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Extension of Transient Resolvents

VALENTIN GRECEA
Institute of Mathematics of the Romanian Academy, P.O. 1-764 70700 Bucharest, Romania

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Abstract. For a submarkovian resolvent $\mathcal{U} = (U_\alpha)_{\alpha>0}$ with bounded initial kernel U_0 on a Radon space X, we construct a minstable cone of potentials C on a compact metrizable space $Y \supset X$ such that $\mathcal{U}$ extends to a subordonated (to C) resolvent on Y.

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1. Introduction

It is an old problem in Potential Theory, to find the most convenient axiomatic system containing the gist of the potentialist structure, permetting an elegant development of the most classical results. In 1970, Mokobodzky introduced the "cones of potentials" [2], a very simple and general setting, but full in resources: developping the classical carrier theory due to M-me Hervé, Mokobodzky considered the subordonated resolvent of kernels assigned to a cone of potentials, and to a fixed element of this cone. Also, the theory of resolvent (with or without reference measure) have an amazing development. But resolvents come also from other areas of mathematics: Markov processes, analysis, etc. The aim of this note is to establish the converse of the Mokobodzky's result: under very general conditions of regularity, any submarkovian resolvent of positive kernels on a measurable space with *proper* initial kernel, comes from a subordonated resolvent assigned to a *minstable* cone of potentials (Theorem 1).

2. Preliminaries and Main Result

Let A be a set and let M be a set of numerical functions on A. For any $s_1, s_2 \in M$, we consider the numerical function (inf denotes pointwise infimum)

$$R^M(s_1 - s_2) \overset{\text{def}}{=} \inf\{t \in M : t + s_2 \geqslant s_1\}$$

supposing the set in braces nonvoid.

A positive convex cone C of finite functions on a set A is called a cone of potentials ([2]) if the following axioms are satisfied:

(1) $R^C(s_1 - s_2) \in C$, for any $s_1, s_2 \in C$.

(2) $s_1 - R^C(s_1 - s_2) \in C$, for any $s_1, s_2 \in C$.

Let now K be a Hausdorff compact space (not necessary metrizable) and consider a cone of potentials C consisting of continuous functions on K, containing the constants, separating the points of K, and closed with respect to uniform norm on $\mathcal{C}(K)$.

Fix an element $c \in C$. Under these conditions, it can be shown ([2]) that there exists an unique positive regular kernel V on $\beta(K)$ with following properties:

(a) $V1 = c$
(b) $f \in \mathcal{C}_+(K) \Rightarrow Vf \in C$
(c) $\delta_C(Vf) \subset \operatorname{supp} f$, for any $f \in \mathcal{C}_+(K)$, where $\operatorname{supp} f = \overline{\{f > 0\}}$,

where for any upper semicontinuous, upper bounded function φ on K, we denote by $\delta_C(\varphi)$ the Choquet boundary of K with respect to the convex cone of lower semicontinuous, lower bounded functions on K:

$$C_\varphi = \{s - \alpha\varphi : s \in C, \alpha \geqslant 0\}.$$

Moreover, the elements of C are V dominant (we recall that a positive finite Borel function u on K is called V dominant if for any two bounded positive Borel functions on K, the following implication is valid:

$$u + Vf \geqslant Vg \text{ on } \{g > 0\} \Rightarrow u + Vf \geqslant Vg \text{ on } K).$$

This follows from (c). The kernel V is called *subordonated* to C, and the unique submarkovian resolvent of continuous, regular positive kernels on $(K, \beta(K))$ possesing V as initial kernel is called the *subordonated resolvent* assigned to C and c.

Given an arbitrary resolvent $\mathcal{V} = (V_\alpha)_{\alpha>0}$ on a measurable space $(X, \mathcal{X})$, we say that $\mathcal{V}$ posseses a *Ray basis* if there exists a *countable* set D consisting of bounded supermedian functions on X, which separates the points of X, and such that $\mathcal{X} \subset (\sigma(D))^*$ (the universal completion of the σ algebra $\sigma(D)$). We recall that a set $A \in \mathcal{X}^*$ is called *absorbent* if the following implication is valid:

$$x \in A \Rightarrow V_\alpha(x, X \backslash A) = 0, \quad \text{for any } \alpha > 0$$

(it suffices for one $\alpha > 0$). In this situation, it is well known that the family of kernels $\mathcal{U}^A = (\mathcal{U}^A_\alpha)_{\alpha>0}$ defined on the measurable space $(A, \mathcal{X}_A)$ by the relations

$$x \in A, \quad U^A_\alpha f(x) = U_\alpha \bar{f}(x), \qquad \text{for any } \alpha > 0$$

(where $\mathcal{X}_A$ is the trace of $\mathcal{X}$ on A and $\bar{f}$ is a measurable extension of f on X), is a resolvent on A, called the restriction of $\mathcal{U}$ on the absorbent set A.

