), \tag{8} $$

where we have applied the definition of μ'_n and the set $b(A^*_n)$ in order to obtain the second equality, the first and the last resulting from the properties of the energy functional L. From (7) and (8) we get

$$ L({}^*R^{A_n}_{\mu'_n U}, s) = L(\mu'_n U, s), $$

for any $s \in S$, and hence we get (6), $s \in S$ being arbitrary. In this manner (3), (5) imply the relations

$$ \eta_1^n \leqslant \mu'_n U \leqslant \eta_1^n + \nu_n, \qquad \eta_1^n + \eta_2^n = \eta. \tag{9} $$

But the sequence $(\mu'_n U)_{n\in N}$ is obviously decreasing to $(\mu|_E)U$, and we have shown that the sequence $(\eta_1^n)_{n\in N}$ is decreasing too, and hence the sequence $(\eta_2^n)_{n\in N}$ is increasing. Let us put

$$ \eta_1 = \wedge_n \eta_1^n, \qquad \eta_2 = \vee \eta_2^n, \tag{10} $$

where $\wedge$ and $\vee$ denote the infimum and the supremum in the H-cone Exc. From [2] follows immediately

$$ \eta_1 + \eta_2 = \eta. $$

But if we pass to the infimum $\wedge$ in the inequalities in (9) we get

$$ \eta_1 = \wedge_n \mu'_n U = (\mu|_E)U. \tag{11} $$

If we make use of the uniqueness of the decomposition of the excessive measure η in potential part and harmonic part, valid under our hypotheses (see [7]), it follows from (11) that there exists a positive measure μ' on X such that

$$\lambda U = (\mu|_E)U + \mu' U = (\mu|_E + \mu')U.$$

From the uniqueness of the charge of a potential measure, valid under our hypotheses conditions (see [7]), it follows then that

$$\lambda = \mu|_E + \mu',$$

and since $\mu = \mu|_E$ on E we get now the desired inequality

$$\mu \leqslant \mu|_E + \mu' = \lambda \quad \text{on } E,$$

and the theorem is proved. □

REMARK. It can be seen that the method of proof, which generalizes that in [6], eludes the harmonic parts of the considered excessive measures, whose role is inexistent.

We show now how the classical Grishin lemma can be deduced from above theorem. In fact we deal with Fuglede's version of the Grishin lemma, a refined version which is close to us.

We recall that an extended real function w defined on a domain $\Omega \subset R^n$ is called δ-subharmonic if w can be written as the difference of two superharmonic functions u and v, that is $w = u - v$ (u and v can be assumed positive, at least locally) and then $\mu[w]$ denotes the Riesz charge of the function w, that is the unique Borel signed measure on Ω, such that locally $\mu[w]$ coincides with the difference of the charges of the potential parts of u and v in the Riesz decomposition of the superharmonic functions u and v respectively.

COROLLARY. *For any* positive *δ-subharmonic function w on $\Omega \subset R^n$, if we consider the set*

$$E = \{x \in \Omega;\ \underset{y\to x}{\text{fine}} \liminf_{y\neq x} w(y) = 0\},$$

we have

$$\mu[w]|_E \leqslant 0.$$

We should remark that all can be localized and then, using standard arguments (for example like in [6]) we can reduce the problem to the case where Ω posesses a Green function $G(x, y)$ which is simmetric in x and y (the Laplace operator is self-adjoint) and then the standard H-cone S consisting of all superharmonic positive functions on Ω is autodeal (see [2]). For example, we can suppose that Ω is a ball Ω' such that the closure of Ω' is contained in the original domain Ω. Also the resolvent $\mathcal{V} = (V_\alpha)_{\alpha>0}$ such that $S = \{s \in \xi(\mathcal{V}) : [s = \infty] \text{ is negligible}\}$ is autodual with respect to the Lebesque measure l (restricted to Ω of course) which is excessive (and coexcessive too) relative to $\mathcal{V}$. Then it is well-known that we can identify

$$S \simeq \text{Exc}(\mathcal{V}) \tag{12}$$

through the isomorphism of (standard) H-cones

$$S \ni s \to s \cdot (l|_\Omega) \in \text{Exc}(\mathcal{V}).$$

We remark also that the set E defined above (like in [6]) admits now the description

$$E = \bigcap_{n \in N} b\left(\left\{u \leqslant v + \frac{1}{n}\right\}\right),$$

where $w = u - v, u, v \in S$. If we identify the potential excessive measures with the Green potentials with the same charge (from (12))

$$G_\mu \to \mu U,$$

it follows that the above corollary is a consequence of our theorem if we take

$$\nu_n = (1/n) \cdot l.$$

Now observing that if we denote for any $n \in N$

$$A'_n = \left\{u < v + \frac{1}{n}\right\},$$

A'_n being finely open (with respect to S), it follows from [1] that $A'_n = (A'_n)^*$ and hence the set E defined above coincides with the set E, considered before in our theorem.

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