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On Some Results Concerning the Reduite and Balayage

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Abstract. If J is an analytic, saturated gambling house with compact sections, and $\mu \underset{J}{\leq} \lambda$, we show that there exists a (submarkovian) borel kernel P permitted in J such that $\mu = \lambda P$. If $\mathcal{V} = (V_\alpha)_{\alpha>0}$ is a proper submarkovian resolvent on a Lusin space X, we study the regularity of the reduite R_s^A of an excesive function s on a set $A \subset X$.

0. Introduction

In this paper we propose an improvement of some more or less recent results concerning the theory of reduite and balayage in Potential Theory. First, we consider the "discrete" frame given by a gambling house on a compact metric space (see [6, X, XI]); we give an abstract version of an old, and well – known result due to Cartier for the case of compact convex sets in locally convex spaces, and extended then by Strassen to the case of convex cones of continuous functions on arbitrary compact spaces (see Theorem 1.6). Next, we pass to the "continuous" frame given by a proper submarkovian resolvent $\mathcal{V} = (V_\alpha)_{\alpha>0}$) on a Lusin measurable space X. We improve several recent results due to Beznea and Boboc concerning the reduite. Improvement refers to regularity of sets and functions in question; in the particular case where there exists a borel (right) process on X whose resolvent is $\mathcal{V}$, that is X is semisaturated (as it is the case in [4]), both Theorem 2.2 and its analogous from [2] are well – known consequences of the fundamental theorem of Hunt (see [6]). We do not hesitate to use probabilistic and analytic methods as much, looking for generality of the statements and rapid proofs: Proposition 2.5 appears as a consequence of the fundamental lemma of Hunt for Ray processes, whereas Theorem 2.6 is proved using mostly analytic tools, as an application of Capacity Theory (we point out here a bicapacitary

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operator), gambling houses, and a beautiful result concerning the characterization of reduite due to Mokobodzky (taken from [3]).

The first part of Section 1 is devoted to recalling the minimal vocabulary concerning the gambling houses. In exchange we asuume the reader sufficiently familiar with the general theory of resolvents and right processes (for which we refer to [6] and [7]).

1.

Let $(E, \mathcal{E})$ be a measurable space. We denote by $\mathcal{M}_+(E)$ the set of all *positive bounded measures* on E, and by $\Pi' = \{\mu \in \mathcal{M}_+(E),\ \mu(E) \leq 1\}$ the set of all *subprobabilities* on E. We also denote by $p\,\mathcal{E}$ (resp. $b\,\mathcal{E}$) the set of positive (resp. bounded) $\mathcal{E}-$measurable functions on E. Given $\mu \in \mathcal{M}_+(E)$, let $\mathcal{E}^\mu$ be the completion of $\mathcal{E}$ with respect to μ. We shall consider the universal completion $\mathcal{E}^u$ of $\mathcal{E}$, the $\sigma-$field defined by $\bigcap_{\mu \in \mathcal{M}_+(E)} \mathcal{E}^\mu$. For any $\mu \in \mathcal{M}_+(E)$ and for any $f \in p\,\mathcal{E}^\mu$ (or $b\,\mathcal{E}^\mu$), we denote by $\mu(f)$ the Lebesque integral of f with respect to μ. We endow $\mathcal{M}_+(E)$ with the $\sigma-$field $\mathcal{E}'$ generated by the functions $\mu \to \mu(f),\ f \in b\,\mathcal{E}$ (the trace of $\mathcal{E}'$ on Π' is still denoted by $\mathcal{E}'$).

Remark 1.1. Let $\mathcal{A}$ be a subset of $b\,\mathcal{E}$ such that the $\sigma-$field on E generated by $\mathcal{A}$ is equal to $\mathcal{E}'$, $\mathcal{A}$ is closed under multiplication (or is a $\Lambda-$stable vector space), $1 \in \mathcal{A}$. Then for any set $\mathcal{A}_1 \subset b\,\mathcal{E}$ such that the uniform closure of $\mathcal{A}_1$ contains $\mathcal{A}$ (in particular for $\mathcal{A}_1 = \mathcal{A}$) it follows that $\mathcal{E}'$ is generated by the functions $\mu \to \mu(f),\ f \in \mathcal{A}_1$.

Indeed, denoting by $\mathcal{E}'_1$ this $\sigma-$field, obviously $\mathcal{E}'_1 \subset \mathcal{E}'$. Let

$$\mathcal{H} \ = \ \left\{ f \in b\,\mathcal{E} : \text{the function } \mu \to \mu(f) \text{ is } \mathcal{E}'_1 \text{ measurable on } \mathcal{M}_+(E) \right\}.$$

Then $\mathcal{H}$ contains $\mathcal{A}_1$, and hence $\mathcal{A}$ by uniform density. A typical application of the monotone class theorem ([6, I, 2.1, 22.3]) leads now to conclusion.