THEOREM 1. *Let X be a Radon space, and $\mathcal{V} = (V_\alpha)_{\alpha>0}$ be a submarkovian resolvent of positive kernels on $(X, (\beta(X))^*)$ with* bounded *initial kernel $V = V_0$. Then $\mathcal{V}$ possesses a Ray basis iff $\mathcal{V}$ is the restriction of a subordonated resolvent $\mathcal{U}$ relatively to a minstable cone of potentials on a compact metrizable space K that contains X as universally measurable absorbent subset. Moreover we may suppose X dense in K with respect to the topology on K.*

Proof. We begin with the simple implication "$\Leftarrow$". Let C be a cone of potentials on a compact metrizable space K such that C is minstable, separates the points of K and contains the constants.

Then C-C is dense in $\mathcal{C}(K)$ (Weierstrass–Stone theorem). Since K is metrizable, C-C is separable and we choose a countable dense subset of C-C, say $D_1 = \{s_n - t_n\}_{n\in N}$, where s_n, t_n belong to C. But the elements of C are supermedian for $\mathcal{U}$, and since X is absorbent, it follows that the set

$$D = \{s_{n|X} : n \in N\} \cup \{t_{n|X} : n \in N\}$$

forms a Ray basis for $\mathcal{V}$. In fact D-D is uniformly dense in $bC_d(X) =$ the linear space of all bounded uniformly continuous functions on X (we consider the metric induced on X).

For the converse, let $\mathcal{V} = (V_\alpha)_{\alpha>0}$ be a resolvent on $(X, (\beta(X))^*)$ as in the statement, possesing a Ray basis D. Let us denote by $b\mathcal{S}(\mathcal{V})$ the cone of all bounded $\mathcal{V}$ supermedian functions, which is known to be a cone of potentials. We define recurrently the sequence of sets of functions, all contained in $b\mathcal{S}(\mathcal{V})$:

$$P_0 = D \cup \{1\},$$
$$P^1_{n+1} = \{as + bt : s, t \in P_n, a, b \geqslant 0\},$$
$$P^2_{n+1} = \{s \wedge t : s, t \in P_n\},$$
$$P^3_{n+1} = \{V_\alpha s : s \in P_n, \alpha > 0\},$$
$$P^4_{n+1} = \{R(s-t) : s, t \in P_n\} \cup \{s - R(s-t) : s, t \in P_n\},$$
$$P_{n+1} = \bigcup_{i=1}^{4} P^i_{n+1},$$

where $R(s-t)$ is considered in the cone of potentials $b\mathcal{S}(\mathcal{V})$. We consider now

$$P = \bigcup_{n=0}^{\infty} P_n \subset b\mathcal{S}(\mathcal{V}).$$

It is easy to see that P is a Ray cone ([1, XII, 85]) possesing the additional property:

(i) $s, t \in P \Rightarrow R(s-t), s - R(s-t) \in P$,

where $R(s-t)$ is considered as above. In particular P is a cone of potentials, such that

$$R^P(s-t) = R(s-t), \quad \text{for any } s, t \in P.$$

We show now that P is separable with respect to uniform norm, proceeding by induction. If P_n is supposed to be separable, it is easily seen that P^i_{n+1} are separable too, for $i = 1, 2, 3$. For $i = 4$, it suffices to recall that the operation

$$b\mathcal{S}(\mathcal{V}) \times b\mathcal{S}(\mathcal{V}) \ni (s,t) \to R(s-t) \in b\mathcal{S}(\mathcal{V})$$

is continuous with respect to uniform norm. Therefore P_{n+1} is separable. Since n is arbitrary it follows that P is separable, that is a true Ray cone, possesing moreover the property (i) as above. We consider now the Ray compactification K of X with respect to this Ray cone ([1, XII, 87]) and we denote by $\mathcal{U} = (U_\alpha)_{\alpha>0}$ the Ray resolvent extending $\mathcal{V}$ on K. It follows from [1, XII, 87], that X is universally measurable in K and is moreover absorbent since $\mathcal{U}$ extends $\mathcal{V}$. Of course X is dense in its Ray compactification K. For any $s \in P$ we denote by $\bar{s}$ the unique extension of s to a continuous, $\mathcal{U}$ supermedian function on K. If we denote $\mathcal{S}_c = \mathcal{S}(\mathcal{U}) \cap \mathcal{C}(K)$, we show now the following relation

$$s, t \in P \Rightarrow R^{\mathcal{S}_c}(\bar{s} - \bar{t}) = \overline{R(s - t)} \tag{1}$$

