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A family of L^2-spaces associated to the jumps of a Markov process

Research Article

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Abstract: Given the (canonical) Markov process associated with a sufficiently general semigroup (P_t), we establish a result concerning the uniform completeness of a family of L^2-spaces naturally associated with the jumps of the process. An application of this result is presented.

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1. Introduction

Square integrable martingales play a fundamental role in probability. In the frame given by a sufficiently regular Markov process, it happens that square integrable martingales that are also (nonincreasing) additive functionals "generate" all square integrable martingales by means of an "elementary" stochastic calculus.

The study of this special class of martingales, started by Dynkin [3] and Venttsel [6], was later developed by other authors in connection with Lévy systems and *carré du champ* operators; see [2, 5].

In this paper, we deal with another feature of this class of martingales in connection with a representation of *increasing* additive functionals with totally inaccessible jumps, in the general frame given by a Ray Markov process on a compact metric space.

More precisely, our main result, Theorem 3.1, presents a property of "uniform" completeness for a family of L^2-spaces explicitly given, naturally associated with jumps of the Ray process considered. As an immediate application, in Corollary 3.2 (and its proof) we establish in this general frame a representation of pure discontinuous part of these martingales, considered in [2, Chapter XV, Theorem 46] for the case of Feller processes.

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2. General preliminaries

We consider the notions from general processes theory and martingale theory well-known. Given a compact metric space F, we consider a Ray Markov semigroup (P_t) on $(F, \beta(F))$ and the canonical Ray process $X = \big(\Omega, \mathcal{F}, \mathcal{F}_t, \Theta_t, P^x\big)$ with transition semigroup (P_t). For the definition and properties of Ray semigroups and Ray processes we refer the reader to [2] and [4]. We recall here only a minimum of facts necessary for comprehension of the paper.

Let N denote the set on nonbranching points for the resolvent (U_p) associated to (P_t), namely

$$N = \left\{ x \in F \,\Big|\, \lim_{p\to\infty} p\, U_p f(x) = f(x),\ f \in \mathcal{C}(F) \right\};$$

we denote by B the set $F \setminus N$ consisting of branching points. Then $\Omega = \big\{\omega\colon [0,\infty) \to N \mid \omega(t)$ is right continuous and $\omega(t-)$ exists in F for any $t > 0\big\}$, and $X_t(\omega) = \omega(t)$ for any $t \geq 0$. On Ω we consider the natural filtration $\mathcal{F}^0_t = \sigma(X_s,\, s \leq t)$, $\mathcal{F}^0_\infty = \sigma(X_s,\, s < \infty)$. For any law μ on N, we denote by $\mathcal{F}^\mu_t$ the completion of $\mathcal{F}^0_t$ in $\mathcal{F}^0_\infty$ with respect to P^μ, where P^μ is the unique law on $\mathcal{F}^0_\infty$ such that (X_t) is Markov under P^μ with transition semigroup (P_t) and initial law μ; we note that each $(\mathcal{F}^\mu_t)$ is right continuous. Finally, $\mathcal{F}_t = \bigcap_\mu \mathcal{F}^\mu_t$, $\mathcal{F} = \mathcal{F}_\infty$ by convention, and Θ_t denotes the shift operator on Ω, $(\Theta_t\omega)(s) = \omega(t+s)$ for any $s \geq 0$.

If A is a subset of $R_+ \times \Omega$, then A is called *evanescent* if it is P^μ-evanescent (that is, $P^\mu\{\omega \mid (t,\omega) \in A$ for some $t \geq 0\} = 0$) for *any* law μ on N.

We recall that if $(W, \mathcal{G}_t, \mathcal{G}, Q)$ is a general filtered probability space with $(\mathcal{G}_t)$ satisfying the usual conditions, then a stopping time T is accessible (totally inaccessible) iff its graph is contained in a countable union of graphs of predictable stopping times (respectively, $Q\{T = S < \infty\} = 0$ for any predictable stopping time S).

In the sequel a stopping time is meant relative to $(\mathcal{F}_t)$, if it is not specified, and in addition it is called *predictable* (*totally inaccessible*) if it is so relative to $(\mathcal{F}^\mu_t, P^\mu)$ for *any* law μ on N (in the above sense). The following fact is fundamental for the sequel; it is essentially proved in [2, Chapter XV, 48].