Let P be a positive, submarkovian kernel on $(E, \mathcal{E})$, that is a measurable mapping $x \to P_x$ from $(E, \mathcal{E})$ to $(\Pi', \mathcal{E}')$. If E is a topological space and $\mathcal{E}$ is the borel $\sigma-$field, we say that P is *borel*. If (in this context) the mapping $x \to P_x$ is measurable only from $(E, \mathcal{E}^u)$ $\big($resp. $\big(E, \mathcal{E}^\lambda\big), \lambda \in \mathcal{M}_+(E)$ fixed$\big)$ to $(\Pi', \mathcal{E}')$, we say that P is universally measurable (resp. λ measurable). Here a simple consequence of above considerations.

Remark 1.2. Suppose E metrizable, separable, endowed with the borel $\sigma-$field. If $\lambda \in \mathcal{M}_+(E)$ is fixed, then P is $\lambda-$borel iff there exists P' borel (positive submarkovian) such that $P_x = P'_x\ \lambda$ a.s.

Following Dubins and Savage, we call *gambling house* any subset J of $E \times \Pi'$, such that $\pi_1(J) = E$. In the sequel we consider J such that $(x, \varepsilon_x) \in J$ for any $x \in E$ (ε_x denotes the Dirac mass at x), and moreover J is an $\mathcal{E} \times \mathcal{E}'-$analytic subset of $E \times \Pi'$ (for the definition of analytic sets and their theory, see [6]). For any $x \in E$, we denote by J_x the section $\{\mu \in \Pi' : (x, \mu) \in J\}$. A kernel P is *permitted* in J if $P_x \in J_x$ for any $x \in E$.

Example 1.3. Given a positive submarkovian kernel P on E, the subset J of $E\times\Pi'$ defined by $J_x = \{\varepsilon_x, P_x\}$ for any $x \in E$, is even a $\mathcal{E}\times\mathcal{E}'-$measurable gambling house if E is metrizable and separable.

We recall (from [6]) that a lower bounded function (not necessarily measurable) f on E is called $J-$*supermedian*, if for any $x \in E$ and $\mu \in J_x$, we have

$$\mu^*(f) \ \leq \ f(x)$$

(μ^* denotes the upper integral with respect to μ). Here we have changed a little the definition from [6] in order to be more appropiate for our purpose. Let $\bar{J}$ be the saturated of J, that is the gambling house defined by

$$\bar{J}_x \ = \ \{\mu \in \Pi' : \mu(f) \leq f(x) \text{ for any } f \text{ universally measurable } J-\text{supermeridian}\}\,.$$

We say that J is *saturated*, if $J = \bar{J}$.

Example 1.4. Given a set Γ of lower bounded, universally measurable functions on E, the (not necessarily analytic) gambling house J defined by $J_x = \{\mu \in \Pi' : \mu(f) \leq f(x)$, for any $f \in \Gamma\}$ is automaticly saturated, and the elements of Γ are $J-$supermedian. (In fact any saturated gambling house is of such kind, by taking Γ to be the convex cone of universally measurable $J-$supermedian functions).

Also from [6], given f lower bounded on E, we consider the function J^*f on E, defined by $J^*f(x) = \sup_{\mu\in J_x}\mu^*(f)$, for any $x \in E$, and recurently $J^{*n}f$ for any $n \in N$. Finally, we see that the sequence $J^{*n}f$ is increasing, and denote $Rf = \sup_n J^{*n}f$, the *reduite* of f with respect to J. It is easy to see that Rf is the smallest $J-$supermedian function dominating f. Again we note the difference from the definition given in [5], where one consider only positive f. We return now to a set Γ as above, and for any μ, $\lambda \in \mathcal{M}_+(E)$, we consider the relation

$$\mu \underset{\Gamma}{\leq} \lambda \quad \overset{\text{def}}{\Longleftrightarrow} \quad \mu(f) \ \leq \ \lambda(f)\,, \quad \text{for any} \quad f \in \Gamma\,,$$

which defines obviously a preorder relation on $\mathcal{M}_+(E)$. If in particular Γ is the convex cone of $J-$supermedian, universally measurable functions with respect to a given gambling house J, the above relation is called the *balayage* preorder ([6]).

Proposition 1.5. *Let Γ be a convex cone of continuous functions on a compact space K, such that $1 \in K$, and denote by Γ' the $\wedge-$stabilised of Γ, that is* $\Gamma' = \{s_1 \wedge s_2 \wedge \ldots \wedge s_n : n \in N,\ s_i \in \Gamma \text{ for } i = 1, 2 \ldots n\}$. *Let $\mu, \lambda \in \mathcal{M}_+(K)$ be Radon measures. Then the following statements are equivalent:*

1) $\mu \underset{\Gamma'}{\leq} \lambda$.