$(R(s - t) \in P!)$. Let $u \in \mathcal{S}_c$, $u \geqslant \bar{s} - \bar{t}$. Then

$$u_{|X} \geqslant s - t$$

and hence

$$u_{|X} \geqslant R(s - t) \quad (u_{|X} \in b\mathcal{S}(\mathcal{V})).$$

Since u is continuous on K, it follows

$$u \geqslant \overline{R(s - t)}.$$

Since $u \in \mathcal{S}_c$ is arbitrary, we have established the inequality "$\geqslant$" in (1). For the opposite inequality we remark that $\overline{R(s - t)} \in \mathcal{S}_c$, and moreover

$$\overline{R(s - t)}|_X = R(s - t) \geqslant s - t.$$

The continuity implies

$$\overline{R(s - t)} \geqslant \bar{s} - \bar{t}$$

and hence

$$\overline{R(s - t)} \geqslant R^{\mathcal{S}_c}(\bar{s} - \bar{t}).$$

If we denote $\overline{P} = \{\bar{s} : s \in P\}$, it follows from (1) and (i) that $\overline{P}$ is a cone of potentials consisting of continuous $\mathcal{U}$ supermedian functions on K.

We denote now by C the closure (in $\mathcal{S}_c$) of $\overline{P}$ with respect to uniform norm. Using again the continuity of the "reduite" operation mentioned above, it then follows easily that C is a cone of potentials on K, consisting of continuous, $\mathcal{U}$ supermedian functions, and moreover C is minstable, contains the constants and separates the points of K (since $\overline{P}$ does, by construction). In addition, from (1) it follows the relation

$$R^{\mathcal{S}_c}(f) \in \mathcal{C}, \quad \text{for any } f \in \mathcal{C}_+(K). \tag{2}$$

It follows that $\mathcal{C} = \mathcal{S}_c$. If U_0 denotes the initial kernel of $\mathcal{U}$, then U_0 is a continuous kernel since V_0 is bounded by hypothesis, and hence we have

$$f \in \mathcal{C}_+(K) \Rightarrow U_0(f) \in C. \tag{3}$$

Finally, we show the relation

$$f \in \mathcal{C}_+(K) \Rightarrow \delta_c(U_0 f) \subset \operatorname{supp} f = \overline{\{f > 0\}}.$$

To this end we use the fact that the elements of C are $\mathcal{U}$ supermedian, and hence U_0 dominant. Therefore we have the implication

$$f \in \beta(K),\ f \geqslant 0,\ f \in C,\ s \geqslant U_0 f \text{ on } \{f > 0\} \Rightarrow s \geqslant U_0 f.$$

If then follows that for any $f \in \mathcal{C}_+(K)$, the set $\overline{\{f > 0\}}$ is a boundary set for the convex cone

$$C_{U_0 f} = \{s - \alpha U_0 f : s \in C, \alpha \geqslant 0\}$$

and therefore it contains the Choquet boundary of K (with respect to this convex cone) which is known to be the smallest boundary set, and the proof is finished. □

REMARK 1. The existence of a Ray basis consisting of supermedian functions is rather theoretical. In practice, the Ray basis consists of excessive functions, and if we suppose moreover that the cone of excessive functions is minstable and contains the constants, it follows that in our proof the cone P consists of excessive functions (it is not the case for $\overline{P}$) and consequently the original space X is contained in the set of nonbranch points for $\mathcal{U}$.

REMARK 2. In the theory of resolvents, one works with proper resolvents, and we have considered bounded resolvents. Let $\mathcal{U} = (U_\alpha)_{\alpha>0}$ be a proper resolvent, that is there exists $0 < g \leqslant 1$ such that $U_0(g)$ is bounded. It is well known that the (bounded) kernel

$$V_0(f) \stackrel{\text{def}}{=} U_0(g \cdot f)$$

possesses the complete maximum principle, and therefore exists a unique submarkovian resolvent $\mathcal{V} = (V_\alpha)_{\alpha>0}$, whose initial kernel is V_0. It is also well known that $\mathcal{U}$ and $\mathcal{V}$ have the same excessive functions; but we remark moreover that the mapping

$$\mu \circ U_0 \Rightarrow \mu \circ V_0$$

extends to a natural ordered isomorphism between the excessive measures. Therefore nothing is lost from the potentialist structure by this transform which allows us to reduce to bounded resolvents.

References

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