Proposition 2.1 ([2]).
If μ is a law on N, then a stopping time T is totally inaccessible relative to $(\mathcal{F}^\mu_t, P^\mu)$ iff the following relations hold:

$$X_{T-} \neq X_T, \quad X_{T-} \in N \qquad \text{on} \quad \{T < \infty\},$$

except on a P^μ-negligible set.

Next, we consider the set

$$\mathcal{J} = \big\{(s,\omega) \mid s > 0,\ X_{s-}(\omega) \neq X_s(\omega),\ X_{s-}(\omega) \in N\big\}, \tag{1}$$

which can be expressed as a disjoint countable union of graphs of stopping times S^n (in a canonical manner). Then each S^n is totally inaccessible, and some kind of converse is also true.

Corollary 2.2.
Let (G_t) be a (real) r.c.l.l. process adapted to $(\mathcal{F}_t)$, with totally inaccessible jumps. Then except on an evanescent set the jumps of (G_t) are contained in $\mathcal{J}$.

Proof. Indeed, it suffices to recall that the set of jumps of (G_t) can be expressed as a *countable* union of graphs of stopping times, and apply the above result. □

Finally, if (N_t) is a local martingale on some filtered probability space, we denote by $(N^d)_t$ the purely discontinuous part of (N_t), see [2, Chapter VIII, 43], and if (H_t) is a (bounded) predictable process, then $\int_0^t H_s\, dN_s = (H \cdot N)_t$ denotes the usual stochastic integral.

3. The main result

Let $H^{2,x}_{\text{loc}}$ denote the space of r.c. martingales (M_t) relative to $(\Omega, \mathcal{F}^x_t, \mathcal{F}^x, P^x)$, such that $E^x[M_t^2] < \infty$ for any $t > 0$, endowed with the locally convex topology given by the seminorms $p^x_t(M) = (E^x[M_t^2])^{1/2}$.

We recall that an *additive functional* is a real r.c. process (G_t), adapted to $(\mathcal{F}_t)$, such that $G_0 = 0$, and for any fixed $s, t \geq 0$ the following relation holds:

$$G_{t+s} = G_t + G_s \circ \Theta_t,$$

except on an evanescent set depending on s, t. In the sequel $\mathcal{M}$ denotes the space of processes (M_t) which are both elements of $H^{2,x}_{\text{loc}}$ for any $x \in N$ and additive functionals, endowed with the topology given by the above seminorms p^x_t.

We denote by $\mathcal{E}$ the σ-algebra generated by universally measurable p-excessive functions on F, where $p > 0$ is fixed, but $\mathcal{E}$ does not depend on p. For any $x \in N$ and $t \geq 0$ we consider the positive measures on $(F \times F, \mathcal{E} \times \mathcal{E})$ defined by

$$\mu^{x,t}(f) = E^x\left[\sum_{s\leq t,\, s\in\mathcal{J}} f(X_{s-}, X_s)\right],$$

where $\mathcal{J}$ is the set considered in (1).

Clearly, $\mu^{x,t}$ depends only on the restriction to f on $(N \times N) \cap ((F \times F) \setminus \Delta)$, where Δ denotes the diagonal of $F \times F$, and therefore we consider in the sequel the space $(N \times N) \setminus \Delta$ instead of $F \times F$.

For $p > 0$ we denote $\Lambda^p = \left\{ f \colon (N \times N) \setminus \Delta \to R \mid f \in L^p(\mu^{x,t}) \text{ for any } x \in N,\ t \geq 0 \right\}$. If (f^n) is a sequence in Λ^p, we say that (f^n) is Cauchy in Λ^p if it is Cauchy in $\Lambda^p(\mu^{x,t})$ for any $x \in N$, $t \geq 0$.

According to [2, Chapter XV, 45], we denote by (S^f_t) *the compensed sum of jumps associated to f.*

Theorem 3.1.

For any Cauchy sequence $(f^n) \subset \Lambda^2$, there exists $f \in \Lambda^2$ such that $f^n \to f$ in Λ^2, where Λ^2 is endowed with the family of seminorms

$$q^x_t(f) = \left(E^x\left[\sum_{s\leq t,\, s\in\mathcal{J}} f^2(X_{s-}, X_s)\right]\right)^{1/2}.$$

Proof. Let (f^n) be a Cauchy sequence in Λ^2. The positive (negative) part of f^n is also a Cauchy sequence in Λ^2, so we may suppose $f^n \geq 0$ for any $n \in \mathbb{N}$, without loss of generality.