2) *For any system of Radon measures $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathcal{M}_+(K)$ such that $\lambda = \sum_{i=1}^n \lambda_i$ there exists a system of Radon measures $\mu_1, \mu_2, \ldots, \mu_n \in \mathcal{M}_+(K)$ such that $\mu = \sum_{i=1}^n \mu_i$ and moreover*

$$\mu_i \ \underset{\Gamma}{\leq} \ \lambda_i \quad \textit{for} \quad i \ = \ 1\,, 2\,, \ldots\,, n\,.$$

Proof. The implication 1) $\Rightarrow$ 2) follows using repeatedly the Riesz decomposition property in the convex cone of Radon measures from $\mathcal{M}_+(K)$ endowed with the preorder $\underset{\Gamma'}{\leq}$, that is for any Radon measures $\mu_1, \lambda_1, \lambda_2 \in \mathcal{M}_+(K)$ such that $\mu \underset{\Gamma'}{\leq} \lambda_1 + \lambda_2$ there then exists Radon measures $\mu_1, \mu_2 \in \mathcal{M}_+(K)$ such that $\mu = \mu_1 + \mu_2$ and $\mu_1 \underset{\Gamma'}{\leq} \lambda_1$, $\mu_2 \underset{\Gamma'}{\leq} \lambda_2$. Indeed, if for $\mu \in \mathcal{M}_+(K)$ and $f \in \mathcal{C}(K)$ we define $Q_\mu(f) = \inf\{\mu(s), s \in \Gamma'\}$ known as the Bauer functional, and we denote by $Q(f)$ the function

$$x \longrightarrow Q_{\varepsilon_x}(f)$$

the minstability of Γ' implies that $Q_{\lambda_1+\lambda_2}(f) = (\lambda_1+\lambda_2)(Q(f)) = \lambda_1(Qf)+\lambda_2(Qf) = Q_{\lambda_1}(f) + Q_{\lambda_2}(f)$. Hence, for any $f \in \mathcal{C}(K)$ we have

$$\mu(f) \leq Q_\mu(f) \leq Q_{\lambda_1+\lambda_2}(f) = Q_{\lambda_1}(f) + Q_{\lambda_2}(f) .$$

The desired spliting of μ as above follows using a well−known application of Hahn−Banach theorem, the Bauer functional being subadditive and positive homogeneous on $\mathcal{C}(K)$. For the converse, let $s = s_1 \wedge s_2 \wedge \ldots \wedge s_n \in \Gamma'$. Consider the sets $A_1 = \{s = s_1\}$, $A_2 = \{s = s_2\} \setminus A_1, \ \ldots, \ A_n = \{s = s_n\} \setminus \bigcup_{i=1}^{n-1} A_i$. Then (A_i) form a (borel) partition of E. Put $\lambda_i = \lambda\big|_{A_i}$ for $i = 1, \ldots n$, and let μ_i be as in 2) for $i = 1, 2, \ldots, n$. We have

$$\mu(s) = \sum_{i=1}^{n} \mu_i(s) \leq \sum_{i=1}^{n} \mu_i(s_i) \leq \sum_{i=1}^{n} \lambda_i(s_i) = \sum_{i=1}^{n} \lambda_i(s) = \lambda(s) . \qquad \square$$

In the sequel E is a compact metric space and Π' is endoved with the narrow (vague) topology.

Theorem 1.6. *Let J be an analytic, saturated gambling house with compact sections (J_x compact, for any $x \in E$) and let λ, $\mu \in \mathcal{M}_+(K)$ be such that $\mu \underset{J}{\leq} \lambda$. Then there exists a borel submarkovian kernel P on E, permitted in J, such that $\mu = \lambda P$.*

Proof. First, we remark that the relation $\mu \underset{J}{\leq} \lambda$ implies the relation

$$\mu(f) \leq \lambda(Rf)$$

for any f analytic, lower bounded on E, Rf being analytic by [6, X, 14], in particular for any $f \in \mathcal{C}(E)$. If we consider the sublinear functional on $\mathcal{C}(E)$:

$$f \longrightarrow \lambda(Rf) = \int Rf(x)\, \lambda(dx)$$

as integral of sublinear functionals, and since the functions $Rf (f \in \mathcal{C}(E))$ are analytic, hence λ measurables on E, we see that the hypotheses from the well−known Strassen's disintegration theorem are satisfied (see the version of STRASSEN's result from [6, X, 33] applied here to the separable Banach space $\mathcal{C}(E)$, the bounded functional $f \to \mu(f)$, and the family of sublinear functionals $f \to Rf(x)$, $x \in E$. There exists then a borel

(E is metrizable and we apply the introducing remark) kernel P on E such that $\mu = \lambda P$ and

$$P_x(f) \;\leq\; Rf(x) \tag{1.1}$$

for any $f \in \mathcal{C}(E)$, λ almost surely. Let $x \in E$ be such that (1.1) holds for any $f \in \mathcal{C}(E)$. Since J is saturated, hence stable under composition of permitted kernels, it follows from [6, X, 22] that

$$Rh(x) \;=\; J^*h(x) \;=\; \sup\{\mu(h) : \mu \in J_x\}$$

for any h analytic, lower bounded. Let g be upper semicontinuous, bounded. There exists a sequence $(f_n) \subset \mathcal{C}(E)$ converging decreasingly to g. Since J_x is compact, by hypothesis, we deduce from (1.1) applied to each f_n, and the Dini – Cartan lemma by passing to the limit, that (1.1) holds for any upper semicontinuous, bounded function g.

Let now f be universally measurable, bounded. If $\varepsilon > 0$ is given, there exists g upper semicontinuous, bounded, such that $g \leq f$ and $P_x(f) \leq P_x(g) + \varepsilon$. Then

$$P_x(f) \;\leq\; P_x(g) + \varepsilon \;\leq\; Rg(x) + \varepsilon \;\leq\; Rf(x) + \varepsilon\,. \tag{1.2}$$

Since ε is arbitrary, (1.2) implies obviously that

$$P_x \underset{J}{\leq} \varepsilon_x\,. \tag{1.3}$$

Hence $P_x \in J_x$, since J is assumed to be saturated. If we modify eventually P_x on a borel set of measure null (by taking $P_x = \varepsilon_x$) we may get the kernel P permitted in J, and the proof is finished. □

Corollary 1.7. (Cartier – Strassen). *Let Γ be a minstable convex cone of continuous functions on E, containing the positive constants. If we consider the gambling house J defined by*

$$J_x \;=\; \left\{\mu \in \mathcal{M}_+(K) : \mu \underset{\Gamma}{\leq} \varepsilon_x\right\} \tag{1.4}$$

for any $x \in E$, and if the measures $\mu, \lambda \in \mathcal{M}_+(K)$ satisfy the relation $\mu \underset{\Gamma}{\leq} \lambda$, then there exists a borel kernel P permitted in J such that $\mu = \lambda P$.

In order to "deduce" this "classical" result from our result, it suffices to note that the gambling house defined by (1.4) is saturated (follows from definition), that J is even a compact subset of $E \times \Pi'$, and finally the nontrivial fact that the order of balayage on measures is given by the elements of Γ. This result is quite similar to [6, XI, 38, 39].

Theorem 1.8. a) *If f is upper semi – continuous and bounded and if the function $\hat{f}$ is defined by*

$$\hat{f}(x) \;=\; \inf\{g(x) : g \in \Gamma, g \geq f\}$$

then $\hat{f} = Rf = J^*f$.

b) *Let λ and μ be positive measures. Then*

$$\left(\mu \underset{J}{\leq} \lambda\right) \iff \left(\mu \underset{\Gamma}{\leq} \lambda\right).$$

We omit the proof which is essentially the same as for the result quoted above, noting only that it leans on the following "classical" consequence of Hahn – Banach theorem (and Dini – Cartan lemma): if for an arbitrary function f and a positive measure λ, we denote $p_\lambda(f) = \inf\{\lambda(g) : g \in \Gamma,\ g \geq f\}$, then for any upper semicontinuous, upper bounded function f, we have

$$p_\lambda(f) = \sup\left\{\mu(f) : \mu \underset{\Gamma}{\leq} \lambda\right\}.$$

Remark 1.9. Returning to the theorem, if we remove the compactness of sections from hypothesis, the theorem fails (for example, if we consider the "elementary" gambling house J defined in [6, XI, 40 bis a)] as follows: $E = [0, \infty]$ endowed with the usual compact topology; for $x = \infty$ we take $J_x = \varepsilon_x$, and for any $x < \infty$ we take $J_x = \{\mu \in \Pi'; \mu \text{ is carried by } \{x\} \cup (x+1, \infty)\}$. Then J is analytic, saturated, even in the stronger sense considering the relation $\mu \underset{J}{\leq} \varepsilon_x$ given by positive supermedian functions; if λ and μ are respectively the restrictions of Lebesque measure to the intervals $[0, 1]$ and $[1, 2]$, there is no kernel P permitted in J such that $\mu = \lambda P$).

Finally, we note that if in the statement of our theorem the gambling house J is supposed moreover to be compact in $E \times \Pi'$, then it follows from [6, XI, 26, 28] that J is generated by a convex cone Γ of continuous functions on E (Γ is just the cone of continuous J – supermedian functions on E) and in this case our theorem is equivalent to the "classical" case.

Remark 1.10. Theorem 1.6 extends immediately to the case where E is a Lusin topological space. Indeed, imbed E in a compact metric space $\widehat{E}$, as borel subset. Define

$$\widehat{J}_x = \begin{cases} J_x & \text{for} \quad x \in E, \\ \varepsilon_x & \text{for} \quad x \in \widehat{E} \backslash E. \end{cases}$$

Then $\widehat{J}$ is an analytic gambling house in $\widehat{E} \times \Pi'(\widehat{E})$, with compact sections ([6, III, 58]), and one checks easily that it is saturated. Apply Theorem 1.6 for $\widehat{J}$, and finally restrict the obtained kernel to E.