For any fixed $x \in N$, let f^x be a limit of (f^n) in $L^2(\mu^{x,t})$ for any $t \geq 0$; this is clearly possible since the family of positive measures $(\mu^{x,t})$ is increasing in $t \geq 0$ (we may reduce it by considering a sequence (t^x_p) increasing to ∞). We remark now that the measures $\mu^{x,t}$, $x \in N$, $t \geq 0$, are "uniformly" σ-finite on $(N \times N) \setminus \Delta$; moreover, if we denote

$$U_k = \left\{(x, y) \in F \times F \mid 2^{-k-1} < d(x, y) \leq 2^{-k}\right\},$$

it follows from [2, Chapter XV, 27] that there exists a partition $(A^b_j)_{j\in N}$ of N, $A^b_j \in \mathcal{E}$, such that

$$\mu^{x,t}\big(U_k \cap \big(N \times A^k_j\big)\big) = E^x\left[\sum_{s\leq t,\, s\in J} I_{U_k\cap(N\times A^k_j)}(X_{s-}, X_s)\right] < \infty.$$

We fix $k, j \in \mathbb{N}$ and for any real function f on $(N \times N) \setminus \Delta$ we denote $f_{k,j} = f \cdot I_{U_k\cap(N\times A^k_j)}$. Let us consider the increasing additive functionals defined by

$${}^{k,j}A^n_t = \sum_{s\leq t} f^n_{k,j}(X_{s-}, X_s), \qquad {}^{k,j}A^x_t = \sum_{s\leq t} f^x_{k,j}(X_{s-}, X_s).$$

Using the Schwarz inequality, the following estimations are derived:

$$\begin{aligned}\left\|{}^{k,j}A_t^n - {}^{k,j}A_t^x\right\|_{L^1(P^x)} &= E^x\left[\left|\sum_{s\le t}\left(f_{k,j}^n - f_{k,j}^x\right)(X_{s-},X_s)\right|\right] \le E^x\left[\sum_{s\le t}\left|f_{k,j}^n - f_{k,j}^x\right|(X_{s-},X_s)\right] \\ &= \int \mu^{x,t}(dy)\left(|f^n - f^x| I_{U_k\cap(N\times A_j^k)}\right)(y) \le \|f^n - f^x\|_{L^2(\mu^{x,t})}\left(\mu^{x,t}\{U_k\cap(N\times A_j^k)\}\right)^{1/2},\end{aligned} \tag{2}$$

and similarly we have

$$\left\|{}^{k,j}A_t^x\right\|_{L^1(P^x)} \le ||f^x||_{L^2(\mu^{x,t})}\left(\mu^{x,t}\{U_k\cap(N\times A_j^k)\}\right)^{1/2}.$$

Therefore, for any *fixed* $t \ge 0$, the sequence $({}^{k,j}A_t^n)$ is convergent to ${}^{k,j}A_t^x$ in $L^1(P^x)$. Using now [2, Chapter XV, Lemma 2, b)], for each $t \ge 0$, we find an $\mathcal{F}_t$-measurable, positive random variable ${}^{k,j}C_t \in L^1(P^x)$ such that

$${}^{k,j}C_t = {}^{k,j}A_t^x \qquad P^x \text{ a.s.}$$

for *any* $x \in N$. Next, we regularize ${}^{k,j}C_t$ by defining

$${}^{k,j}B_t = \inf_{s>t,\, s\in Q} {}^{k,j}C_t,$$

and, finally,

$${}^{k,j}A_t = \begin{cases} {}^{k,j}B_t & \text{if } {}^{k,j}B_0 = 0, \\ 0 & \text{if } {}^{k,j}B_0 \ne 0. \end{cases}$$

It follows then that $({}^{k,j}A_t)$ is r.c.l.l., increasing, zero at 0, such that

$${}^{k,j}A_t = {}^{k,j}A_t^x \qquad P^x \text{ a.s.} \tag{3}$$

for any $x \in N$, and because of right continuity of the paths, the above relation holds for *any* $t \ge 0$.