2.

Let $(X, \mathcal{X})$ be a measurable space and let $\mathcal{V} = (V_\alpha)_{\alpha>0}$ be a submarkovian resolvent on $(X, \mathcal{X})$. We denote by $\mathcal{S}$ (resp. $\mathcal{E}$) the convex cone of $\mathcal{V}$ – supermedian (resp. $\mathcal{V}$ – excessive) functions on X. We say that a set $A \in \mathcal{X}$ is $\mathcal{V}$ – *regular* if for any $s \in \mathcal{E}$ we have the relation

$$\lim_n n V_n(s \cdot 1_A)(x) = s(x)$$

for any $x \in A$. For example, if $\mathcal{X}$ is generated by the excessive functions and A is an arbitrary set from $\mathcal{X}$, it is shown in [1, proof of Th. 1.2] that the set

$$A^* \;=\; \Big\{x \in A : \liminf_{n\to\infty} nV_n 1_A(x) = 1\Big\}$$

is $\mathcal{V}-$regular. Moreover if $\mathcal{E}$ is minstable and $1 \in \mathcal{E}$, then for any fine open set A we have $A = A^*$, and hence A is $\mathcal{V}-$regular. Also if A is absorbent for $\mathcal{V}$ $\big($that is $V_\alpha(1_{X\setminus A})(x) = 0$, for any $\alpha > 0$ and $x \in A\big)$, then A is obviously $\mathcal{V}-$regular. If $\mathcal{X}^u$ denotes the universal completion of $\mathcal{X}$, then $\mathcal{V}$ extends naturally to a resolvent on $\mathcal{X}^u$, still denoted by $\mathcal{V}$. The notations $\mathcal{S}^u$ (resp. $\mathcal{E}^u$) are clear. In the following result R denotes the reduite with respect to the convex cone $\mathcal{S}$, that is for any $h \geq 0$ (not necessary measurable) we consider (following MOKOBODZKY) $Rh = \inf\{s\in\mathcal{S} : s\geq h\}$; if h is $\mathcal{X}-$measurable, it is known that $Rh \in \mathcal{S}$.

Proposition 2.1. *Let $\mathcal{V} = (V_\alpha)_{\alpha>0}$ be a submarkovian resolvent on a measurable space $(X, \mathcal{X})$, let $A \in \mathcal{X}$ be $\mathcal{V}$ –regular and let $s \in \mathcal{E}$. Then $R(s\cdot 1_A) \in \mathcal{E}$, and moreover we have:*

$$\begin{aligned} R(s\cdot 1_A) \;&=\; \inf\{t \in \mathcal{E} : t \geq s \text{ on } A\} \\ &=\; \inf\{t \in \mathcal{E}^u : t \geq s \text{ on } A\} \\ &=\; \inf\{t \in \mathcal{S}^u : t \geq s \text{ on } A\}\,. \end{aligned}$$

P r o o f. It is well−known from the general theory of resolvent that the reduite $R(s \cdot 1_A)$ is given by the relation

$$R(s\cdot 1_A) \;= \sup_n R^{nV_n}(s\cdot 1_A) \tag{2.1}$$

where for any $n \in N$, the operator R^{nV_n} denotes the discrete reduite with respect to the (submarkovian) kernel nV_n, and moreover we have $nV_n\big(R^{nV_n}(s\cdot 1_A)\big) = R^{nV_n}(s\cdot 1_A)$ on $X\setminus A$; hence it follows that allways $R(s\cdot 1_A) = R(\widehat{s\cdot 1}_A)$ on $X\setminus A$. For $x \in A$, we use the fact that A is $\mathcal{V}-$regular to deduce that

$$R(\widehat{s\cdot 1}_A)(x) \overset{\text{def}}{=} \lim_n nV_n(R(s\cdot 1_A))(x) \;\geq\; \lim_n nV_n(s\cdot 1_A)(x) \;=\; s(x) \;=\; R(s\cdot 1_A)(x)\,.$$

Since the converse inequality is obvious, the first assertion and the first equality in the statement are clear. Further, fix $n \in N$, and denote $P = nV_n$. Since $R^P(h) = \sup_m J^{*m}(h)$ (where J is the gambling house given by $J_x = \{\varepsilon_x, P_x\}$ for any $x \in X$) for any $n.n.m.$ function $h \geq 0$), it follows for $h = s\cdot 1_A$, that $R^P(s\cdot 1_A)$ is also the smallest $\mathcal{X}^u-$measurable $P-$supermedian function dominating $s \cdot 1_A$. If we consider s (resp. A) as $\mathcal{X}^u-$measurable function (resp. set) and we take $n \to \infty$, it follows from (2.1) the equality of $R(s\cdot 1_A)$ to the third brace in the statement, and hence the equality to the second brace follows immediately. □

For the remainder of the paper, we suppose that the initial kernel $V = V_\circ$ is proper, the cone $\mathcal{E}$ is minstable, $1 \in \mathcal{E}$, and the $\sigma-$algebra $\sigma(\mathcal{E})$ generated by $\mathcal{E}$ on X coincides with $\mathcal{X}$. We denote by $\widetilde{\mathcal{E}}$ the $\sigma-$algebra $\sigma(\mathcal{E}^u)$ generated by the universally measurable excessive functions. Obviously $\mathcal{X} \subset \widetilde{\mathcal{E}} \subset \mathcal{X}^u$. We recall that a subset A of X is called a *Lusin set* if the measurable space A endowed with the trace of $\mathcal{X}$ on A, is a Lusin measurable space. It is well–known that if $(X,\mathcal{X})$ is Lusin, then a subset A of X is Lusin, iff $A \in \mathcal{X}$.

Theorem 2.2. *Suppose that* $(X,\mathcal{X})$ *is Lusin* (*resp. Radon*), *and let* $A \in \widetilde{\mathcal{E}}$ (*resp. A be a Lusin set*). *Then for any* $s \in \mathcal{E}^u$, *the function*

$$R_s^A \stackrel{\text{def}}{=} \inf\{t \in \mathcal{E}^u : t \geq s \text{ on } A\}$$

(*called the reduite of s on A*) *is* $\widetilde{\mathcal{E}}$ (*resp.* $\mathcal{X}^u$) –*measurable,* $\mathcal{V}$ –*supermedian, and if we denote by* B_s^A *its excessive regularization,* $\lim_{\alpha\to\infty}\alpha V_\alpha\left(R_s^A\right)$ *we have*

$$B_s^A = R_s^A \quad \text{on} \quad X\setminus A\,.$$

Proof. First, we remark that we may suppose $V_\circ$ bounded, using a standard transform of resolvent preserving the excessive and supermedian functions. The above hypothesis permit us to choose a Ray cone $\mathcal{R}$ for $\mathcal{V}$ (see [6]) consisting of $\mathcal{X}$–measurable bounded excessive functions. Considering the Ray compactification Y of X with respect to $\mathcal{R}$ (see [6]) and the Ray resolvent $\overline{\mathcal{V}}$ on the metrizable compact space Y, we have that $\overline{\mathcal{V}}$ extends $\mathcal{V}$, and X is absorbent for $\overline{\mathcal{V}}$.

Moreover the initial space X is contained in the set D of nonbranch points for $\overline{\mathcal{V}}$, since $\mathcal{R}$ consists of excessive functions (not only supermedian). It follows that the initial kernel $\overline{V}_\circ$ of $\overline{\mathcal{V}}$ (which extends $V_\circ$) is continuous, and hence bounded on Y (a fortiori on D). We can consider the Ray process $(X_t)_{t\geq 0}$ on D, whose associated resolvent is the restriction of $\overline{\mathcal{V}}$ on D, still denoted in the sequel by $\overline{\mathcal{V}}$. For any $\mathcal{V}$–supermedian function u on X, the function $\bar{u}$ defined on D by (X is universally measurable in D)

$$\bar{u}(x) = \begin{cases} u(x) & x \in X\,, \\ \infty & x \in D\setminus X\,, \end{cases}$$

is obviously $\overline{\mathcal{V}}$–supermedian. Moreover if u is excessive, then the excessive regularization $\hat{\bar{u}}$ of $\bar{u}$ coincides with u on X (since X is absorbent in D).

Therefore the trace on X of the σ–algebra $\mathcal{E}^{\simeq}$ on D generated by the set $\bar{\mathcal{E}}^u$ of universally measurable $\overline{\mathcal{V}}$–excessive functions coincides with $\widetilde{\mathcal{E}}$. It follows that: if X is Lusin and $A \in \mathcal{E}$, then X is borel in D (by Lusin's theorem), in particular $X \in \bar{\mathcal{E}}$, and hence $A \in \bar{\mathcal{E}}$; if (X is Radon and) A is a Lusin set, we have that A is borel in D (since D is a Lusin space) and hence $A \in \mathcal{E}^{\simeq}$.

If $s \in \mathcal{E}^u$, we can consider the function

$$\overline{R}_{\hat{\bar{s}}}^A \stackrel{\text{def}}{=} \inf\left\{\bar{t} \in \bar{\mathcal{E}}^u : \bar{t} \geq \hat{\bar{s}} \text{ on } A\right\}\,.$$