The set $\left\{\omega \in \Omega \mid t \mapsto {}^{k,j}A_t(\omega) \text{ is not purely discontinuous}\right\}$ is then null, and after a standard modification as above, we consider that $({}^{k,j}A_t)$ is purely discontinuous, and, moreover, with totally inaccessible jumps. Indeed, since $({}^{k,j}A_t)$ is $(\mathcal{F}_t)$-adapted, it follows that the set of its jumps may be expressed by $\bigcup_n T^n$, where T^n are (disjoint) stopping times. Then from (3) it follows that for each n we have for any $x \in N$,

$$P^x\left\{T^n < \infty,\, X_{T^n_-} = X_{T^n} \text{ or } X_{T^n_-} \notin N\right\} = 0.$$

But the set in braces is $\mathcal{F}$-measurable, hence the above relation still holds with P^x replaced by P^μ, where μ is an arbitrary law on N, and hence the jumps of $({}^{k,j}A_t)$ are totally inaccessible, from Proposition 2.1.

Next we remark that ${}^{k,j}A_t^n$ converges to ${}^{k,j}A_t$ in probability (for fixed t) with respect to any P^μ, where μ is a law on N. Indeed, given μ, there exists ν equivalent to μ, such that $\nu(h) < \infty$, where $h(x) = \sup_n E^x[{}^{k,j}A_t^n] < \infty$, for any $x \in N$, from (2). Then clearly ${}^{k,j}A_t^n \to {}^{k,j}A_t$ in $L^1(P^\nu)$, and the convergence is still in probability with respect to P^μ.

Finally, passing to the limit in the relation

$${}^{k,j}A_{t+s}^n = {}^{k,j}A_t^n + {}^{k,j}A_s^n \circ \Theta_t \qquad P^\mu \text{ a.s.},$$

for any law μ (in fact, the above relation holds identically) we establish that $({}^{k,j}A_t)$ is an additive functional. Given μ, let $\Theta = \mu \circ P_t$, and since ${}^{k,j}A_s^n \to {}^{k,j}A_s$ in probability with respect to P^Θ, it follows that ${}^{k,j}A_s^n \circ \Theta_t \to {}^{k,j}A_s \circ \Theta_t$ in probability with respect to P^μ from a simple Markov property.

By [1, Theorem 4.2] there exists an $(\mathcal{E}\times\mathcal{E})$-measurable positive function $f_{k,j}$ on $(N\times N)\setminus\Delta$, such that for any $x\in N$

$$\sum_{s\le t} f_{k,j}(X_{s-},X_s) = {}^{k,j}A_t = {}^{k,j}A^x_t = \sum_{s\le t} f^x_{k,j}(X_{s-},X_s), \tag{4}$$

except on an P^x-null set, $t\ge 0$.

Let (S^n) be a sequence of "disjoint" stopping times such that $J=\bigcup_n S^n$. Then from (4) it follows that for any $n\in\mathbb{N}$,

$$f_{k,j}(X_{S^n_-},X_{S^n}) = f^x_{k,j}(X_{S^n_-},X_{S^n}) \qquad P^x \text{ a.s.},$$

$x\in N$. By summation, we deduce for any $t\ge 0$,

$$\int |f_{k,j}-f^x_{k,j}|(y)\,\mu^{x,t}(dy) = E^x\left[\sum_{s\le t,\,s\in\mathcal{J}} |f_{k,j}-f^x_{k,j}|(X_{s-},X_s)\right] = E^x\left[\sum_n I_{\{S^n\le t\}}|f_{k,j}-f^x_{k,j}|(X_{S^n_-},X_{S^n})\right]=0,$$

$x\in N$, and consequently we may take $f_{k,j}$ to be zero outside the set $U_k\cap(N\times A^k_j)$. If we put $f=\sum_{k,j} f_{k,j}$, then clearly f is the limit of (f^n) in Λ^2, and the proof is complete. □

Corollary 3.2.

For any $(M_t)\in\mathcal{M}$, there exists $f\in\Lambda^2$ such that $(I_{\mathcal{N}}\cdot M)^d_t = S^f_t$ with respect to any probability P^x, where $\mathcal{N}=\{(s,\omega)\mid X_{s-}(\omega)\in N\}$.