By the celebrated theorem of Hunt on balayage (see [6, XIV, 97]) and its immediate consequences we know that the function $\bar{R}_{\hat{\bar{s}}}^A$ is $\mathcal{E}^{\simeq}$ measurable, $\overline{\mathcal{V}}$–supermedian, and it conincides with its $\overline{\mathcal{V}}$–excessive regularization $\overline{B}_{\hat{\bar{s}}}^A$ on $D\setminus A$. From above considerations, the restriction of $\overline{R}_{\hat{\bar{s}}}^A$ on X coincides with R_s^A, also the restriction of $\overline{B}_{\hat{\bar{s}}}^A$ on X

coincides with B_s^A, and the proof is finished. □

It is convenient to choose the variant $(X, \mathcal{X})$ Lusin and $A \in \widetilde{\mathcal{E}}$. We say that $A \subset \widetilde{\mathcal{E}}$ is *polar* if $B_1^A = 0$; A is *thin* at x if there exists $s \in \mathcal{E}^u$ such that $B_s^A(x) < s(x)$; A is *totally thin* if A is thin at any $x \in X$; A is *semipolar* if A is a countable union of totally thin sets. For any $A \in \widetilde{\mathcal{E}}$ we denote $b(A) = \{x \in X : A \text{ is not thin at } x\}$; on the other hand, if $X_{t\,(t \geq 0)}$ is the Ray process considered in the above proof, let $A^r = \{x \in D : P^x\{T_A = 0\} = 1\}$ be the set of regular points for A, where T_A is the hitting time of A.

Proposition 2.3. $b(A) = X \cap A^r$.

Proof. Denoting by $\overline{B}^A$ the above "balayage" operator considered on D, we have $(A \subset X)$

$$\begin{aligned} b(A) &= \big\{x \in X : B_s^A(x) = s(x), \text{ for any } s \in \mathcal{E}^u\big\} \\ &= X \cap \big\{x \in D : \overline{B}_{\bar{s}}^A(x) = \bar{s}(x), \text{ for any } \bar{s} \in \mathcal{E}^u\big\} \\ &= X \cap A^r\,, \end{aligned}$$

by the Hunt's theorem on balayage: $\overline{B}_{\bar{s}}^A(x) = P^x\big\{\bar{s} \circ X_{T_A}; T_A < \infty\big\}$ for any $\bar{s} \in \bar{\mathcal{E}}^u$; if $x \in A^r$ then obviously $\overline{B}_{\bar{s}}^A(x) = \bar{s}(x)$ for any $\bar{s} \in \bar{\mathcal{E}}^u$, and conversely, if $\overline{B}_{\bar{s}}^A(x) = \bar{s}(x)$ for any $\bar{s} \in \bar{\mathcal{E}}^u$, it suffices to take $\bar{s} = \overline{V}\,\bar{h}$, where $\bar{h} > 0$ on D with $\overline{V}\,\bar{h}(x) < \infty$, to get

$$E^x\left(\int_{T_A}^{\xi} h \circ X_t\,dt\right) \;=\; \overline{B}_{\overline{V}\,\bar{h}}^A(x) \;=\; \overline{V}\,\bar{h}(x) \;=\; E^x\left(\int_0^{\xi} h \circ X_t\,dt\right),$$

and hence $T_A = 0$ a.s. P^x, that is $x \in A^r$. □

It is well−known that $A \backslash A^r$ is semipolar in D. By arguing as above, we can complete the statement of the theorem:

Corollary 2.4. *For any $A \in \widetilde{\mathcal{E}}$ and $s \in \mathcal{E}^u$, we have*

$$B_s^A \;=\; R_s^A$$

off the semipolar set $A \backslash b(A)$.

Indeed, since $s \geq R_s^A \geq B_s^A$ anywhere, it follows that on $A \cap b(A)$ we have $s = R_s^A = B_s^A$, and we have that $A \backslash b(A) = A \backslash A^r$, which is semipolar in D and hence in X.

We could continue in this manner indefinitely; it is clear that a substantial part of the "analytic" Potential Theory associated to a transient resolvent may be deduced from the "probabilistic" Potential Theory associated to a borel (right) process. We end this idea with a nice consequence of Hunt's fundamental lemma.

Let μ be a $\sigma-$finite measure on X. For any set $A \in \widetilde{\mathcal{E}}$ and any $s \in \mathcal{E}^u$, we denote (according to [2])

$$R_s^A(\mu) \;=\; \inf\{\mu(t); t \in \mathcal{E}^u, t \geq s \text{ on } A\}\,.$$

Suppose for the moment that $\mathcal{V}$ comes from a Ray resolvent restricted to the set X of nonbranch points and let $(X_t)_{t\geq 0}$ be the canonical Ray process whose resolvent is $\mathcal{V}$. Let $s = Vh$ be a bounded potential such that $\mu(s) < \infty$. Then we have the relation $R_s^A(\mu) = \mu(R_s^A)$. Indeed, by Hunt's fundamental lemma (see [6, XIV, 94]), there exists a decreasing sequence $(H_n) \in \widetilde{\mathcal{E}},\ A \subset H_n \subset H_n^r$ for any $n \in N$, such that P^μ a.s. D_{H_n} (D denotes the enter time) tends to D_A, stationarilly on $\{D_A < \infty\}$. Therefore this convergence happens P^x a.s. for $\mu-$almost all $x \in X$. Hence $E^x\{s \circ X_{D_{H_n}}\} \searrow E^x\{s \circ X_{D_A}\}$ for μ almost all $x \in X$. Using Hunt's theorem, we have $\mu\big(R_s^A\big) = E^\mu\{s \circ X_{D_A}; D_A < \infty\}, \mu(R_s^{H_n}) = E^\mu\{s \circ X_{D_{H_n}} < \infty\}$ for any $n \in N$. If we denote $t_n = R_s^{H_n} = B_s^{H_n} \in \mathcal{E}^u$, it follows by dominated convergence that $\mu(t_n) \searrow \mu\big(R_s^A\big)$; also $t_n = s$ on $H_n \supset A$, hence $R_s^A(\mu) \leq \mu\big(R_s^A\big)$ and since the opposite inequality is obvious, the desired relation is clear. We return now to our general X and $\mathcal{V}$.

Proposition 2.5. *$R_s^A(\mu) = \mu\big(R_s^A\big)$, for any set $A \in \widetilde{\mathcal{E}}$ and any $s \in \mathcal{E}^u$ such that there exists $t \in \mathcal{E}^u,\ t > 0$ with $\mu(t) < \infty$.*

Proof. Since the mapping $s \to R_s^A(\mu)$ is increasing (that is $s_n \nearrow s \Rightarrow R_{s_n}^A(\mu) \nearrow R_s^A(\mu)$) as is noted in [2], we may reduce to the case where s is a bounded potential with $\mu(s) < \infty$. We imbed X in D and we extend $\mathcal{V}$ to $\overline{\mathcal{V}}$ as in the proof of theorem; we apply the above remarks and we pass to the original space by restriction. □

Theorem 2.6. *For any analytic set $A \subset X$, and for any potential $s = Vh$, where $h \geq 0$ is analytic on X, the reduite R_s^A is analytic.* (X *may be taken Souslin here*).

Proof. As usual now, we may reduce to the case of a bounded Ray resolvent $\mathcal{V}$ on a compact metric space Y, such that $X = D$ — the set of nonbranch points. Let us denote by $\mathcal{S}_c$ the convex cone of (positive) continuous $\mathcal{V}-$supermedian functions on Y. We consider the gambling house J associated as above to this ministable convex cone of positive functions on Y and the corresponding reduite R. It can be easily checked (using the notations from Section 1) that the following mapping

$$(A, f) \;\longrightarrow\; T(A, f) \;\stackrel{\text{def}}{=}\; J^*((Vf)\cdot 1_A) \;=\; \big(\widehat{(Vf)\cdot 1_A}\big)$$

where $A \subset Y$, and $f \geq 0$ on Y, is a bicapacitary operator (see [6, XI, 12] for definition) on $Y \times Y$, (each) Y being endowed with the paving of compact sets). Using now a well−known result of Mokobodzky (also used in [2] and [3]), it follows that if A is compact and contained in D, we have

$$R^A\big(V\big(f|_D\big)\big) \;=\; T(A, f)\big|_D \tag{2.2}$$

at least for any continuous positive function f on Y. But for any fixed $x \in D$, the Radon measure

$$f \;\longrightarrow\; R^A\big(V\big(f\big|_D\big)\big)(x)$$

extends obviously to a capacity on Y (argument function) Hence (2.2) holds also for any upper−semicontinuous positive f on Y. Now, using Choquet capacitability theorem, it follows that (2.2) holds for any analytic set $A \subset D$ $\big($we also have $R^A s(x) = \sup\left\{R^K s(x); K \subset A, K \text{ compact}\right\}$ for any $s \in \mathcal{E}\big)$ and for any analytic $f \geq 0$ on Y. Using [6, XI, 14], we conclude from (2.2) that R_s^A is analytic for $s = Vh$, where $h \geq 0$ is analytic on D. □

Remark 2.7. If s is an arbitrary (univ. measurable) excessive function on X, it follows by Hunt's approximation theorem that $s = \sup_n Vh_n$, where $h_n = n(s \wedge n - nV_n(s \wedge n))$. Therefore if s is borel, it follows from above theorem that R_s^A is analytic ([3]); if s is assumed to be analytic, R_s^A is measurable with respect to the σ-field generated by analytic subsets of X.

Notes 2.8. Proposition 2.1, suggested by [1, Th. 1.2] says that for regular sets the "reduite" and the "balayage" coincide and are somewhat elementary operations, closely connected to the discrete theory. Theorem 2.2 and Proposition 2.5 extend similar results from [2], using a quite different approach. The consideration of the σ-field $\widetilde{\mathcal{E}}$, instead of working with "nearly borel" or "nearly analytic" sets, is justified by the fact that it contains almost all the sets which naturally occur in the theory. Therefore, Theorem 2.6 should be considered a theoretic result; it refines a similar result from [3], where it is established for the case of a borel excessive function.

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