Proof. Let (M_t) be of the form $M_t=\int_0^t g(X_{s-})\,dC^\varphi_s$, where φ belongs to the (extended) domain $D^e(L)$ of (U_t) and

$$C^\varphi_t = \varphi(X_t)-\varphi(X_0)-\int_0^t L\varphi\circ X_s\,ds.$$

Then, as in [2, Chapter XV, Theorem 46], we may take $f(x,y)=(g\cdot I_N)(x)(\varphi(y)-\varphi(x))$ and the desired relation holds. Since the mapping $f\to S^f$ is additive, it follows that the conclusion holds if (M_t) is a *finite* sum of martingales of the above form.

From [2, Chapter XV, 49] it follows that for any $M\in\mathcal{M}$ there exists a sequence $M^n\to M$ in $H^{2,x}_{\text{loc}}$, $x\in N$, such that M^n is a finite sum as above, $n\in\mathbb{N}$. Consequently we can write for any fixed $x\in N$, $t\ge 0$:

$$\|f^n-f^m\|_{L^2(\mu^{x,t})} = p^x_t\left(S^{f^n}-S^{f^m}\right)\le p^x_t(M^n-M^m),$$

and therefore the sequence (f^n) is Cauchy in Λ^2.

From Theorem 3.1, there exists $f\in\Lambda^2$ such that $f^n\to f$ in Λ^2 and hence we have

$$(I_{\mathcal{N}}\cdot M)^d = \lim_n (I_{\mathcal{N}}\cdot M^n)^d = \lim_n S^{f^n} = S^f,$$

with respect to any P^x, where the above limits are understood in $H^{2,x}_{\text{loc}}$. □

We recall that a general process (Z_t) on some filtered probability space is called *left quasi-continuous* if $Z_S=Z_{S-}$ a.s. for any predictable stopping time S.

Proposition 3.3.

Let (M_t) be a local martingale with respect to $(\mathcal{F}^\mu_t,P^\mu)$, where μ is a fixed law on N. Then (M_t) is left quasi-continuous iff $M^d_t=(I_{\mathcal{N}}\cdot M)^d_t$ except on a P^μ-evanescent set.

Proof. (⇒) Choosing a suitable $(\mathcal{F}_t^\mu)$ stopping time and removing the initial value, we may assume that $M \in H^1(P^\mu)$, $M_0 = 0$. We recall (see [2, Chapter VIII, 44]) that $(M^d)_t = \lim_n (A_t^n) - (\tilde{A}^n)_t$ in H^1, where

$$A_t^n = \sum_{s \le t} (\Delta M_s)\, I_{\{|\Delta M_s| > 1/n\}} \tag{5}$$

and $(\tilde{A}_t^n)$ is the compensator of (A_t^n) with respect to $(\mathcal{F}_t^\mu, P^\mu)$. Since the jumps of (M_t) are totally inaccessible, it follows that $(\tilde{A}_t^n)$ is continuous, and, from Proposition 2.1, we may replace (ΔM_s) by $\Delta(I_{\mathcal{N}} \cdot M)_s = I_{\{X_{s-} \in N\}}(\Delta M_s)$ in (5).

(⇐) Let S be a predictable stopping time with respect to $(\mathcal{F}_t^\mu, P^\mu)$. Then outside a P^μ-null set we have the following relations:

$$\Delta M_S = (\Delta M^d)_S = \Delta(I_{\mathcal{N}} \cdot M)_S^d = \Delta(I_{\mathcal{N}} \cdot M)_S = I_{\{X_{S-} \in N\}} \Delta M_S.$$

Moreover, it is known that $\Delta M_S = 0$ on $\{X_{S-} \in N\}$, since on this set we have $X_S = X_{S-}$, a fundamental consequence of strong Markov property, and we can apply e.g. [2, Chapter XV, 48]. □

4. Remarks and comments

At a first glance, our main result looks rather as an axiom satisfied by the considered Ray process, by means of the probabilities (P^x), since no martingale or additive functional appears in its statement. The announced connection (in Introduction) with the representation of increasing additive functionals with totally inaccessible jumps is somewhat hidden in its proof, where [1, Theorem 4.2] is used essentially. Finally, we do not know if Corollary 3.2 can be proved without a completeness result (as our main result is, for example).

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